A Method for Proving L^p Boundedness of Singular Radon Transforms in Codimension 1

Michael Greenblatt

1. Set-up

Singular Radon transforms are a type of operator combining characteristics of both singular integrals and Radon transforms. They are important in a number of settings in mathematics. In recent work of Christ, Nagel, Stein, and Wainger [1], L^p boundedness of singular Radon transforms for $1 < p < \infty$ is proven under a general finite-type condition using the method of lifting to nilpotent Lie groups. In this paper, an alternate approach is presented, building on the paper of Phong and Stein [5] and the thesis of Cuccagna [2]. L^p bounds are proved in codimension 1 under a similar curvature condition using a real-variable argument not involving vector fields or lifting.

We assume we have an operator T taking Schwartz functions on R^{n+1} to Schwartz functions on R^{n+1} given by

$$Tf(t,x) = \int_{\mathbf{R}^n} f(t + S(t,x,y), y)K(t,x,y)dy \tag{1.1}$$

Here x , $y \in \mathbf{R}^n$, $t \in \mathbf{R}$. We assume $K(t,x,y)$ is a distribution supported on a compact set U and is equal to a function at points where $x \neq y$. $K(t,x,y)$ is to be viewed as a a singular integral kernel in x and y, so we stipulate that for $0 \leq |\alpha| \leq 1$ $K(t,x,y)$ satisfies

$$|\partial_x^\alpha K(t,x,y)| < M|x-y|^{-n-|\alpha|}\,,\ |\partial_y^\alpha K(t,x,y)| < M|x-y|^{-n-|\alpha|}$$

$$|\partial_t^\alpha K(t,x,y)| < M|x-y|^{-n} \tag{$K1$}$$

We also need a cancellation condition on K. Although the methods presented here can be applied to K's satisfying more general cancellation conditions, for the purpose of this paper we stipulate that for all $\epsilon > 0$ K satisfies

$$|\int_{|y-x|=\epsilon} K(t,x,y)dy|\,,\,|\int_{|y-x|=\epsilon} K(t,x,y)dx| < M\epsilon \qquad (K2)$$

Here the integrals are taken with respect to surface measure. Thus T is an averaging operator along the surfaces defined by S; $Tf(t,x)$ is the average of f on the graph of $y \to (t+S(t,x,y),y)$ with respect to the measure $K(t,x,y)$. We will have that $S(t,x,x) = 0$ so that (t,x) is on this surface. The conditions $(K1)$ - $(K2)$ are a way of describing $K(t,x,y)$ as being a singular integral kernel; the operator at (t,x) is a singular integral over a surface containing (t,x), "centered at (t,x)", and these surfaces may vary with (t,x). The curvature condition will be a way of saying that the surfaces "change to at most finite order" in at least one direction tangent to the surfaces.

We now turn to our assumptions on S. We will define a positive integer d below, the "order of degeneracy" of T. We assume S is a C^d function satisfying

$$S(t,x,x) = 0\ \ ,\ \ \sup_{(t,x,y)\in U, |\alpha|,|\beta|,|\gamma|\leq d+1} |\partial_x^\alpha \partial_y^\beta \partial_t^\gamma S(t,x,y)|\ <\ M, |\partial_t S| < 1/2 \quad (S1)$$

The curvature condition is a finite-type condition closely related to that of [1]. There are a few ways to state it; a convenient way for the purposes of this paper is as follows. At each (t_0,x_0), we consider the function $t^S_{t_0,x_0}(x,y)$ defined to be the solution t^* to $t^* + S(t^*,x,y) - t_0 - S(t_0,x_0,y) = 0$. This function exists and is as smooth as S is by the implicit function theorem. Define $\tau^S_{\alpha,\beta}(t_0,x_0) = \partial_x^\alpha \partial_y^\beta t^*_{t_0,x_0}(x_0,x_0)$. Our curvature condition is:

$$\forall (t_0,x_0) \in U\ \exists\ |\alpha| \geq 1 \text{ and } |\beta| \geq 1 \text{ such that } \tau^S_{\alpha,\beta}(t_0,x_0) \neq 0 \qquad (S2)$$

Due to compactness, can assume for our d

$$\exists \epsilon > 0\ \exists |\alpha|,|\beta| \geq 1 \text{ with } |\alpha| + |\beta| \leq d \text{ such that } |\partial_x^\alpha \partial_y^\beta \tau^S_{\alpha,\beta}(t_0,x_0)| \geq \epsilon \quad (S2')$$

Assume d has been chosen to be minimal.

To help see what this means, consider the coordinate change given by $C_{t_0,x_0} : (t,y) \to (t - S(t_0,x_0,y), y)$. Then this is evidently a smooth coordinate change with a Jacobian determinant near 1 taking (t_0, x_0) to itself. Thus L^p boundedness of T is equivalent to L^p boundedness of $C_{t_0,x_0}^{-1}\ T\ C_{t_0,x_0}$. However, $C_{t_0,x_0}^{-1}\ T\ C_{t_0,x_0}$ is also a singular Radon transform (although now $K(t,x,y)$ is replaced by something which may not satisfy (K2)), with "S-function" given by $S_1(t,x,y) = S(t_0,x_0,x)+S(t+S(t_0,x_0,x),x,y)-S(t_0,x_0,y)$. Note that S_1 has the property that $S_1(t_0,x_0,y) = 0$ for all y. We will see that the curvature condition at (t_0, x_0) is just

$$\exists\ |\alpha| \geq 1, |\beta| \geq 1 \text{ with } |\alpha| + |\beta| \leq d \text{ such that } |\partial_x^\alpha \partial_y^\beta S_1(t_0,x_0,x_0)| \geq \epsilon \quad (S2'')$$

This statement of the curvature condition has appeared before (see Phong [4] for example). In the translation invariant case (where $S(t,x,y) = \tilde{S}(x-y)$ for some $\tilde{S}$) the curvature condition appearing here is equivalent to S being of finite type.

