A Direct Resolution of Singularities for Functions of Two Variables with Applications to Analysis

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1. Introduction

In their seminal paper [PS], Phong and Stein consider operators T on $L^2(\mathbf{R})$ of the form

$$Tf(x) = \int_{\mathbf{R}} e^{i\lambda S(x,y)}\phi(x,y)f(y)dy \tag{1.1}$$

Here $S(x,y)$ is a real-analytic function defined in a neighborhood of $(0,0)$, $\phi(x,y)$ is a C^∞ function supported in an appropriately small neighborhood of the origin with $\phi(0,0) \neq 0$, and λ is a real parameter. The goal is to characterize the maximum ϵ such that for some constant C, $||T||_{L^2\rightarrow L^2} < C|\lambda|^{-\epsilon}$ for all λ.

When the second derivative $\partial^2_{xy}S(x,y)$ is nonzero near the origin, ϵ is known to be equal to $\frac{1}{2}$ by a relatively straightforward TT^* argument. Therefore, the situation of interest is the *degenerate* situation, namely when $\partial^2_{xy}S(0,0) = 0$. [PS] deals with the degenerate case by dividing the support of ϕ outside the zeroes of $\partial^2_{xy}S(x,y)$ into rectangular pieces on which $\partial^2_{xy}S(x,y)$ is within a constant factor of a fixed value. By a careful summation, using almost-orthogonality when necessary, the sharp ϵ is determined in terms of the Newton polygon of $\partial^2_{xy}S(x,y)$. A major tool used in [PS], as well as its simplification and elaboration in [Ry], is the Weierstrass preparation theorem (see [W])

Weierstrass Preparation Theorem:
Suppose $f(z_1,z_2)$ is a (complex) analytic function in a neighborhood of $(0,0)$. Then there exists a neighborhood U of $(0,0)$ and an analytic function $h(z_1,z_2)$ defined and nonzero on U, such that on U we have a factorization:

$$f(z_1,z_2) = z_1^k\prod_{i=1}^{q}(z_2 - Y_i(z_1))^{l_i}h(z_1,z_2) \tag{1.2}$$

Here each l_i is an integer and Y_i is an analytic function of $z_1{}^{\frac{1}{M}}$ for some large M. (1.2) can to be taken to hold on any fixed branch of $\log z_1$.

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It is natural to attempt to extend the method of [PS] to higher-dimensional operators, as well as to stability questions relating to the operator T. For example, if we have a 1-parameter family of phase functions $S_t(x,y)$, we might ask how the behavior of the operator T changes with t. Since factorizations like that of (1.2) are restricted to two dimensions and furthermore are not very well-behaved under perturbations of the function $f(x,y)$, it makes sense to try to find a substitute for these Weierstrass factorizations when trying to extend the Phong-Stein method. One natural thing to try to do is to directly resolve the singularities of the phase function S, for such resolutions of singularities are known to exist in all dimensions, such as in the famous work of Hironaka. Furthermore, using direct resolutions of singularities, Varchenko showed in [V] that if one considers the *scalar* analogues of (1.1), namely

$$I_\lambda = \int e^{i\lambda S(x,y)} \phi(x,y) dx\, dy$$

Then the largest ϵ for which we have an estimate of the form

$$|I_\lambda| < C|\lambda|^{-\epsilon}$$

can be expressed in terms of the Newton polygon of S in many situations.

The purpose of this paper is first to define from scratch a direct resolution of the singularities of a locally defined real-analytic function in the plane; this resolution and its properties may be of independent interest. The existence of Puiseux series will be a consequence. Next, complex analytic analogues will be described and the Weierstrass preparation theorem will fall out easily. Lastly, we will use the real-analytic resolution in conjunction with many of the analytic methods of [PS] and its predecessors (e.g. operator Van der Corput lemmas) to reprove the results of [PS]. No facts from algebraic geometry, including the Weierstrass Preparation Theorem, will be quoted; the real-analytic resolution of this paper supplies all the necessary information.

Many of the ideas behind the resolution procedure here are ancient; Isaac Newton himself did some of the important early work (see [BK].) Over the centuries a number of different ways of resolving singularities of plane curves have been developed, by Puiseux, M. Noether, and many others. The resolution procedure here naturally is related to these, but is tailored to analyzing operators such as (1.1). Our arguments are relatively direct and only involve finitely many coordinate changes of the form $(x,y) \rightarrow (x, y - g(x))$. This allows us to break up the support of $\phi(x,y)$ in (1.1), in the *original* coordinates, into finitely many curved triangles (e.g. sets of the form $\{(x,y) : a(x) < y < b(x), 0 < x < x_0\}$ where $a(0) = b(0) = 0$). On each of these curved triangles, $\partial^2_{xy} S(x,y)$ and its first two y-derivatives will satisfy appropriate bounds for using the analytic tools of [PS], again in the *original* coordinates. It is very important that our resolution algorithm give estimates in the original coordinates and not in a transformed coordinate system arising from the resolution process; the almost-orthogonality and operator Van der Corput lemmas used in [PS] and here depend critically on the role played by the coordinate axes. This is in contrast to the scalar oscillatory integral denoted by I_λ above, where the coordinate axes play no special role. This issue is discussed in more depth in [PSSt], where another proof of the main theorem of [PS] (Theorem 1.1 below) is also given.

We now provide some relevant definitions. Suppose $f(x,y)$ is real-analytic in a neighborhood of the origin with Taylor expansion $\sum_{m,n} a_{m,n}x^m y^n$. For each (m,n) in $\mathbf{R}^2$, let $Q_{m,n}$ be the set $\{(x,y) \in \mathbf{R}^2 : x \geq m, y \geq n\}$.

Definition The *Newton polygon* $N(f)$ of f is defined to be the convex hull of the $Q_{m,n}$ for which $a_{m,n} \neq 0$.

Definition The *reduced Newton polygon* $N_r(f)$ of f is defined to be the convex hull of the $Q_{m,n}$ for which $a_{m,n} \neq 0$, $m \neq 0$, and $n \neq 0$.

Newton polygons have been extensively used by algebraic geometers in their study of the zeroes of polynomials in two variables. It is well-known that $N(f)$ (and variants like $N_r(f)$) has finitely many extreme points. As a result, the boundary of $N(f)$ consists of an infinite vertical ray, an infinite horizontal ray, and finitely many (possibly zero) slanted segments between the two rays, the slopes of which get larger (less negative) as one goes counterclockwise from the vertical ray to the horizontal ray of the polygon.

Definition: The *Newton distance* of f is defined to be the least $\delta > 0$ such that $(\delta, \delta) \in N(f)$.

Definition: The *reduced Newton distance* of f is defined to be the least $\delta > 0$ such that $(\delta, \delta) \in N_r(f)$.

Geometrically, the Newton distance of f is the x-coordinate of the intersection of the line $\{y = x\}$ with $N(f)$, and the reduced Newton distance is the x-coordinate of the intersection of the line $\{y = x\}$ with $N_r(f)$.

The main result of [PS] as well as section 5 of this paper is as follows:

Theorem 1.1 Let T be the operator (1.1). Then if ϵ is equal to one-half of the reciprocal of the reduced Newton distance of $\partial^2_{xy}S(x,y)$, we have the sharp estimate

$$||T||_{L^2 \rightarrow L^2} < C|\lambda|^{-\epsilon} \tag{1.3}$$

The organization of this paper is as follows. In section 2 the real-analytic resolution of singularities procedure is defined and some consequences of the resolution are proven. In section 3, the (minor) modificiations needed to resolve the singularities of complex analytic functions are indicated, and the Weierstrass Preparation Theorem will be shown to follow quickly. In section 4, several lemmas concerning the L^2 behavior of oscillatory integral operators are proven. These are very similar to analogous lemmas in [PS], such as their operator Van der Corput lemmas, and have also been influenced to some degree by [PSSt]. In section 5, the results of sections 2 and 4 are combined to reprove the main result of [PS].

2. The Resolution Algorithm

In this section, $f(x, y)$ is assumed to be a real-analytic function of two variables, defined in a neighborhood of $(0, 0)$ such that $f(0, 0) = 0$. The following resolution procedure, as well the lemmas of this section, should be construed to hold in a small disk U containing the origin even if this is not made explicit. This procedure will divide U into "curved triangles" of the form $\{(x, y) \in U : h_1(x) < y < h_2(x)\}$, where $h_1(0) = h_2(0)$, as well as 4 similar curved triangles on either side of the positive and negative y-axis. On each curved triangle, f will be well-behaved enough to apply the lemmas of section 4. To give some idea how to go about this, first write $f(x, y)$'s Taylor expansion as

$$f(x, y) = \sum_{m,n} f_{m,n} x^m y^n \tag{2.1}$$

Suppose $(x, y) \in U$ with $x, y < 1$ is in the upper right quadrant; an analogous discussion is valid for the other three quadrants. Then for $d = \frac{\ln y}{\ln x}$, we have that (x, y) can be written in the form (x, x^d) with $d > 0$. For a given d, we look at the curve C_d, where for some fixed but small x_0 we define C_d by

$$C_d = \{(x, x^d) : 0 < x < x_0\}$$

On the curve C_d, $f(x, y)$ can be written as

$$f(x, y) = f(x, x^d) = \sum_{m,n} f_{m,n} x^{m+nd} \tag{2.2}$$

Hence if e denotes the minimum value of $m + nd$ over all nonzero $f_{m,n}$, then (2.2) becomes

$$f(x, x^d) = \sum_{m+dn=e} f_{m,n} x^e + o(x^e) \tag{2.3}$$

To relate this to the Newton polygon $N(f)$, suppose (a, b) is a vertex of $N(f)$ between two sides of slopes $-\frac{1}{m_i}$ and $-\frac{1}{m_{i+1}}$, where $0 \leq m_i < m_{i+1} \leq \infty$. Then if $m_i < d < m_{i+1}$, the number e of (2.3) is equal to $a + db$, and furthermore there are no (m, n) other than (a, b) for which $m + dn = e$. Thus on C_d, (2.3) becomes

$$f(x, y) = f_{a,b} x^{a+db} + o(x^{a+db}) \tag{2.4}$$

Hence on C_d, the behavior of $f(x, y)$ is well approximated by the monomial $f_{a,b} x^{a+db}$. These heuristics are formalized by the following lemma.