This is the main result:

Main Theorem. Suppose T is as in equation (1.1) and K satisfies (K1)-(K2) , S satisfies (S1)-(S2'). Then there exists an $r > 0$ depending on ϵ, n, d, and M such that if K is supported in a set of the form $\{(t,x,y) : |t-t_0|, |x-y| < r\}$, then T is bounded on each L^p for $1 < p < \infty$ with bound depending on ϵ, n, d, p, and M.

Note: In what follows, any constant C, C', C'', C_0, C_1, C_2, etc. or δ, δ', δ'', δ_0, δ_1, δ_2, etc. that appears denotes a positive constant depending only on M, d, n, ϵ, and p. Sometimes I will use the notation C several times within a single argument in which case C denotes the maximum of several constants appearing there.

The Main Theorem suffices to prove L^p boundedness of our original singular Radon transform; we just chop up the function K into 2 parts. The first part K_o is supported on $\{(t,x,y) : |x-y| > r\}$ and the resulting singular Radon transform T_o is easily seen to be bounded on any L^p since K_o is now a bounded compactly supported function:

$$\int |T_o f(t,x)|^p dt\, dx = \int |\int_{\mathbf{R}^n} f(t+S(t,x,y),y) K_o(t,x,y) dy|^p dt\, dx$$

$$\leq C \int_U |f(t+S(t,x,y),y)|^p dt\, dx\, dy$$

But since $|\nabla_t S| < 1/2$, in the t-y integration we may make the coordinate change $(t', y') = (t + S(t,x,y), y)$, and the above is at most

$$C \int_{\mathbf{R}^{2n+2}} |f(t',y')|^p dt'\, dy' < C||f||_p^p$$

Thus this first part of our singular Radon transform is bounded on L^p.

The second part can be written as the sum of at most $Cr^{-(2n+1)}$ kernels each of which is supported in a set of the form $\{(t,x,y) : |t - t_0|, |x-y| < r\}$ and thus the theorem applies to the resulting singular Radon transforms. Hence the norm of the original singular Radon transform on L^p is bounded in terms of ϵ, n, d, M, and p.

The study of singular Radon transforms has proceeded along 2 main lines. The first starts with the translation-invariant situation, where the proof of L^p boundedness of T is less complicated since T is now a convolution operator. The idea is to then reduce the study of an arbitrary T to one which is almost translation-invariant on a nilpotent Lie group by a lifting technique: Applying the lifted operator to a function independent of certain coordinates is essentially equivalent to applying the unlifted operator to a "slice" of this function. This method was developed by several workers, culminating in [1]. In [1], L^p boundedness of singular Radon transforms is proven for all $1 < p < \infty$ in any codimension, under a natural curvature condition which reduces in the translation-invariant case to the submanifolds integrated over being of finite type.

The second main line of attack, exemplified by Phong and Stein [5] and Greenleaf and Uhlmann [3], involves writing $Tf(t,x)$ as

$$\int_{\mathbf{R}^{n+m}} e^{i\langle \lambda, t+S(t,x,y)\rangle} K(t,x,y)\hat{f}(\lambda,y)\, d\lambda\, dy$$

Here $t \in \mathbf{R}^m$, x, $y \in \mathbf{R}^n$, $\hat{f}$ denotes Fourier transform in the first component only. In [5], T is then written as the sum of 2 parts, the "inside portion" where $|x-y| < C_1|\lambda|^{1/2}$ and the "outside" portion where $|x-y| > C_2|\lambda|^{1/2}$. The "outside" portion is dealt with via a TT^* argument, while the "inside" portion is viewed as

$$\int_{\mathbf{R}^{n+m}} e^{i\langle \lambda, t\rangle} a(t,\lambda)\, \hat{f}(\lambda,.)\, d\lambda$$

where $a(t,\lambda)$ is an oscillatory integral operator taking functions of y to functions of x given by

$$a(t,\lambda)f(x) = \int_{\mathbf{R}^n} e^{i\langle\lambda,S(t,x,y)\rangle}\phi(|\lambda||x-y|^2)K(t,x,y)\,f(y)\,dy$$

So in this sense the "inner" portion is a pseudodifferential operator with an operator valued symbol, and in [5] pseudodifferential operator methods are used to prove L^2 boundedness. The results in [5] are in the nondegenerate case $d = 2$, in codimension 1, when S satisfies a condition called rotational curvature at every point (t,x). This means that if at some (t,x) we are in coordinates such that $S(t,x,y) = 0$ for all y, the Hessian of S in x and y has full rank. It is clear from the arguments in [5] that the results can be extended to some higher codimension situations when S has an analogous nondegeneracy condition. This is taken a large step further in the thesis of Cuccagna [2] where the the Hessian of S is merely required to have rank at least 1, with appropriate generalizations to higher codimensions. The purpose of this paper is to build on these arguments in the degenerate case $d > 2$ and give a theorem like [1] in codimension 1.

Acknowledgement.

I would like to thank my advisor Eli Stein for the many useful suggestions he made throughout the course of my doctoral research, which resulted in the current paper. I am certain it could never have been done without his generous assistance.

2: Definition of Balls and Outline of Proof.

We now will define a system of non-Euclidean balls. They will be used not only to apply the Calderon-Zygmund theory towards proving our L^p estimates, but also to make relevant estimates used in the L^2 theory. Simpler non-Euclidean balls were used for the L^p theory in [5] and [2]; they were the inspiration for the non-Euclidean balls and the function $\sigma(t,x;r)$ used here. Define the ball $B^S(t,x;r)$ by $B^S(t,x;r) =$

$$\{(s,y) : |y-x| < r, |s-t-S(t,x,y)| < (\sum_{1\leq|\alpha|,|\beta|\leq d}(\tau^S_{\alpha,\beta}(t,x)(cr)^{|\alpha|+|\beta|})^2)^{1/2}\}$$

Here c is a large constant to be chosen later. We use $\sigma^S(t,x;r)$ to denote the sum $(\sum_{0<|\alpha|,|\beta|,|\alpha|+|\beta|\leq d}(\tau^S_{\alpha,\beta}(t,x)(cr)^{|\alpha|+|\beta|})^2)^{1/2}$; we suppress the superscript S when there is no ambiguity as to which S-function is being considered.

Let $\gamma_0(t)$ be a radial nonnegative bump function on R supported on $|t|<1$, and equal to 1 for $|t|<1/2$. Let $\gamma=\gamma_0(t)-2\gamma_0(2t)$, and $\gamma_1=\gamma_0(t)-\gamma_0(2t)$ First, for $i>0$, we define $T_{i,j}f(t,x)=$

$$\int_{\mathbf{R}^{n+1}} 2^i\sigma(t,x,2^{-j})^{-1}\gamma(2^i\sigma(t,x;2^{-j})^{-1}(t+S(t,x,y)-s))$$
$$\gamma_1(2^j(x-y))K(t,x,y)f(s,y)dsdy \tag{2.1}$$

$T_{0,j}$ is defined the same way with $i=0$, except with the γ replaced by γ_0. We define $T_i=\sum_j T_{i,j}$. Thus $T=\sum_{i\geq 0}T_i$.

Thus $T_{i,j}$ is T_i with $K(t,x,y)$ localized to $|x-y|\sim 2^{-j}$. If we write $K_j(t,x,y)=K(t,x,y)\gamma_1(2^j(x-y))$, then K_j satisfies the same estimates that K does and in addition is supported in $|x-y|\sim 2^{-j}$ so we can rewrite (3.1) as

$$\int_{\mathbf{R}^{n+1}} 2^i\sigma(t,x;2^{-j})^{-1}\gamma(2^i\sigma(t,x;2^{-j})^{-1}(t+S(t,x,y)-s))K_j(t,x,y)f(s,y)ds\,dy$$

For the purposes of our arguments $T_if(t,x)$ could have been defined as

$$\int_{\mathbf{R}^{n+1}} 2^i\sigma(t,x;|x-y|)^{-1}\gamma(2^i\sigma(t,x;|x-y|)^{-1}(t+S(t,x,y)-s))K(t,x,y)f(s,y)ds\,dy \tag{2.1$'$}$$

(with γ replaced by γ_0 if $i=0$) instead of writing T_i as a sum over j and fixing the radius in the σ's on a fixed $T_{i,j}$. However for the arguments appearing here a dyadic decompostion over these annuli would be necessary anyhow, and in addition the analysis becomes slightly less cumbersome with a fixed radius in the σ's. But in the form (2.1$'$) it is perhaps more apparent that T_0 is basically a singular integral operator with a metric deriving from the balls defined above, and that T_i is in some sense a hybrid of T_0 and the original operator T; with increasing i, T_i may be viewed as getting "closer" to T, but with an additional cancellation condition arising from the fact that $\int\gamma=0$.

One way of proving L^2 boundedness of many singular integrals, such as Riesz transforms, involves the use of almost-orthogonality, and due to the resemblance of the T_i to singular integral operators we will be able to use almost-orthogonality here too. We will see that we have $||T_{i,j_1}T^*_{i,j_2}||_{2,2}, ||T^*_{i,j_1}T_{i,j_2}||_{2,2} < C2^{3i/2}2^{-|j_1-j_2|/2}$.

However, we will also show for $i > 0$ that there is a $\delta > 0$ such that as an operator on L^2,$||T_{i,j}|| < C2^{-\delta i}$. We will do this by applying the following lemma to the kernel of $T_{i,j}T^*_{i,j}$.

Lemma 2.1: Suppose $T(f)(x) = \int K(x,y)f(y)dy$ and suppose $\int |K(x,y)|dx < C_1$, $\int |K(x,y)|dy < C_2$. Then as an operator on L^2, $||T|| < (C_1C_2)^{1/2}$.