Lemma 2.1: Suppose (a,b) is a vertex of $N(f)$, between edges of slopes $-\frac{1}{m_i}$ and $-\frac{1}{m_{i+1}}$, where $m_i < m_{i+1}$.

If $m_i > 0$ and $m_{i+1} < \infty$ there are δ_i, $M_i > 0$ and constants C, C' such that if V_i is defined by

$$V_i = \{(x,y) \in U : M_i|x|^{m_{i+1}} < |y| < \delta_i|x|^{m_i}\} \tag{2.5}$$

Then for $(x,y) \in V_i$ we have the relation

$$C|f_{a,b}x^a y^b| < |f(x,y)| < C'|f_{a,b}x^a y^b| \tag{2.6a}$$

Also, if α, $\beta \leq 2$, we have

$$|\partial_x^\alpha \partial_y^\beta f(x,y)| < C''|f_{a,b}x^{a-\alpha}y^{b-\beta}| \tag{2.6b}$$

In the case that $m_i = 0$, $(2.6a) - (2.6b)$ holds if we define V_i by

$$V_i = \{(x,y) \in U : M_i|x|^{m_{i+1}} < |y|\}$$

In the case that $m_{i+1} = \infty$, $(2.6a) - (2.6b)$ holds with

$$V_i = \{(x,y) \in U : |y| < \delta_i|x|^{m_i}\}$$

If both $m_i = 0$ and $m_{i+1} = \infty$, then V_i can be taken to be all of U.

Proof: We assume $x > 0$, as the case $x < 0$ is done exactly the same way. We write $f(x,y) = S_1 + S_2 + S_3$, where

$$S_1 = \frac{1}{2}f_{a,b}x^a y^b + \sum_{m \leq a,\ n > b} f_{m,n}x^m y^n$$

$$S_2 = \frac{1}{2}f_{a,b}x^a y^b + \sum_{n \leq b,\ m > a} f_{m,n}x^m y^n$$

$$S_3 = \sum_{m > a,\ n > b} f_{m,n}x^m y^n$$

In S_1, we change coordinates $y = x^{m_i}y'$. The condition $|y| < \delta_i|x|^{m_i}$ becomes $|y'| < \delta_i$, and S_1 becomes

$$S_1 = x^{a+m_i b}[\frac{1}{2}f_{a,b}(y')^b + O((y')^{b+1})] + O(x^{a+m_i b+\eta}(y')^{b+1}) \tag{2.7}$$

Here $\eta = \min(m_i, 1) > 0$. Hence if δ_i sufficiently small to deal with the $O((y')^{b+1})$ and $O(x^{a+m_i b+\eta}(y')^{b+1})$ terms, we have

$$|S_1 - \frac{1}{2}f_{a,b}x^a y^b| < \frac{1}{6}|f_{a,b}x^a y^b| \tag{2.8}$$

Similarly, if we exchange the roles of the x and y axes and M_i is large enough, we have that

$$|S_2 - \frac{1}{2}f_{a,b}x^a y^b| < \frac{1}{6}|f_{a,b}x^a y^b| \tag{2.9}$$

As for S_3, each term $f_{m,n}x^m y^n$ appearing there satisfies $m > a$ and $n > b$. So if x and y are small enough, which we may assume, shrinking U if necessary, we have that

$$|S_3| < \frac{1}{6}|f_{a,b}x^a y^b| \tag{2.10}$$

Combining (2.8)-(2.10) gives (2.6a). As for (2.6b), one applies the above argument to $\partial_x^\alpha \partial_y^\beta f(x,y)$ in place of $f(x,y)$. If $\alpha \leq a$ and $\beta \leq b$ the exact same argument works. Otherwise, (2.10) holds as before, while in place of (2.8) – (2.9) one has

$$|S_1| < \frac{1}{6}|f_{a,b}x^{a-\alpha}y^{b-\beta}| \tag{2.8$'$}$$

$$|S_2| < \frac{1}{6}|f_{a,b}x^{a-\alpha}y^{b-\beta}| \tag{2.9$'$}$$

Adding (2.8$'$), (2.9$'$) and (2.10) gives (2.6b). This completes the proof of Lemma 2.1.

In view of Lemma 2.1, the regions V_i define a collection of curved triangles on which f is well-behaved. Each V_i corresponds to four such triangles, one in each quadrant. We will now divide $U - \cup_i V_i$ into more curved triangles. To motivate how we will do this, observe that each (x,y) in $U - \cup_i V_i$ satisfies $\delta_i |x|^{m_i} < |y| < M_{i-1}|x|^{m_i}$ for some $0 < m_i < \infty$. Thus it makes sense to examine the behavior of $f(x,y)$ on curves D_i^c, where for some small x_0

$$D_i^c = \{(x, cx^{m_i}) : 0 < x < x_0\}$$

On the curve D_i^c, if e_i denotes the minimal value of $m + nm_i$ over all (m,n) with $f_{m,n} \neq 0$, f satisfies

$$f(x,y) = f(x, cx^{m_i}) = \sum_{m+nm_i=e_i} (f_{m,n}c^n)x^{e_i} + o(x^{e_i})$$

Hence so long as c is not a root of the polynomial $g_i = \sum_{m+nm_i=e_i} f_{m,n}c^n$, $f(x,y)$ is comparable to x^{e_i} on the curve D_i^c for small values of x. In this case $f(x,y)$ can be viewed as staying away from its zeroes on the curves D_i^c, and being of roughly maximal magnitude. This suggests dividing the portion of $U - \cup_i V_i$ where $x > 0$ into several large curved triangles away from the D_i^c for which c is a root of g_i, and several small curved triangles where f is small and which we will further analyze later.

Making these heuristics precise, we define

$$T_i = \{(x,y) \in U - \cup_i V_i : \delta_i |x|^{m_i} < |y| < M_{i-1}|x|^{m_i}\}$$

Observe that $\cup_i T_i = U - \cup_i V_i$. We will now divide the portion of a given T_i on which $x > 0$ into curved triangles; the $x < 0$ portion is done the same way with $f(x,y)$ replaced by $f(-x,y)$ and $g_i(c)$ replaced by $\sum_{m+nm_i=e_i} f_{m,n}(-1)^m c^n$.

If g_i has no roots, we don't subdivide T_i at all and define W_i^1 to be T_i. Otherwise, let c_i^0,...,c_i^l denote the roots of $g_i(c)$. For each j, we let $[r_j, r'_j]$ be an interval with c_i^j in its interior, such that neither g_i nor any of its derivatives has any zeroes in $[r_j, r'_j]$ other than c_i^j. Define X_i^j by

$$X_i^j = \{(x, y) : r_j x^{m_i} \leq |y| \leq r'_j x^{m_i}\}$$

Then the part of $T_i - \cup_j X_i^j$ where $x > 0$ can be written as the union of regions W_i^j of the form

$$W_i^j = \{(x, y) : s_j x^{m_i} < |y| < s'_j x^{m_i}\}$$

Confirming the heuristics that $f(x, y)$ should be roughly of magnitude x^e on W_i^j is the following lemma.

Lemma 2.2: Let (a, b) be either vertex of the Newton polygon $N(f)$ on a bounded edge of slope $-\frac{1}{m_i}$. Then there exist constants C and C' such that for each j, each (x, y) in W_i^j we have

$$C|x^a y^b| < |f(x, y)| < C'|x^a y^b| \tag{2.11a}$$

Furthermore, if $|\alpha|, |\beta| < 2$, we have

$$|\partial_x^\alpha \partial_y^\beta f(x, y)| < C'|x^{a-\alpha} y^{b-\beta}| \tag{2.11b}$$

Proof: We do the variable change $y = x^{m_i} y'$ on W_i^j. In the new coordinates W_i^j is the portion of U where $s_j < y' < s'_j$, and we have the Taylor expansion

$$f(x, y) = f(x, x^{m_i} y') = g_i(y') x^e + o(x^e) \tag{2.12}$$

By definition of W_i^j, $g_i(y')$ does not have any zeroes on $s_j \leq y' \leq s'_j$, so in particular there is some $\epsilon > 0$ such that $|g_i(y')| > \epsilon$ on $[s_j, s'_j]$. Thus by (2.12), if $|x|$ is sufficiently small, which we may assume, then for some constants δ and δ' we have

$$\delta x^e < |f(x, y)| < \delta' x^e \tag{2.13}$$

By the definition of W_i^j, for any (a, b) with $a + m_i b = e$ we have $x^a y^b$ is within a constant factor of x^e on W_i^j, and thus (2.13) implies (2.11a). (2.11b) is proved the same way as (2.11a), applied to $\partial_x^\alpha \partial_y^\beta f(x, y)$. The only possible difference is that the analog to $g_i(y')x^e$ might be have zeroes or even be a zero term; however since (2.11b) only requires upper bounds this will not cause any problems. This completes the proof of Lemma 2.2.

We have now divided U into various curved triangles V_i and W_i^j where the behavior of $f(x, y)$ is well-controlled by Lemmas 2.1 and 2.2, as well as some curved triangles X_i^j which contain all possible zeroes of $f(x, y)$. Consider a given X_i^j. By the heuristics above, we expect any zero of $f(x, y)$ with $x > 0$ to be located near the curve $\{(x, y) : 0 < x < x_0,\ y = c_i^j x^{m_i}\}$. The traditional way of further analyzing the zeroes near this curve would be to do a coordinate change $y' = y - c_i^j x^{m_i}$, and to consider the function

$f(x,y)$ in the (x,y') coordinates, namely $\tilde{f}(x,y') = f(x, y - c_i^j x^{m_i})$. Then the zeroes of $\tilde{f}$ lie near $y' = 0$, and thus doing the previous analysis on $\tilde{f}(x,y')$ in place of $f(x,y)$ would further specify the location of the zeroes of $f(x,y)$, using the Newton polygon of $\tilde{f}(x,y')$. (The vertices of the Newton polygon do not have to be integers for the above analysis to work.) We could then do the procedure on the resulting analogs to the X_i^j, and so on, ad infinitum. At the kth stage of this procedure, we would be in $(x, y^{(k)})$ coordinates, where $(x, y^{(k)}) = (x, y - p_k(x^{\frac{1}{N_k}}))$, p_k being a polynomial and N_k a large integer. The end result of all of this would be an infinite collection of curved triangles, and to do the type of analysis we will do in later sections, we would have to keep very good track of all constants appearing in the various stages and hope to be able to add up the resulting estimates. Thus there would be a significant advantage in trying to determine an iteration scheme that converges in finitely many steps. This in fact can be done; the key is instead of doing the coordinate change $y' = y - c_i^j x^{m_i}$, one must do a coordinate change of the form $y' = y - c_i^j x^{m_i} - q_{ij}(x) x^{m_i + \mu}$, where $\mu > 0$ and $q_{ij}(x)$ is a particular real-analytic function of $x^{\frac{1}{M}}$ for some large M. To see how one chooses this $q_{ij}(x)$, we use the following lemma. An analogous lemma will hold for $x < 0$.

Lemma 2.3: Let (a,b) and (a',b') be the vertices of a bounded edge of $N(f)$ of slope $-\frac{1}{m_i}$, with $b < b'$ and $a > a'$. Denote the order of a zero c_i^j of $g_i(c)$ by q_i^j.

1) $q_i^j \leq b' - b$ and $q_i^j m_i \leq e_i$ There are constants C and C' such that on the set X_i^j we have

$$C x^{e_i - q_i^j m_i} < |\partial_y^{q_i^j} f(x,y)| < C' x^{e_i - q_i^j m_i} \tag{2.14}$$

Furthermore q_i^j is the minimal value of q for which an identity of the following form holds on X_i^j for any $f \geq 0$:

$$C x^f < |\partial_y^q f(x,y)| < C' x^f \tag{2.14$'$}$$

2) For each sufficiently small x, there is a unique $h_i^j(x)$ with $\partial_y^{q_i^j - 1} f(x, h_i^j(x)) = 0$. The function $h_i^j(x)$ is a real-analytic function of $x^{\frac{1}{b'-b}}$ whose leading term is given by $c_i^j x^{m_i}$. In addition, on X_i^j we have

$$C x^{e_i - q_i^j m_i} |y - h_i^j(x)| < |\partial_y^{q_i^j - 1} f(x,y)| < C' x^{e_i - q_i^j m_i} |y - h_i^j(x)| \tag{2.15}$$

Proof: The highest and lowest powers of c appearing in $g_i(c)$ are $c^{b'}$ and c^b respectively, so any zero of $g_i(c)$ other than $c = 0$ must be of order at most $b' - b$. Thus each q_i^j is at most $b' - b$. Therefore we have

$$m_i q_i^j \leq m_i(b' - b) \leq m_i b' \leq a' + m_i b' = e_i$$

Next, we do the coordinate change $(x,y) = (x, x^{m_i} y')$. Then for some constant $\sigma \neq 0$ we have

$$f(x,y) = f(x, x^{m_i} y') = x^{e_i}[\sigma(y' - c_i^j)^{q_i^j} + O(y' - c_i^j)^{q_i^j + 1}] + o(x^{e_i}) \tag{2.15}$$

If $q \leq q_i^j$, taking y derivatives of $f(x,y)$ q times corresponds to taking y' derivatives of $f(x, x^{m_i}y')$ q times and multiplying the result by $x^{-m_i q}$. Thus we have for some constant $\rho_q \neq 0$

$$\partial_y^q f(x,y) = x^{e_i - qm_i}[\rho_q(y' - c_i^j)^{q_i^j - q} + O(y' - c_i^j)^{q_i^j - q + 1}] + o(x^{e_i - qm_i}) \tag{2.16}$$

In particular, if we let $q = q_i^j$, we get

$$\partial_y^{q_i^j} f(x,y) = x^{e_i - q_i^j m_i}[\rho_{q_i^j} + O(y' - c_i^j)] + o(x^{e_i - q_i^j m_i}) \tag{2.17}$$

We can assume that X_i^j was chosen small enough that the $O(y' - c_i^j)$ term is dominated by the constant $\rho_{q_i^j}$, and that x is small enough that the $o(x^{e_i - q_i^j m_i})$ term suffers the same fate. This gives (2.14).

Furthermore, we cannot have any relation (2.14′) for $q < q_i^j$. For in this case, (2.16) tells us that on the line $y' = c_i^j$, $f(x,y)$ is $o(x^{e_i - qm_i})$, while on any other line $y' = c$, there is some $\delta > 0$ with $|f(x,y)| > \delta x^{e_i - qm_i}$ if x is sufficiently small. This completes the proof of the first statement of this lemma.

Moving to the second statement, we examine (2.16) for $q = q_i^j - 1$. The function $p(x,y') = x^{-e_i}\partial_{y'}^{q_i^j - 1} f(x, x^{m_i}y')$ has a zero at $(x,y') = (0, c_i^j)$, while there is a $\delta > 0$ such that $|\partial_{y'} p(x,y')| > \delta$. Hence by the implicit function theorem, there is a function $H_i^j(x)$ such that $p(x, H_i^j(x)) = 0$ if x is sufficiently small. Furthermore, we have that $m_i = \frac{a' - a}{b' - b}$, so $p(x,y')$ is a real-analytic function of $x^{\frac{1}{b'-b}}$ and y. As a result, $H_i^j(x)$ is a real-analytic function of $x^{\frac{1}{b'-b}}$ with leading term c_i^j. Defining $h_i^j(x) = x^{m_i}H_i^j(x)$, we have that $h_i^j(x)$ is a real-analytic function of $x^{\frac{1}{b'-b}}$ with leading term $c_i^j x^{m_i}$, such that $\partial_y^{q_i^j - 1} f(x, h_i^j(x)) = 0$. (2.15) now follows directly from integrating (2.14) in the y-variable. This completes the proof of Lemma 2.3.

We are now in a position to define the full decomposition of U into finitely many curved triangles on which $f(x,y)$ is well-behaved.

Decomposition of U into curved triangles

We start with our small neighborhood U of the origin, and we let V_i, W_i^j, and X_i^j be as in the above discussion. Lemmas 2.1 and 2.2 give all the information we need concerning f's behavior on the the V_i and W_i^j, so we do no further subdivisions on them. We use the notation $(V_i)_1 = V_i$ and $(W_i^j)_1 = W_i^j$ to signify that the V_i and W_i^j come from the first stage of the resolution process.

Next, on a given X_i^j, we make the variable change $(x,y') = (x, y - h_i^j(x))$, creating a region $\tilde{X}_i^j$ in the new coordinates. The function $f(x,y)$ becomes $\tilde{f}(x,y') = f(x, y - h_i^j(x))$.

We next create the V_k, W_k^l, and X_k^l corresponding to the function $\tilde{f}$, which we denote by $(V_k)_2$, $(W_k^l)_2$, and $(X_k^l)_2$ to signify that they arise in the second stage of the resolution process. We continue to iterate the procedure until we get to a stage p where no $(X_k^l)_p$ are created; we will see by Lemma 2.4 that such a stage p exists. As a result, we have divided U into a collection of $(V_k)_n$ and $(W_k^l)_n$ on which the behavior of $f(x,y)$ is well-described by Lemmas 2.1 and 2.2. Observe that by Lemmas 2.1 and 2.2, $f(x,y)$ can not have any zeroes in the closure of a $(V_k)_n$ or a $(W_k^l)_n$ corresponding to a $0 < m_i < \infty$ in its derived coordinates, except at $(0,0)$. As a result, any possible zeroes of $f(x,y)$ must be either on the $y-$axis, or along the curve corresponding to the x-axis in some nth stage. Note that this fact implies the existence of Puiseux series.

We now prove the resolution takes finitely many stages. There is a corresponding lemma for $x < 0$.

Lemma 2.4: Let p be the maximum of q_i^j over all i and j. Then the above-defined resolution procedure terminates in at most $n = p + 1$ steps.