Proof: For any functions f and g $|\langle Tf, g\rangle| =$

$$|\int K(x,y)f(y)g(x)dy\,dx| = |\int K(x,y)(C_2/C_1)^{1/4}f(y)(C_1/C_2)^{1/4}g(x)dy\,dx|$$

$$\leq .5(C_2/C_1)^{1/2}\int |f(y)|^2|K(x,y)|dydx + .5(C_1/C_2)^{1/2}\int |g(x)|^2|K(x,y)|dydx$$

$$\leq \frac{1}{2}(C_1C_2)^{1/2}(\int |f(y)|^2dy + \int |g(x)|^2dx)$$

Taking the supremum over all f and g with L^2 norm 1 gives us the Lemma.

We will show that the kernel of $T_{i,j}T^*_{i,j}$ satisfies the conditions of Lemma 2.1 by an integration by parts argument somewhat related to the proof of the Van der Corput lemma, using the curvature condition plus the fact that $\int \gamma = 0$. We will then have that $||T_{i,j}||_{2,2} \leq \bar{C}2^{-\delta i}$ for all i and j. As a result,

$$||T_{i,j_1}T^*_{i,j_2}||, ||T^*_{i,j_1}T_{i,j_2}|| < \min(C2^{3i/2}2^{-|j_1-j_2|/2}, C2^{-2\delta i}) < C'2^{-\delta'|j_1-j_2|-\delta'' i}$$

We can then apply almost orthogonality to conclude that $||T_i|| \leq C''2^{-\delta''' i}$. Summing over i gives $||T|| < C'''$.

Our use of almost orthogonality has its origins in a related use of almost-orthogonality in the thesis of Cuccagna [2], which in turn is derived from the almost-orthogonality arguments implicitly used in the pseudodifferential operator-like methods of the original paper of Phong-Stein [5].

We now give a rough idea of how the arguments of this paper are related to those of the above sources. In the nondegenerate case $d = 2$, we may replace $\sigma(t, x; r)$ by r^2 for any t and x. As a result (2.1) may be rewritten as

$$\int_{\mathbf{R}^{n+1}} e^{i\langle\lambda, t+S(t,x,y)\rangle}\hat{\gamma}(2^{-i}2^{-2j}\lambda)\gamma_1(2^j(x-y))K(t,x,y)\hat{f}(\lambda,y)\,d\lambda\,dy$$

If we let $j' = i + 2j$, this becomes

$$\int_{\mathbf{R}^{n+1}} e^{i\langle\lambda, t+S(t,x,y)\rangle}\hat{\gamma}(2^{-j'}\lambda)\gamma_1(2^{-i/2}2^{j'/2}(x-y))K(t,x,y)\hat{f}(\lambda,y)\,d\lambda\,dy$$

which is basically the same as

$$\int_{\mathbf{R}^{n+1}} e^{i\langle\lambda, t+S(t,x,y)\rangle}\hat{\gamma}(2^{-j'}\lambda)\gamma_1(2^{-i}|\lambda||x-y|^2)K(t,x,y)\hat{f}(\lambda,y)\,d\lambda\,dy$$

In Cuccagna's thesis [2], when $i = 0$, the sum of these operators over j is considered. (i.e. the $\hat{\gamma}$ disappears), and a 1/2-1/2 pseudodifferential operator argument is used analogous to the "inside" operators mentioned above. As for the rest of the operator, for a fixed j' he considers the sum over $i > 0$ of these operators (i.e. there is no decomposition in i), call it $T'_{j'}$. Almost-orthogonality is used over the different j''s, and then for a fixed j' uniform estimates $|T'_{j'}| < C$ are proved using a TT^* argument with an appropriate integration by parts that uses the nondegeneracy condition. Although not done this way in [2], one way to prove these uniform estimates is to write $T'_{j'}$ as the sum of $T_{i,j'}$'s like above and prove exponential decay in i of $||T_{i,j'}||$. This approach to the L^2 estimates in [2], suggested to me by E. Stein, grows out of the paper Ricci-Stein [6] where an analogous method is used for analyzing oscillatory integral operators with polynomial phase. This idea was the behind the format for an earlier rendition of this paper, where operators $U_{i,j}$ were defined like $U_{i,j} =$

$$\int_{\mathbf{R}^{n+1}} e^{i\langle\lambda, t+S(t,x,y)\rangle}\hat{\gamma}(2^{-j}\lambda)\gamma(2^{-i}2^{j}\sigma(t,x;|x-y|))K(t,x,y)\hat{f}(\lambda,y)\,d\lambda\,dy$$

For $i > 0$ almost orthogonality was again used over the different j's, and an argument like that of Section 5 was used to show exponential norm decay in i. For $i = 0$ a separate argument somewhat resembling that of Section 4 was used. However it became clearer as time went on that these Fourier transform definitions were not natural in some respects, and they were eventually supplanted by the operators used in this paper.

As for the L^p theory for the singular Radon transforms of this paper, the generalization of the Calderon-Zygmund theorem for non-Euclidean balls will show that on any L^p, $\infty > p > 1$, we have $||T_i|| \leq C'i$. Thus by simple interpolation with the L^2 situation we will have $||T_i|| \leq C''\, 2^{-\delta' i}$. Summing then gives $||T|| < C'''$.