Proof: Let $(X_k^l)_n$ and $(X_{k'}^{l'})_{n+1}$ be such that $(X_{k'}^{l'})_{n+1} \subset (X_k^l)_n$. Let $q_{k'}^{l'}$ and q_k^l be orders of the relevant zeroes of the corresponding polynomials $g_{k'}^{l'}$ and g_k^l. Then it suffices to show that $q_{k'}^{l'} < q_k^l$. Let $(x,\tilde{y})$ be the coordinates after the nth stage y-shift is performed on $(X_k^l)_n$, and let $\tilde{f}(x,\tilde{y})$ be f in the transformed coordinates. Then by Lemma 2.3, part 2, there are constants C, $C' > 0$ such that for some $e > 0$ we have

$$Cx^e|\tilde{y}| < |\partial_{\tilde{y}}^{q_k^l - 1}\tilde{f}(x,\tilde{y})| < C'x^e|\tilde{y}| \tag{2.18}$$

By definition of $(X_{k'}^{l'})_{n+1}$, there are constants $C'', C''', m > 0$ such that for any $(x,\tilde{y})$ in $(X_{k'}^{l'})_{n+1}$ we have

$$C''x^m < |\tilde{y}| < C'''x^m \tag{2.19}$$

As a result, on $(X_{k'}^{l'})_{n+1}$, we have

$$CC''x^{e+m} < |\partial_{\tilde{y}}^{q_k^l - 1}\tilde{f}(x,\tilde{y})| < C'C'''x^{e+m} \tag{2.20}$$

By the minimality condition of Lemma 2.3, part 1, we therefore have that $q_{k'}^{l'} \leq q_k^l - 1$. So $q_{k'}^{l'} < q_k^l$ and we are done with the proof of Lemma 2.4.

We have now proved everything we need from the resolution procedure. We summarize what we need in the following. The natural analogues will hold for $x < 0$.

Theorem 2.1: Let $f(x,y)$, U, $(V_i)_n$, and $(W_i^j)_n$ be as in the above arguments. For a function $p(x)$ defined in an interval $I = [0,\gamma)$, let $z(p)$ denote the greatest z for which $|p(x)| < Cx^z$ on I. If such a C exists for any z, let $z(p) = \infty$.

Suppose $n > 1$. Then there exists a collection of functions $r_{k_1}(x)$, $r_{k_1,k_2}(x)$,..., $r_{k_1,\ldots,k_l}(x)$, $s_{k_1}(x)$, $s_{k_1,k_2}(x)$,..., $s_{k_1,\ldots,k_l}(x)$, $t_{k_1}(x)$, $t_{k_1,k_2}(x)$,...,$t_{k_1,\ldots,k_l}(x)$, each defined on an interval

$I = [0, \gamma)$, such that each $Z = (V_i)_n$ or $(W_i^j)_n$ is expressible in the form

$$Z = \{(x,y) \in U : \sum_{m=1}^{n-1} r_{k_1,...,k_m}(x) + s_{k_1,...,k_{n-1}}(x) < y < \sum_{m=1}^{n-1} r_{k_1,...,k_m}(x) + t_{k_1,...,k_{n-1}}(x)\}$$

Here l is at most the p in lemma 2.4, and each k_n varies over an index set $\{1, ..., M_n\}$ for some M_n. These functions satisfy the following equations for each m.

$$z(r_{k_1}) < z(r_{k_1,k_2}) < ... < z(r_{k_1,...,k_l})$$

$$z(s_{k_1,...,k_m}),\ z(t_{k_1,...,k_m}) \geq z(r_{k_1,...,k_m}) \tag{2.21}$$

In addition, there is a positive integer M such that each $r_{k_1,...,k_m}(x)$, $s_{k_1,...,k_l}(x)$, and $t_{k_1,...,k_l}(x)$ is a real analytic function of $x^{\frac{1}{M}}$ on its respective domain. In fact, each $s_{k_1,...,k_l}(x)$ and $t_{k_1,...,k_l}(x)$ is a monomial.

Each r_{k_1} satisfies $z(r_{k_1}) = m_i$, where $-\frac{1}{m_i}$ is the slope of a bounded edge of f's Newton polygon. In fact, $r_{k_1}(x)$ is of the form $c_i^j x^{m_i} + o(x^{m_i})$, where c_i^j is a zero of the function $g_i(c)$.

On Z, the function $f(x,y)$ satisfies the following equation, where (a,b) is a vertex of the Newton polygon of $\tilde{f}(x,\tilde{y}) = f(x, y - \sum_{m=1}^{n-1} r_{k_1,...,k_m}(x))$ on an edge of slope $-\frac{1}{z(t_{k_1,...,k_m})}$ and an edge of slope $-\frac{1}{z(s_{k_1,...,k_m})}$ (these may be the same edge since $z(t_{k_1,...,k_m})$ can equal $z(s_{k_1,...,k_m})$.)

$$C|x|^a|y - \sum_{m=1}^{n-1} r_{k_1,...,k_m}(x)|^b < |f(x,y)| < C'|x|^a|y - \sum_{m=1}^{n-1} r_{k_1,...,k_m}(x)|^b \tag{2.22a}$$

If $\beta \leq 2$, we also have

$$|\partial_y^\beta f(x,y)| < C''|x|^a|y - \sum_{m=1}^{n-1} r_{k_1,...,k_m}(x)|^{b-\beta} \tag{2.22b}$$

The functions $s_{k_1,...,k_{n-1}}(x)$ or $t_{k_1,...,k_{n-1}}(x)$ in (2.22a)-(2.22b) are allowed to be the zero function. In fact, the zeroes of $f(x,y)$ for $x > 0$ not on the x axis comprise a subset of the curves

$$\{(x,y) : y = \sum_{m=1}^{n-1} r_{k_1,...,k_m}(x)\} \tag{2.23}$$

Proof: Everything above follows directly from the resolution procedure defined here and Lemmas 2.1 - 2.2. The details are omitted. It should be pointed out that with an additional argument, one can show that each $r_{k_1,...,k_m}(x)$, $s_{k_1,...,k_m}(x)$ and $t_{k_1,...,k_m}(x)$ are real-analytic functions of $x^{\frac{1}{(b'-b)!}}$, where (a,b) and (a',b') are the vertices of the edge of the Newton polygon of f of slope $-\frac{1}{m_i}$ such that $r_{k_1}(x) = c_i^j x^{m_i} + o(x^{m_i})$. So for example, if $b' - b = 1$, $r_{k_1,\ldots,k_m}(x)$, $s_{k_1,\ldots,k_m}(x)$ and $t_{k_1,\ldots,k_m}(x)$ become real-analytic functions of x, regardless of how far into the resolution procedure we are.

We also need the following lemma for the arguments of section 5.

Lemma 2.5: Let $\tilde{f}(x, \tilde{y}) = f(x, y - \sum_{m=1}^{n-1} r_{k_1,\ldots,k_m}(x))$ be as in Theorem 2.1. Suppose the edge of the Newton polygon of $f(x,y)$ with slope $-\frac{1}{m_i}$ has equation $\frac{x}{m_i} + y = e$. Then any edge of the Newton polygon of $\tilde{f}$ of slope $-\frac{1}{t}$ with $t \geq m_i$ has equation of the form $\frac{x}{t} + y = e'$, where $e' \leq e$.

Proof: Each term of $\sum_{m=1}^{n-1} r_{k_1,\ldots,k_m}(x)$ is of the form cx^d with $d \geq m_i$, so since any vertex (p,q) of the Newton polygon $N(f)$ of f satisfies $\frac{p}{m_i} + q \geq e$, any vertex (p', q') of the Newton polygon $N(\tilde{f})$ of $\tilde{f}$ will satisfy $\frac{p'}{m_i} + q' \geq e$ as well. Furthermore, if (p,q) is the left vertex of the edge of slope $-\frac{1}{m_i}$ of $N(f)$ then all edges of $N(f)$ to the left of (p,q) are edges of $N(\tilde{f})$ and vice versa. In addition, (p,q) is the left vertex of an edge of $N(\tilde{f})$ of slope $-\frac{1}{m'}$ where $m' \geq m$. As a result, $\frac{p}{m'} + q \leq e$. Starting with the edge of $N(\tilde{f})$ of slope $-\frac{1}{m'}$ and moving counterclockwise, the y-intercepts of the lines containing the edges of $N(\tilde{f})$ decrease, and the lemma follows.

3. Complex-analytic analogues, proof of Weierstrass Preparation Theorem

When $f(z_1, z_2)$ is a complex-analytic function of two variables in a neighborhood U of the origin in $\mathbf{C}^2$ with $f(0,0) = 0$, the arguments of the previous section hold with little change. We pick some θ_0, and let

$$U' = \{(z_1, z_2) \in U : \arg(z_1) \neq \theta_0\}$$

Fractional powers of z_1 are well-defined on U', and we can apply the arguments of section 2 on U'. There now only needs to be one W_i^j for a given i, which we denote by W_i. The regions V_i and W_i are now

$$V_i = \{(z_1, z_2) \in U' : M_i |z_1|^{m_{i+1}} < |z_2| < \delta_i |z_1|^{m_i}\}$$

$$W_i = \{(z_1, z_2) \in U' : \delta_i |z_1|^{m_i} < |z_2| < M_{i-1} |z_1|^{m_i},\ \ |z_2 - c_i^j z_1^{m_i}| > \epsilon_i |z_1|^{m_i}\ \ \forall j\}$$

Here c_i^j are roots of a polynomial $g_i(z)$ analogous to the g_i^j from before. Theorem 2.1 still holds in the current setting, with the functions $r_{k_1,\ldots,k_m}(z_1)$, $s_{k_1,\ldots,k_m}(z_1)$, and $t_{k_1,\ldots,k_m}(z_1)$ now being analytic functions of $z_1^{\frac{1}{M}}$, defined for $\arg(z_1) \neq \theta_0$. In particular one obtains the Puiseux expansions in the complex analytic setting as well.

How to go from Puiseux's theorem to the Weierstrass Preparation Theorem was essentially already known by Newton (see [BK]), and the same sort of argument will work here.

Proof of Weierstrass Preparation Theorem: Let $f(z_1, z_2)$ and U' be as above; we will factorize $f(z_1, z_2)$ on U' in the required fashion. We first factor out all the powers of z_1 we can, writing $f(z_1, z_2) = z_1^k F(z_1, z_2)$. As a result $F(0, z_2)$ has at most an isolated zero at $z_2 = 0$; otherwise we would be able to factor out an additional z_1 as can be seen from the fact that $F(z_1, z_2) = F(0, z_2) + z_1 F_1(z_1, z_2)$ for some analytic F_1. We now apply the resolution procedure to $F(z_1, z_2)$. By section 2, the zeroes of $F(z_1, z_2)$ not on $z_2 = 0$ are curves of the following form

$$\{(z_1, z_2) \in U' : z_2 = \sum_{m=1}^{q} r_{k_1,\ldots,k_m}(z_1)\} \tag{3.1}$$

In the coordinates $(z_1, \tilde{z_2}) = (z_1, z_2 - \sum_{m=1}^{q} r_{k_1,\ldots,k_m}(z_1))$ of the corresponding V_i, the function $\tilde{F}(z_1, \tilde{z_2}) = F(z_1, z_2 - \sum_{m=1}^{q} r_{k_1,\ldots,k_m}(z_1))$ can be factored as

$$\tilde{F}(z_1, \tilde{z_2}) = \tilde{z_2}^{\,l} \tilde{F}_1(z_1, \tilde{z_2}) \tag{3.2}$$

If the number l in (3.2) is chosen to be maximal, and U is appropriately small, then $\tilde{F}_1(z_1, 0)$ has at most one zero, at $z_1 = 0$. Translating in terms of $F(z_1, z_2)$, we have the factorization

$$F(z_1, z_2) = (z_2 - \sum_{m=1}^{q} r_{k_1,\ldots,k_m}(z_1))^l F_1(z_1, z_2) \tag{3.3}$$

Here $F_1(z_1, z_2)$ is an analytic function of $z_1^{\frac{1}{M}}$ and z_2, and has no zeroes along the curve (3.1) except possibly the origin. We can then iterate the argument leading to (3.3) since the resolution procedure applies to analytic functions of z_2 and a fractional power of z_1 exactly as it does for analytic functions of z_2 and z_1. In this fashion we can remove all the zeroes of $F(z_1, z_2)$ not on $\{z_2 = 0\}$. After possibly factoring a final factor of $z_2^{k'}$, we obtain a factorization of the form

$$f(z_1, z_2) = z_1^k F(z_1, z_2) =$$

$$z_1^k \prod_{i=1}^{q} (z_2 - Y_i(z_1))^{l_i} h(z_1, z_2) \tag{3.4}$$

Here $Y_i(z_1)$ are analytic functions of $z_1^{\frac{1}{M}}$ and $h(z_1, z_2)$ is an analytic function of $z_1^{\frac{1}{M}}$ and z_2 with at most an isolated zero at the origin. As a reader versed in several complex variables might realize, $h(z_1, z_2)$ should not be able to have any isolated zeroes. To see why, suppose $h(z_1, z_2)$ did have an isolated zero at $(0, 0)$. Then since h is analytic in the second variable, by Cauchy's integral formula we have that the following expression is a nonzero integer, for a small $\epsilon > 0$:

$$\frac{1}{2\pi i} \int_{|z_2|=\epsilon} \frac{\frac{\partial h}{\partial z_2}}{h}(0, z_2)\; dz_2 \tag{3.5}$$

But then by continuity, if $|z_1| \neq 0$ is small enough, the same nonzero integer would equal

$$\frac{1}{2\pi i} \int_{|z_2|=\epsilon} \frac{\frac{\partial h}{\partial z_2}}{h}(z_1, z_2)\; dz_2 \tag{3.6}$$

This would imply that $h(z_1, z_2)$ has zeroes where $z_1 \neq 0$, a contradiction. We conclude that $h(z_1, z_2)$ has no zeroes at all in a small enough neighborhood of the origin.

To complete this proof of the Weierstrass Preparation Theorem, we must show that $h(z_1, z_2)$ is an analytic function of z_1 and z_2, not just an analytic function of $z_1^{\frac{1}{M}}$ and z_2. To this end, we write $h(z_1, z_2) = H(z_1^{\frac{1}{M}}, z_2)$, and $Y_i(z_1) = Z_i(z_1^{\frac{1}{M}})$, with H and Z_i analytic. Let (z_1, z_2) be in U, and write $z_1 = \rho e^{i\phi}$. We may take $\theta_0 = \phi$ in the definition of U' when obtaining (3.4). In the formula (3.4), we move counterclockwise around the circle $\{(z_1, z_2) : |z_1| = \rho\}$, going from $\theta = \phi + \delta$ to $\theta = \phi - \delta$. Letting δ go to zero, by continuity of (3.4) we have

$$\prod_{i=1}^{q}(z_2 - Z_i(z_1^{\frac{1}{M}}))^{l_i} H(z_1^{\frac{1}{M}}, z_2) = \prod_{i=1}^{q}(z_2 - Z_i(e^{\frac{2\pi i}{M}} z_1^{\frac{1}{M}}))^{l_i} H(e^{\frac{2\pi i}{M}} z_1^{\frac{1}{M}}, z_2) \tag{3.7}$$

Thus the functions $Z_i(z_1^{\frac{1}{M}})$ are a permutation of the functions $Z_i(e^{\frac{2\pi i}{M}} z_1^{\frac{1}{M}})$, and we have

$$H(z_1^{\frac{1}{M}}, z_2) = H(e^{\frac{2\pi i}{M}} z_1^{\frac{1}{M}}, z_2) \tag{3.8}$$

Writing the Taylor expansion of $H(w_1, w_2)$ as $\sum_{m,n} H_{mn} w_1^m w_2^n$, by (3.8) we have for each m and n:

$$e^{\frac{2\pi m i}{M}} H_{mn} = H_{mn} \tag{3.9}$$

As a result, $H_{mn} = 0$ unless m is a multiple of M. Equivalently, $h(z_1, z_2) = H(z_1^{\frac{1}{M}}, z_2)$ is an analytic function of z_1 and z_2. This completes this proof of the Weierstrass Preparation Theorem.

4. Some analytic lemmas

In this section we prove three analytic lemmas needed for the arguments of section 5. The first is a variant of Schur's test.

Lemma 4.1: Suppose R is an operator defined by $Rf(x) = \int f(y)K(x,y)dy$, where $K(x,y)$ is supported in an a by b rectangle and $|K(x,y)| \leq 1$. Then $||R||_{L^2 \to L^2} \leq (ab)^{\frac{1}{2}}$.

Proof: $||R||_{L^2 \to L^2}$ is given by

$$\sup_{||f||_2, ||g||_2 \leq 1} \int K(x,y)g(x)f(y)dx\, dy \tag{4.0}$$

Using that $|g(x)f(y)| \leq \frac{(\frac{a}{b})^{\frac{1}{2}}g(x)^2+(\frac{b}{a})^{\frac{1}{2}}f(y)^2}{2}$ and that $|K(x,y)| \leq 1$, (4.0) is at most

$$\sup_{||f||_2,||g||_2\leq 1}\int \frac{(\frac{a}{b})^{\frac{1}{2}}g(x)^2+(\frac{b}{a})^{\frac{1}{2}}f(y)^2}{2}dx\,dy$$

$$= \sup_{||g||_2\leq 1}\int \frac{(ab)^{\frac{1}{2}}}{2}g(x)^2dx + \sup_{||f||_2\leq 1}\int \frac{(ab)^{\frac{1}{2}}}{2}f(y)^2dy$$

$$= (ab)^{\frac{1}{2}}$$

This completes the proof of Lemma 4.1.

The next lemma is a straightforward orthogonality argument.

Lemma 4.2: Suppose $R = \sum_i R_i$ is an operator, such that each R_i can be written as $R_if(x) = \int f(y)K_i(x,y)\,dy$, where $K_i(x,y)$ is supported on a product of intervals $I_i \times J_i$. Further suppose that if $i \neq j$, $I_i \cap I_j = J_i \cap J_j = \emptyset$. Then $||R||_{L^2\to L^2} = \sup_i ||R_i||_{L^2\to L^2}$.

Proof: $R_iR_j^*$ has kernel $L(x_1,x_2) = \int K_i(x_1,y)K_j(x_2,y)\,dy = 0$ since $I_i \cap I_j = 0$. Similarly, $R_i^*R_j$ has kernel $M(y_1,y_2) = \int K_i(x,y_1)K_j(x,y_2)\,dx = 0$ since $J_i \cap J_j = 0$. So $||R_iR_j^*||_{L^2\to L^2} = ||R_i^*R_j||_{L^2\to L^2} = 0$, and the lemma now follows from the Cotlar-Stein almost-orthogonality lemma, for example.