The main novelty of this paper is in the approach used here to the L^2 estimates of section 5 (e.g. the functions t^* and $\tau_{\alpha,\beta}$) and the geometric properties generated by this approach, such as the definition of the balls of section 2 and the special coordinate changes used to make $S(t,x,y) = 0$ for all y and a fixed (t,x). It is worth stressing that the arguments presented here are real-variable arguments and do not, explicitly at least, use Fourier transform methods.

3. Geometric Lemmas.

The following two facts about our non-Euclidean balls allow us to apply the Calderon-Zygmund method for our L^p estimates, and the second fact is also very important for our L^2 estimates.

Fact 1: Let m be Lebesgue measure. $m(B^S(t,x;2r)) \leq 2^{n+d+1} m(B^S(t,x,r))$

Fact 2:

$$B^S(t_0,x_0;r) \cap B^S(t_1,x_1;r) \neq \emptyset \longrightarrow \exists C\, B^S(t_1,x_1;r) \subset B^S(t_0,x_0;Cr)$$

For the proof of Fact 1 we use the following simple lemma.

Lemma 3.1 If $r_2 \geq r_1$ then $(r_2/r_1)^2 \sigma(t,x;r_1) \leq \sigma(t,x;r_2) \leq (r_2/r_1)^d \sigma(t,x;r_1)$

Proof: Because $2 \leq |\alpha| + |\beta| \leq d$, we have

$$(r_2/r_1)^2 |\tau^S_{\alpha,\beta}(t,x)(cr_1)^{|\alpha|+|\beta|}| \leq |\tau^S_{\alpha,\beta}(t,x)(cr_2)^{|\alpha|+|\beta|}|$$

$$|\tau^S_{\alpha,\beta}(t,x)(cr_2)^{|\alpha|+|\beta|}| \leq (r_2/r_1)^d |\tau^S_{\alpha,\beta}(t,x)(cr_1)^{|\alpha|+|\beta|}|$$

Squaring and adding these over α and β gives us the Lemma.

The proof of Fact 1 is then as follows. $m(B^S(t,x;2r)) =$

$$m(\{(s,y) : |y-x| < 2r, |s-t-S(t,x,y)| < (\sum_{1\leq|\alpha|,|\beta|\leq d}(\tau^S_{\alpha,\beta}(t,x)(2cr)^{|\alpha|+|\beta|})^2)^{1/2}\})$$

which by Lemma 3.1 is at most 2^d times

$$m(\{(s,y) : |y-x| < 2r, |s-t-S(t,x,y)| < (\sum_{1\leq|\alpha|,|\beta|\leq d}(\tau^S_{\alpha,\beta}(t,x)(cr)^{|\alpha|+|\beta|})^2)^{1/2}\})$$

$$= 2^{n+d+1}m(B^S(t,x;r))$$

We therefore have shown Fact 1.

To prove Fact 2, we need a few technical lemmas which will come in handy throughout this paper. But first we note a couple of invariance properties of our balls under certain special coordinate changes. Recall that if C_{t_0,x_0} is the map $C_{t_0,x_0} : (t,y) \to (t - S(t_0,x_0,y), y)$ then $C^{-1}_{t_0,x_0}TC_{t_0,x_0}$ is a singular Radon transform with "S-function" $S_1 = S(t_0,x_0,x)+S(t+S(t_0,x_0,x),x,y)-S(t_0,x_0,y)$. It is easily checked that $t^{S_1}_{(t_1,x_1)}(x,y) = t^S_{(t_1+S(t_0,x_0,x_1),x_1)}(x,y) - S(t_0,x_0,x)$ and hence $\tau^{S_1}_{\alpha,\beta}(t_1,x_1) = \tau^S_{\alpha,\beta}(t_1 + S(t_0,x_0,x_1),x_1)$ for $|\alpha|$ and $|\beta| \geq 0$. This in turn implies that $B^{S_1}(t_1,x_1;r) = \{(s,y) : (s + S(t_0,x_0,y),y) \in B^S(t_1 + S(t_0,x_0,x_1),x_1;r)\}$

Lemma 3.2: If $(t_1,x_1) \in B^S(t_0,x_0;r)$, then $\exists\delta > 0$ such that $\delta\sigma(t_0,x_0;r) \subset \sigma(t_1,x_1;r) \subset 1/\delta\sigma(t_0,x_0;r)$

In what follows, we write $\sigma(t_0,x_0;r) \sim \sigma(t_1,x_1;r)$ as a shorthand for the conclusion of the above lemma.

Proof: We first consider the case where $S(t_0,x_0,y) = 0$ for all y. We differentiate the relation $t^*(t_1,x_1,x,y)+S(t^*(t_1,x_1,x,y),x,y)-t_1-S(t_1,x_1,y) = 0$ with respect to y and x, obtaining

$$\partial_y t^* = -(1 + S_t(t^*,x,y))^{-1}(S_y(t^*,x,y) - S_y(t_1,x_1,y)) \tag{3.1a}$$

$$\partial_x t^* = -(1 + S_t(t^*,x,y))^{-1}S_x(t^*,x,y) \tag{3.1b}$$

Differentiating these expressions with respect to x and y at most d times, substituting (3.1) whenever $\partial_y t^*$ or $\partial_x t^*$ appear, we get

$$\partial_x^\alpha\partial_y^\beta t^* = -(I + S_t(t^*,x,y))^{-1}\partial_x^\alpha\partial_y^\beta S(t^*,x,y) + E_{\alpha,\beta}(t^*,t_1,x_1,x,y) \tag{3.2}$$