The final lemma is an appropriate "operator Van der Corput lemma" (see [PS] and [PS2]).

Lemma 4.3: Suppose $S(x,y)$ and $\chi(x,y)$ are smooth functions on an open set U, with $\chi(x,y)$ supported on an a by b rectangle. Assume for a constant A that $\chi(x,y)$ satisfies

$$0 \leq |\chi(x,y)| \leq 1,\ \ |\partial_y\chi(x,y)| < \frac{A}{b},\ \ |\partial_y^2\chi(x,y)| < \frac{A}{b^2} \tag{4.1}$$

Also assume that for some $\mu > 0$, $S(x,y)$ satisfies

$$|\frac{\partial^2 S}{\partial x\partial y}(x,y)| \geq \mu,\ \ |\frac{\partial^3 S}{\partial x\partial^2 y}(x,y)| < \frac{A\mu}{b},\ \ |\frac{\partial^4 S}{\partial x\partial^3 y}(x,y)| < \frac{A\mu}{b^2} \tag{4.2}$$

For a real parameter λ, define the operator R by

$$Rf(x) = \int e^{i\lambda S(x,y)}\chi(x,y)f(y)dy \tag{4.3}$$

Then for a constant C depending only on A, we have

$$||R||_{L^2\to L^2} < C(\lambda\mu)^{-\frac{1}{2}} \tag{4.4}$$

Proof: Denote the kernel of RR^* by $P(x_1, x_2)$. Then

$$P(x_1, x_2) = \int e^{i\lambda S(x_1,y) - i\lambda S(x_2,y)} \chi(x_1, y)\chi(x_2, y) dy \tag{4.5}$$

We now integrate (4.5) by parts, writing

$$e^{i\lambda S(x_1,y) - i\lambda S(x_2,y)} =$$

$$[(i\lambda\partial_y S(x_1, y) - i\lambda\partial_y S(x_2, y))(e^{i\lambda S(x_1,y) - i\lambda S(x_2,y)})] \times \frac{1}{i\lambda\partial_y S(x_1, y) - i\lambda\partial_y S(x_2, y)}$$

We integrate the bracketed expression. The end result is

$$P(x_1, x_2) = \frac{i}{\lambda} \int e^{i\lambda S(x_1,y) - i\lambda S(x_2,y)} \partial_y [\frac{1}{\partial_y S(x_1, y) - \partial_y S(x_2, y)} \chi(x_1, y)\chi(x_2, y)] dy \tag{4.6}$$

We obtain several terms, depending on which of the factors in the bracketed expression the ∂_y lands on. If it lands on the $\frac{1}{\partial_y S(x_1,y) - \partial_y S(x_2,y)}$ factor, this factor becomes Q, where

$$Q = -\frac{\partial_y^2 S(x_1, y) - \partial_y^2 S(x_2, y)}{(\partial_y S(x_1, y) - \partial_y S(x_2, y))^2}$$

By the mean-value theorem and (4.2), the numerator of Q is at most $C\frac{|x_1 - x_2|\mu}{b}$, while the denominator is at least $C|x_1 - x_2|^2\mu^2$. As a result Q is at most $C'\frac{1}{|x_1 - x_2|\mu b}$, and the resulting term of the integrand in (4.6) is at most $C'\frac{1}{|x_1 - x_2|\mu b}$ as well. On the other hand, if the derivative lands on $\chi(x_1, y)$ or $\chi(x_2, y)$, by (4.2) we gain a factor of $\frac{1}{b}$. Using the lower bounds we had for the denominator of Q, we see that we again get a term of the integrand in (4.6) bounded by $C'\frac{1}{|x_1 - x_2|\mu b}$. Thus the whole integrand in (4.6) is bounded by $C'\frac{1}{|x_1 - x_2|\mu b}$ for some C'.

Similarly, we may integrate by parts again in (4.6), using the higher derivative bounds in (4.1) and (4.2), and we gain an additional factor of $\frac{i}{\lambda}$ coming out, and an integrand now being bounded by $C''\frac{1}{|x_1 - x_2|^2\mu^2 b^2}$ for some C''. We take absolute values and integrate, obtaining that

$$|P(x_1, x_2)| < C''\frac{1}{\lambda^2|x_1 - x_2|^2\mu^2 b} \tag{4.7}$$

Note also that by taking absolute values in (4.5) and integrating, we have

$$|P(x_1, x_2)| \leq b \tag{4.8}$$

We now wish to use (4.7)-(4.8) and apply Schur's test to $P(x_1, x_2)$, by finding bounds for $\int |P(x_1, x_2)| dx_1$ and $\int |P(x_1, x_2)| dx_2$. By symmetry we need only consider the dx_2 integral. We use (4.8) when $\frac{1}{\lambda^2|x_1 - x_2|^2\mu^2 b} > b$, and (4.7) otherwise. Equivalently, we use (4.8) when $|x_1 - x_2| < \frac{1}{\lambda\mu b}$ and (4.7) otherwise. We get two terms of comparable magnitude, and we conclude that

$$\int |P(x_1, x_2)| dx_2 < C_1 \int_{|x_1 - x_2| < \frac{1}{\lambda\mu b}} b \ dx_2 < C_2 \frac{1}{\lambda\mu}$$

Likewise, we have $\int |P(x_1, x_2)| dx_2 < C_1 \frac{1}{\lambda\mu}$, so by Schur's test $||RR^*||_{L^2 \to L^2} < C_2 \frac{1}{\lambda\mu}$. Therefore $||R||_{L^2 \to L^2} < C_3 \frac{1}{(\lambda\mu)^{\frac{1}{2}}}$, and we are done with the proof of Lemma 4.3.

5. Proof of Theorem 1.1

We now let T be the operator of (1.1), and we assume the support of $\phi(x, y)$ is contained in a neighborhood U of the origin on which we can apply the resolution procedure of section 2 on the function $\partial^2_{xy} S(x, y)$. Actually, it simplifies the calculations somewhat to use the following variant of the resolution procedure: Divide a neighborhood of the origin into 8 regions via the lines $x = 0$, $y = 0$, $y = c_1 x$, $y = -c_1 x$ such that the latter two lines stay away from the zeroes of $\partial^2_{xy} S(x, y)$. Then do the resolution on each of the 8 regions taking the x or y axis as horizontal (i.e. rotate the regions adjacent to the y-axis by 90 degrees.) This way all relevant edges of the Newton polygons appearing will have slope between 0 and -1, and all m_i's appearing are at least 1. This will simplify the arguments that follow.

Next, let $\psi(x, y)$ be a function supported in U but which is 1 on the support of $\phi(x, y)$. Enumerate the various $(V_i)_n$ and $(W_i^j)_n$ as $\{Z_l\}_{l=1}^N$. We decompose $\psi(x, y) = \sum_{l=1}^N \psi_l(x, y)$ according to the Z_l as follows. We assume Z_l is a subset of $\{0 < y < c_1 x\}$ (The other 7 regions are dealt with in an analogous fashion.)

Suppose first that Z_l is a $(V_i)_n$ or $(W_i^j)_n$ with $n > 1$. Then as in Theorem 2.1 we can write Z_l in the form

$$Z_l = \{(x, y) \in U : \sum_{m=1}^{n-1} r_{k_1,\ldots,k_m}(x) + s_{k_1,\ldots,k_{n-1}}(x) \leq y < \sum_{m=1}^{n-1} r_{k_1,\ldots,k_m}(x) + t_{k_1,\ldots,k_{n-1}}(x)\} \tag{5.1}$$

Here $s_{k_1,\ldots,k_{n-1}}(x)$ or $t_{k_1,\ldots,k_{n-1}}(x)$ may be the zero function. We define ψ_l to be equal to 1 on Z_l, and to be supported in $\tilde{Z}_l$, where $\tilde{Z}_l$ is defined by

$$\tilde{Z}_l = \{(x, y) \in U :$$

$$\sum_{m=1}^{n-1} r_{k_1,\ldots,k_m}(x) + (1-\delta) s_{k_1,\ldots,k_{n-1}}(x) \leq y < \sum_{m=1}^{n-1} r_{k_1,\ldots,k_m}(x) + (1+\delta) t_{k_1,\ldots,k_{n-1}}(x)\} \tag{5.2}$$

We also stipulate that if $t_{k_1,\ldots,k_{n-1}}$ is not the zero function, then on the upper portion of $\tilde{Z}_l - Z_l$ we have for $\alpha = 1, 2$ that

$$|\partial_y^\alpha \psi_l(x, y)| < C \frac{1}{|t_{k_1,\ldots,k_{n-1}}(x)|^\alpha} \tag{5.3}$$

Similarly, if $s_{k_1,\ldots,k_{n-1}}(x) \neq 0$, on the lower portion of $\tilde{Z}_l - Z_l$ we make the stipulation that for $\alpha = 1, 2$ we have

$$|\partial_y^\alpha \psi_l(x,y)| < C \frac{1}{|s_{k_1,\ldots,k_{n-1}}(x)|^\alpha} \tag{5.4}$$

In the case that Z_l is a $(V_i)_1$ or $(W_i^j)_1$, then for some positive η_1 and η_2, $m = m_i$ or m_{i+1} we have

$$Z_l = \{(x,y) : \eta_1 x^m < y < \eta_2 x^{m_i}\}$$

In this case, we define $\psi_l(x,y)$ to be supported on a $\tilde{Z}_l$ of the form

$$\tilde{Z}_l = \{(x,y) : (\eta_1 - \delta)x^m < y < (\eta_2 + \delta)x^{m_i}\}$$

The analogues of (5.3), (5.4) are $|\partial_y^\alpha \psi_l(x,y)| < Cx^{-\alpha m_i}$ and $|\partial_y^\alpha \psi_l(x,y)| < Cx^{-\alpha m}$ respectively.

The constructions of section 2 ensure there are no incompatibilities between the ψ_l's corresponding to adjacent Z_l's. The consequences of the resolution procedure listed in Theorem 2.1 for a given Z_l will still hold on the larger set $\tilde{Z}_l$ containing the support of ψ_l if δ is sufficiently small. This is perhaps most easily seen by observing that there is some leeway in defining the δ_i, M_i, s_j, s_j' in the definitions of the $(V_i)_n$ and $(W_i^j)_n$; for $n > 1$ this translates into being able to replace $s_{k_1,\ldots,k_{n-1}}$ or $t_{k_1,\ldots,k_{n-1}}$ by a $1 \pm \delta$ multiple of itself.

Let $\psi_l(x,y)$ as above such that $\sum_l \psi_l(x,y) = \psi(x,y)$, and let $\phi_l(x,y) = \psi_l(x,y)\phi(x,y)$. Define the operator T_l by

$$T_l f(x) = \int_{\mathbf{R}} e^{i\lambda S(x,y)} \phi_l(x,y) f(y)\,dy \tag{5.5}$$

Then $\sum_l T_l = T$, and on the support of each ϕ_l the function $\partial_{xy}^2 S(x,y)$ is well behaved in the sense it satisfies the conditions of Theorem 2.1, including $(2.22a)$-$(2.22b)$ if $n > 1$. In the following arguments, we always assume T_l arises from a ϕ_l coming from the set $\{x > 0, y < x\}$; T_l coming from the other 7 regions are done in an analogous fashion. In addition, if $n > 1$, we assume that $s_{k_1,\ldots,k_{n-1}}(x) = c_s x^s$ and $t_{k_1,\ldots,k_{n-1}}(x) = c_t x^t$ satisfy c_s, $c_t > 0$; if c_s, $c_t < 0$ the same argument works with the roles of $s_{k_1,\ldots,k_{n-1}}(x)$ and $t_{k_1,\ldots,k_{n-1}}(x)$ reversed. (c_s and c_t are always of the same sign.)

We now write $T_l = \sum_i T_l^i$ as follows. If $n = 1$, for an appropriate bump function $\eta(x)$ supported on $[\frac{1}{3}, 1]$, we define

$$T_l^i f(x) = \int_{\mathbf{R}} e^{i\lambda S(x,y)} \phi_l(x,y) \eta(2^i y) f(y)\,dy \tag{5.6}$$

If $n > 1$, let $r(x)$ be the function $\sum_{m=1}^{n-1} r_{k_1,\ldots,k_m}(x)$ from Theorem 2.1. Define m to be the exponent of the leading term of $r_{k_1}(x)$ and t to be such that $t_{k_1,\ldots,k_{n-1}}(x) = c_t x^t$. We define $T_l^i f(x)$ by

$$T_l^i f(x) = \int_{\mathbf{R}} e^{i\lambda S(x,y)} \phi_l(x,y) \eta(2^i x^{-t}(y - r(x))) f(y) dy \tag{5.7}$$

Observe that there is an i_0 such that $\eta(2^i x^{-t}(y - r(x)))$ is zero on the support of ϕ_l only if $i < i_0$; this follows from the fact that $t_{k_1,\ldots,k_{n-1}}(x) = c_t x^t$.

We now do a further decomposition in the x variable into pieces for which the Hessian $\partial_{xy}^2 S(x,y)$ is within a constant ratio of a fixed value. If $n = 1$, we do a normal dyadic decomposition. Namely, we let ζ_j be the characteristic function of $[2^{-j}, 2^{-j+1})$ and we write $T_l^i = \sum_j T_l^{ij}$, where

$$T_l^{ij} f(x) = \int_{\mathbf{R}} e^{i\lambda S(x,y)} \phi_l(x,y) \zeta_j(x) \eta(2^i y) f(y) dy \tag{5.8}$$

In the case $n > 1$, we need to divide each of these x-dyadic pieces into $2^{i+j(t-m)}$ smaller pieces; this will allow us to fully take advantage of the orthogonality lemma, Lemma 4.2. Namely, we let ξ_{ijk} be the characteristic function of $[2^{-j} + k2^{-i-j(t-m+1)}, 2^{-j} + (k+1)2^{-i-j(t-m+1)}]$ for $0 \leq k \leq 2^{i+j(t-m)}$ and we write $T_l^i = \sum_{j,k} T_l^{ijk}$, where

$$T_l^{ijk} f(x) = \int_{\mathbf{R}} e^{i\lambda S(x,y)} \phi_l(x,y) \xi_{ijk}(x) \eta(2^i x^{-t}(y - r(x))) f(y) dy \tag{5.9}$$

The analysis for $n = 1$ and $n > 1$ are somewhat divergent from now on, so we consider them separately.

Suppose first that $n > 1$. If x is small enough, which we may assume, then $r'(x)$ is between $\frac{1}{2}cmx^{m-1}$ and $2cmx^{m-1}$. As a result, the support of $\xi_{ijk}(x)\eta(2^i x^{-m}(y - r(x)))$ is contained in a rectangle of the form

$$\begin{aligned} &[x_0, x_0 + 2^{-i-j(t-m+1)}] \times [r(x_0), r(x_0) + Ar'(x_0)2^{-i-j(t-m+1)}] \\ &\quad = [x_0, x_0 + 2^{-i-j(t-m+1)}] \times [r(x_0), r(x_0) + A'2^{-i-jt}] \end{aligned} \tag{5.10}$$

Here $x_0 \in [2^{-j}, 2^{-j+1}]$ and A, A' are constants. As a result, for a given i there is a $B > 0$ such that if $T_l^{ij'k'}$ is at least B operators to the right of T_l^{ijk} (ordering the operators with respect to the x-supports of their kernels), then the x-projections and y-projections of the kernels of T_l^{ijk} and $T_l^{ij'k'}$ are disjoint. Hence, if we write $T_l^i = \sum_{q=1}^{B} U_q$ such that a given U_q takes every Bth T_l^{ijk} under this ordering, then we can apply lemma 4.2 to each U_q, obtaining that $||U_q||_{L^2 \to L^2}$ is the supremum of the corresponding $||T_l^{ijk}||_{L^2 \to L^2}$. Adding these up over all q, we conclude that

$$||T_l^i||_{L^2 \to L^2} \leq B \sup_{j,k} ||T_l^{ijk}||_{L^2 \to L^2} \tag{5.11}$$

Hence we focus our attention on determining bounds for a given $||T_l^{ijk}||_{L^2\to L^2}$. By (5.10) and Lemma 2.1, we have

$$||T_l^{ijk}||_{L^2\to L^2} < C2^{-i-j(t-\frac{m-1}{2})} \tag{5.12}$$

On the other hand, if (a,b) is as in (2.22), by Lemma 4.3 we have

$$||T_l^{ijk}||_{L^2\to L^2} < C|\lambda|^{-\frac{1}{2}}2^{\frac{(i+tj)b+ja}{2}} \tag{5.13}$$

Observe that we need the derivative conditions in (2.22) to apply Lemma 4.3 here. We conclude that

$$||T_l^{ijk}||_{L^2\to L^2} < C\min(2^{-i-j(t-\frac{m-1}{2})}, |\lambda|^{-\frac{1}{2}}2^{\frac{(i+tj)b+ja}{2}}) \tag{5.14}$$

The right-hand side of (5.12) decreases when j increases while the right-hand side of (5.13) increases when j increases. As a result, for fixed i the right-hand side of (5.14) is maximum when the two quantities are equal, which occurs when

$$2^{-j} = |\lambda|^{-\frac{1}{2t-m+1+a+tb}}2^{i\frac{b+2}{2t-m+1+a+tb}} \tag{5.15}$$

In this case, (5.14) becomes

$$||T_l^{ijk}||_{L^2\to L^2} < C|\lambda|^{-\frac{t-\frac{m-1}{2}}{2t-m+1+a+tb}}2^{i\frac{b(1-m)-2a}{2(2t-m+1+a+tb)}} \tag{5.16}$$

Hence by (5.11) we have

$$||T_l^{i}||_{L^2\to L^2} < C'|\lambda|^{-\frac{t-\frac{m-1}{2}}{2t-m+1+a+tb}}2^{i\frac{b(1-m)-2a}{2(2t-m+1+a+tb)}} \tag{5.17}$$

We would like to add (5.17) over all $i \geq i_0$ to obtain a bound for $||T_l||_{L^2\to L^2}$. To do this, we require that $b(1-m)-2a < 0$. Since $m \geq 1$, the only potential problem occurs if $m = 1$ and $a = 0$. This exceptional case must be done separately: By almost-orthogonality we have

$$\begin{aligned}
||\sum_{i,j,k} T_l^{ijk}||_{L^2\to L^2} &< C\sup_j ||\sum_{i,k} T_l^{ijk}||_{L^2\to L^2} \\
&\leq C\sup_j \sum_i ||\sum_k T_l^{ijk}||_{L^2\to L^2} \\
&\leq C\sup_j \sum_i (\sup_k ||T_l^{ijk}||_{L^2\to L^2})
\end{aligned}$$

Applying (5.12) and (5.13) to this expression gives sharp bounds.