Here $E_{\alpha,\beta}(t^*, t_1, x_1, x, y)$ is a constant times a product of functions of the following types: (1)$(I + S_t(t^*, x, y))^{-1}$. (2) $\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^{\alpha_3}S(t^*, x, y)$ for $|\alpha_1| + |\alpha_2| + |\alpha_3| \leq d$. (3) $\partial_y^{\alpha_4}S(t^*, x, y) - \partial_y^{\alpha_4}S(t_1, x_1, y)$ for $|\alpha_4| \leq d$. At least one of the members of this product must be of the form $\partial_x^{\alpha_2}\partial_y^{\alpha_3}S(t^*, x, y)$ for $(\alpha_2, \alpha_3) < (\alpha, \beta)$ or $\partial_y^{\alpha_4}S(t^*, x, y) - \partial_y^{\alpha_4}S(t_1, x_1, y)$ for $\alpha_4 < \beta$.

The above characterization can be proved inductively in a straightforward way; to avoid a technical but unenlightening argument the proof is omitted here.

If we set $t_1 = t_0$ and $y = x = x_1$, then $t^* = t_1 = t_0$, and (3.2) becomes

$$\partial_x^\alpha\partial_y^\beta t^*(t_0, x_1, x_1, x_1) = \partial_x^\alpha\partial_y^\beta S(t_0, x_1, x_1) + E_{\alpha,\beta}(t_0, t_0, x_1, x_1, x_1) \tag{3.3}$$

And since one of the factors in each of the products mentioned above must be of the form $\partial_x^{\alpha_2}\partial_y^{\alpha_3}S(t_0, x_1, x_1)$ for appropriate α_2 and α_3, or of the form $\partial_y^{\alpha_4}S(t_0, x_1, x_1) - \partial_y^{\alpha_4}S(t_0, x_1, x_1) = 0$, we have

$$|E_{\alpha,\beta}(t_0, t_0, x_1, x_1, x_1)| \leq C \sum_{(0,0)\leq(\alpha',\beta')<(\alpha,\beta)} |\partial_x^{\alpha'}\partial_y^{\beta'}S(t_0, x_1, x_1)|$$

Furthermore, we may expand $\partial_x^\alpha\partial_y^\beta S(t_0, x_1, x_1)$ in the last two entries about (t_0, x_0), getting

$$\partial_x^\alpha\partial_y^\beta S(t_0, x_1, x_1) = \sum_{(\alpha,\beta)\leq(\alpha',\beta'),|\alpha'|+|\beta'|\leq d} \partial_x^{\alpha'}\partial_y^{\beta'}S(t_0, x_0, x_0)(x_1 - x_0)^{\alpha'+\beta'-\alpha-\beta}$$

$$+F_{\alpha,\beta}(t_0, x_0, x_1)|x_1 - x_0|^{d+1-|\alpha|-|\beta|} \tag{3.4}$$

where F is bounded in terms of the $d+1$th derivatives of S. Thus for all $|\alpha|, |\beta| \leq d$

$$|\partial_x^\alpha\partial_y^\beta S(t_0, x_1, x_1) - \partial_x^\alpha\partial_y^\beta S(t_0, x_0, x_0)| \leq$$

$$\sum_{(\alpha,\beta)<(\alpha',\beta'),|\alpha'|+|\beta'|\leq d} C|\partial_x^{\alpha'}\partial_y^{\beta'}S(t_0, x_0, x_0)|r^{|\alpha'|+|\beta'|-|\alpha|-|\beta|} + Cr^{d+1-|\alpha|-|\beta|}$$

Subsequently,

$$|\partial_x^\alpha\partial_y^\beta S(t_0, x_1, x_1)(cr)^{\alpha+\beta} - \partial_x^\alpha\partial_y^\beta S(t_0, x_0, x_0)(cr)^{\alpha+\beta}| \leq$$

$$\sum_{(\alpha,\beta)<(\alpha',\beta'),|\alpha'|+|\beta'|\leq d} C|\partial_x^{\alpha'}\partial_y^{\beta'}S(t_0,x_0,x_0)|c^{|\alpha|+|\beta|}r^{|\alpha'|+|\beta'|}+Cc^{|\alpha|+|\beta|}r^{d+1}$$

$$\leq (1/c)\sum_{(\alpha,\beta)<(\alpha',\beta'),|\alpha'|+|\beta'|\leq d} C|\partial_x^{\alpha'}\partial_y^{\beta'}S(t_0,x_0,x_0)|(cr)^{|\alpha'|+|\beta'|}+\frac{C}{c}(cr)^{d+1} \quad (3.5)$$

Also, by (3.3) $|\partial_x^{\alpha}\partial_y^{\beta}t^*(t_0,x_1,x_1,x_1)(cr)^{|\alpha|+|\beta|}-\partial_x^{\alpha}\partial_y^{\beta}S(t_0,x_1,x_1)(cr)^{|\alpha|+|\beta|}|\leq$

$$C\sum_{(0,0)\leq(\alpha',\beta')<(\alpha,\beta)}|\partial_x^{\alpha'}\partial_y^{\beta'}S(t_0,x_1,x_1)|(cr)^{|\alpha|+|\beta|}$$

$$\leq C(cr)\sum_{(0,0)\leq(\alpha',\beta'),|\alpha'|+|\beta'|\leq d}|\partial_x^{\alpha'}\partial_y^{\beta'}S(t_0,x_1,x_1)|(cr)^{|\alpha'|+|\beta'|}$$