Assuming now that we are not in the exceptional case, we may sum (5.17) in i to obtain

$$||T_l||_{L^2\to L^2} < C''|\lambda|^{-\frac{t-\frac{m-1}{2}}{2t-m+1+a+tb}} \tag{5.18}$$

We write the exponent in (5.18) in the form

$$\frac{t-m+\frac{m+1}{2}}{2(t-m)+m+1+t(\frac{a}{t}+b)} \tag{5.19}$$

By Theorem 2.1, the Newton polygon of $\tilde{f}(x,\tilde{y}) = f(x, y - \sum_{m=1}^{n-1} r_{k_1,\ldots,k_m}(x))$ has an edge of slope $-\frac{1}{t}$. So by Lemma 2.5, if (a', b') is the left vertex of the edge of slope $-\frac{1}{m}$ of the Newton polygon of f, then $\frac{a}{t} + b \leq \frac{a'}{m} + b'$. Therefore, the expression in (5.19) is at least

$$\frac{t-m+\frac{m+1}{2}}{2(t-m)+m+1+t(\frac{a'}{m}+b')} = \frac{t-m+\frac{m+1}{2}}{2(t-m)+m+1+\frac{ta'}{m}+b't}$$

$$= \frac{t-m+\frac{m+1}{2}}{(t-m)(1+\frac{a'}{m})+a'+1+(b'+1)t} \tag{5.20}$$

Observe that $\frac{\frac{m+1}{2}}{a'+1+(b'+1)t}$ is one half of the reciprocal of the x- coordinate of the intersection of the line $y=x$ with the line of slope $-\frac{1}{t}$ containing the point $(a'+1, b'+1)$. As a result, we have that $\frac{\frac{m+1}{2}}{a'+1+(b'+1)t}$ is at least the desired ϵ of Theorem 1.1. In addition, the ratio $\frac{t-m}{(t-m)(1+\frac{a'}{m})} = \frac{m}{m+a'}$ satisfies

$$\frac{m}{m+a'} > \frac{m}{m(b'+1)+a'+1} \geq \frac{m+1}{2m(b'+1)+2(a'+1)}$$

This is also at least the ϵ of Theorem 1.1. Thus we conclude that the quantity (5.20) is at least this ϵ. Hence we have

$$||T_l||_{L^2\to L^2} < C''|\lambda|^{-\frac{t-\frac{m-1}{2}}{2t-m+1+a+tb}} < C''|\lambda|^{-\epsilon}$$

This is the desired estimate for $||T_l||_{L^2\to L^2}$.

We now address the case where T_l is from a $(V_i)_1$ or $(W_i^j)_1$. Then T_l^{ij} is as in (5.8). Let a, b, and m_i be as in Lemma 2.1 or 2.2 respectively. By Lemma 4.1, we have $||T_l^{ij}||_{L^2\to L^2} < C2^{\frac{-i-j}{2}}$, and by Lemma 4.3, we have $||T_l^{ij}||_{L^2\to L^2} < C|\lambda|^{-\frac{1}{2}}2^{\frac{ia+jb}{2}}$ so we have

$$||T_l^{ij}||_{L^2\to L^2} < C\min(2^{\frac{-i-j}{2}}, |\lambda|^{-\frac{1}{2}}2^{\frac{ib+ja}{2}}) \tag{5.21}$$

We now define operators $U_n = \sum_{i+j=n} T_l^{ij}$. So we have

$$||T_l||_{L^2\to L^2} \leq \sum_n ||U_n||_{L^2\to L^2} \tag{5.22}$$

Writing $U_n^{\text{odd}} = \sum_{i+j=n,\ i\ odd} T_l^{ij}$ and $U_n^{\text{even}} = \sum_{i+j=n,\ i\ even} T_l^{ij}$, we can apply Lemma 4.2 to U_n^{odd} and U_n^{even} and we conclude that

$$||U_n||_{L^2\to L^2} \leq ||U_n^{\text{odd}}||_{L^2\to L^2} + ||U_n^{\text{even}}||_{L^2\to L^2} \leq 2\sup_{i+j=n} ||T_l^{ij}||_{L^2\to L^2} \tag{5.23}$$

Consequently, using (5.21) we have

$$||U_n||_{L^2 \to L^2} \leq C \sup_j \min(2^{-\frac{n}{2}}, |\lambda|^{-\frac{1}{2}} 2^{\frac{nb+j(a-b)}{2}}) \tag{5.24}$$

We break into cases $a \geq b$ and $b < a$, starting with $a \geq b$. In this case, the right-hand side of (5.24) increases with j. Up to a constant, the j index goes up to the j corresponding to the intersection of $y = x^{m_i}$ with $xy = 2^{-n}$, namely $j = \frac{n}{m_i+1}$. Therefore we have

$$||U_n||_{L^2 \to L^2} \leq C \min(2^{-\frac{n}{2}}, |\lambda|^{-\frac{1}{2}} 2^{\frac{nb}{2} + \frac{n(a-b)}{2(m_i+1)}}) \tag{5.25}$$

$|\lambda|^{-\frac{1}{2}} 2^{\frac{nb}{2} + \frac{n(a-b)}{2(m_i+1)}}$ increases exponentially with n, while $2^{-\frac{n}{2}}$ decreases exponentially with n. Thus in evaluating (5.22), the sum is bounded by a constant times the $||U_n||_{L^2 \to L^2}$ for which the two expressions on the right-hand side of (5.25) are equal. This works out to $2^n = |\lambda|^{\frac{m_i+1}{m_i(b+1)+(a+1)}}$, and we obtain

$$||T_l||_{L^2 \to L^2} \leq C|\lambda|^{-\frac{m_i+1}{2m_i(b+1)+2(a+1)}} \tag{5.26}$$

Since (a, b) is a vertex of the Newton polygon of $\partial^2_{xy} S(x, y)$ on an edge E of slope $-\frac{1}{m_i}$, the exponent in (5.26) is $\frac{1}{2}$ times the reciprocal of the x-coordinate of the intersection of the line containing E with the line $y = x$. This is at least the reduced Newton distance ϵ of Theorem 1.1, so (5.26) gives the desired estimate for $||T_l||_{L^2 \to L^2}$.

We proceed now to the case where $a < b$, $m_{i+1} < \infty$. In this situation, the right-hand side of (5.24) decreases with j. Hence we seek the minimum possible j, which up to a constant corresponds to the intersection of $xy = 2^{-n}$ with $y = x^{m_i}$ or $y = x^{m_{i+1}}$, depending on whether T_l comes from a V_i or a W_i^j. In any event, the same steps we did in the $a \geq b$ case will now lead to either

$$||T_l||_{L^2 \to L^2} \leq C|\lambda|^{-\frac{m_i+1}{2m_i(b+1)+2(a+1)}}$$

or

$$||T_l||_{L^2 \to L^2} \leq C|\lambda|^{-\frac{m_{i+1}+1}{2m_{i+1}(b+1)+2(a+1)}} \tag{5.28}$$

The former situation is what we had in the case $a \geq b$. The latter situation will once again give us an exponent at least the ϵ of Theorem 1.1; in the case of a W_i^j, the point (a, b) is the vertex between sides of slopes $-\frac{1}{m_i}$ and $-\frac{1}{m_{i+1}}$.

Finally, we consider the case where $a < b$, and $m_{i+1} = \infty$. In this case, the minimum possible j is some j_0 depending on the size of the support of the function $\phi(x, y)$ in Theorem 1.1, and (5.24) becomes

$$||U_n||_{L^2 \to L^2} \leq C' \min(2^{-\frac{n}{2}}, |\lambda|^{-\frac{1}{2}} 2^{\frac{nb}{2}}) \tag{5.29}$$

Again we equate the two expressions in (5.29), which occurs when $2^n = |\lambda|^{-\frac{1}{b+1}}$. In this case we have

$$||T_l||_{L^2 \to L^2} \leq C|\lambda|^{-\frac{1}{2(b+1)}} \tag{5.30}$$

Since $m_{i+1} = \infty$, (a, b) is the lowest vertex of the Newton polygon of $\partial^2_{xy} S(x, y)$, and thus its horizontal face has equation $y = b$. So $\frac{1}{2(b+1)}$ is at least the ϵ of Theorem 1.1, and T_l satisfies an estimate at least as good as we need. This concludes the proof of Theorem 1.1.

6. References

[BK] E. Brieskorn, H. Knorrer, *Plane Algebraic Curves*, Birkhauser, (1986).

[Ry] V. Rytchkov, *Sharp L^2 bounds for oscillatory integral operators with C^∞ phases*, Math. Zeitschrift, **236** (2001) 461-489.

[PS] D. H. Phong, E. M. Stein, *The Newton polyhedron and oscillatory integral operators*, Acta Mathematica **179** (1997), 107-152.

[PS2] D. H. Phong, E. M. Stein, *Models of degenerate Fourier integral operators and Radon transforms*, Annals of Mathematics **140** (1994), 703-722.

[PSSt] D. H. Phong, E. M. Stein, J. Sturm, *Multilinear level set operators, oscillatory integral operators, and Newton diagrams*, Math. Annalen, **319** (2001), 573-596.

[V] A. N. Varchenko, *Newton polyhedra and estimates of oscillatory integrals*, Functional Anal. Appl. **18** (1976), no. 3, 175-196.

[W] R. J. Walker, *Algebraic curves*, Princeton Univ. Press, 1950.

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