Hence by (3.5) $|\partial_x^{\alpha}\partial_y^{\beta}t^*(t_0,x_1,x_1,x_1)(cr)^{|\alpha|+|\beta|}-\partial_x^{\alpha}\partial_y^{\beta}S(t_0,x_0,x_0)(cr)^{|\alpha|+|\beta|}|\leq$

$$C(cr+1/c)\sum_{(0,0)\leq(\alpha',\beta'),|\alpha'|+|\beta'|\leq d}|\partial_x^{\alpha'}\partial_y^{\beta'}S(t_0,x_0,x_0)|(cr)^{|\alpha'|+|\beta'|}+C'r^{d+1} \quad (3.6)$$

Since we assume here that $S(t_0,x_0,y)=0$ for all y, we have for each β that $\partial_y^{\beta}S(t_0,x_0,x_0)=0$. By (S1) $S(t_0,y,y)=0$ for all y. Differentiating this in y directions $|\alpha|$ times gives us an expression for $\partial_x^{\alpha}S(t_0,y,y)$ in terms of derivatives of the form $\partial_x^{\alpha'}\partial_y^{\beta'}S(t_0,y,y)$ for $\alpha'+\beta'=\alpha$, $\beta'\neq 0$. If we set $y=x_0$, then all the purely y derivatives in this expression are 0 and our expression for $\partial_x^{\alpha}S(t_0,x_0,x_0)$ contains only mixed derivatives. Thus we can replace (3.6) by

$$|\partial_x^{\alpha}\partial_y^{\beta}t^*(t_0,x_1,x_1,x_1)(cr)^{|\alpha|+|\beta|}-\partial_x^{\alpha}\partial_y^{\beta}S(t_0,x_0,x_0)(cr)^{|\alpha|+|\beta|}|$$

$$\leq C(cr+1/c)\sum_{\alpha,\beta>0,|\alpha'|+|\beta'|\leq d}|\partial_x^{\alpha'}\partial_y^{\beta'}S(t_0,x_0,x_0)|(cr)^{|\alpha'|+|\beta'|}+C'r^{d+1} \qquad (3.7)$$

However, the curvature condition implies that

$$C'r^{d+1}<C''r\sum_{\alpha,\beta>0,|\alpha'|+|\beta'|\leq d}|\partial_x^{\alpha}\partial_y^{\beta}t^*(t_0,x_1,x_1,x_1)|(cr)^{|\alpha|+|\beta|}$$

So if we add the equations (3.7) over the various α and β we get

$$\sum_{\alpha,\beta>0,|\alpha'|+|\beta'|\leq d}|\partial_x^{\alpha}\partial_y^{\beta}t^*(t_0,x_1,x_1,x_1)(cr)^{|\alpha|+|\beta|}-\partial_x^{\alpha}\partial_y^{\beta}S(t_0,x_0,x_0)(cr)^{|\alpha|+|\beta|}|<$$

$$\leq C(cr+1/c)\sum_{\alpha,\beta>0,|\alpha'|+|\beta'|\leq d}|\partial_x^{\alpha'}\partial_y^{\beta'}S(t_0,x_0,x_0)|(cr)^{|\alpha'|+|\beta'|}$$

$$+Cr \sum_{\alpha,\beta>0,|\alpha'|+|\beta'|\leq d} |\partial_x^\alpha \partial_y^\beta t^*(t_0,x_1,x_1,x_1)|(cr)^{|\alpha|+|\beta|} \tag{3.8}$$

Thus if first c is chosen appropriately large and r is appropriately small, we may conclude that

$$0.99 \sum |\partial_x^\alpha \partial_y^\beta t^*(t_0,x_1,x_1,x_1)(cr)^{|\alpha|+|\beta|}| < \sum |\partial_x^\alpha \partial_y^\beta S(t_0,x_0,x_0)(cr)^{|\alpha|+|\beta|}|$$
$$< 1.01 \sum |\partial_x^\alpha \partial_y^\beta t^*(t_0,x_1,x_1,x_1)(cr)^{|\alpha|+|\beta|}| \tag{3.9}$$

(3.9) is true for any x_1 with $|x_1 - x_0| < r$, including x itself. So since $\sigma(t_0,x_1;r)$ is within a factor of C of the right hand sum, we conclude that there is a δ with $\delta < \sigma(t_0,x_1;r)/\sigma(t_0,x_0;r) < \delta^{-1}$ for all such x_1. Note also that by setting $x_0 = x_1$ in (3.9) one can see the equivalence of curvature conditions $(S2')$ and $(S2'')$.

Next, note that for $(t_1,x_1) \in B^S(t_0,x_0;r)$ we have

$$|\sigma(t_1,x_1;r) - \sigma(t_0,x_1;r)| < C \sum |\tau^S_{\alpha,\beta}(t_1,x_1) - \tau^S_{\alpha,\beta}(t_0,x_1)|(cr)^{|\alpha|+|\beta|}$$
$$< C|t_1 - t_0|r^2 < C'r^2\sigma(t_0,x_0;r)$$

Thus as long as $r^2 < 1/2C'^2$, we get $1/2 < \sigma(t_1,x_1;r)/\sigma(t_0,x_1;r) < 2$ and hence we may conclude that there is a δ' with $\delta' < \sigma(t_1,x_1;r)/\sigma(t_0,x_0;r) < \delta'^{-1}$.

So we are done with Lemma (3.2) in the special case that $S(t_0,x_0,y) = 0$ for all y.

For the general case, we replace S by $S_1(t,x,y) = S(t_0,x_0,x) + S(t_0 + S(t_0,x_0,x),x,y) - S(t_0,x_0,y)$ to insure that $S_1(t_0,x_0,y) = 0$. The above argument proves the lemma for S_1 and $B^{S_1}(t_0,x_0;r)$, and the invariance properties of $B^S(t,x;r)$ under this coordinate change (see the discussion prior to the proof of this Lemma) proves the result for $B^S(t_0,x_0;r)$. It is true that the bounds obtained will depend on bounds on the derivatives of S_1 and not S, but the coordinate change above has Jacobian near 1 and as a result the bounds obtained really do depend on the bounds on derivatives of S. Hence we are finished with the proof of Lemma 3.2.

Lemma 3.3: Suppose $(t,x) \in B^S(t_0,x_0;r)$ and $|y - x_0| < 3r$. Then there is C with $|t + S(t,x,y) - t_0 - S(t_0,x_0,y)| \leq C\sigma(t_0,x_0;r)$

Proof: As in the proof of Lemma 3.2, the invariance properties of the σ's and B's under special coordinate changes allows us to assume that $S(t_0, x_0, y) = 0$. Our task is to prove that if $|t - t_0| < \sigma(t_0, x_0; r)$, $|x - x_0| < r$, and $|y - x_0| < 3r$, then there exists C such that $|t + S(t, x, y) - t_0| < C\sigma(t_0, x_0; r)$. But expanding $S(t_0, x, y)$ in x and y about $y = x = x_0$ gives

$$|S(t_0, x, y)| \leq \sum_{|\alpha|+|\beta| \leq d} C_0 |\partial_x^\alpha \partial_y^\beta S(t_0, x_0, x_0)| r^{|\alpha|+\beta|} + C_1 |r|^{d+1} < C_2 \sigma(t_0, x_0; r)$$

Therefore if $|t| < \sigma(t_0, x_0; r)$

$$|t + S(t, x, y) - t_0| \leq |t - t_0| + |S(t, x, y) - S(t_0, x, y)| + |S(t_0, x, y)| \leq C_3 \sigma(t_0, x_0; r)$$

So we are done with the proof of Lemma 3.3. We can now proceed to the proof of Fact 2.

Proof of Fact 2: Suppose $B^S(t_0, x_0; r) \cap B^S(t_1, x_1; r)$ is nonempty, and $(t, x) \in B^S(t_0, x_0; r) \cap B^S(t_1, x_1; r)$. Then $|x - x_0|, |x - x_1| < r$ (and thus $|x_0 - x_1| < 2r$) and

$$|t - S(t_1, x_1, x) - t_1| \leq \sigma(t_1, x_1; r)\,, \; |t - S(t_0, x_0, x) - t_0| \leq \sigma(t_0, x_0; r) \qquad (3.10)$$

Thus by Lemma 3.2 we have $\sigma(t_0, x_0; r) \sim \sigma(t, x; r) \sim \sigma(t_1, x_1; r)$, and by Lemma 3.3 whenever $|y - x_0|$, $|y - x_1 < 3r$ we have

$$|t + S(t, x, y) - t_0 - S(t_0, x_0, y)| \leq C\sigma(t_0, x_0; r)$$

$$|t + S(t, x, y) - t_1 - S(t_1, x_1, y)| \leq C\sigma(t_1, x_1; r)$$

Thus

$$|t_1 + S(t_1, x_1, y) - t_0 - S(t_0, x_0, y)| \leq C\sigma(t_0, x_0; r) + C\sigma(t_1, x_1; r) \leq C_1 \sigma(t_0, x_0; r) \qquad (3.11)$$

Now (3.11) holds whenever $|y - x_0|$, $|y - x_1 < 3r$, in particular when $|y - x_1| < r$. If $(s, y) \in B(t_1, x_1; r)$, then (s, y) satisfies $|y - x_1| < r$, $|s - t_1 - S(t_1, x_1, y)| \leq \sigma(t_1, x_1; r)$, and we have

$$|s - t_0 - S(t_0, x_0, y)| \leq \sigma(t_1, x_1; r) + C_1 \sigma(t_0, x_0; r) \leq C_2 \sigma(t_0, x_0; r) \leq \sigma(t_0, x_0; C_3 r) \qquad (3.12)$$

We use Lemma (3.1) for the last inequality. Furthermore, $|y - x_0| \leq |y - x_1| + |x_1 - x_0| < 3r$. (Assuming $C_3 \geq 3$) We conclude that $(s, y) \in B^S(t_0, x_0; C_3 r)$. Fact 2 is hence proven.

Lastly, we will repeatedly use this little lemma.

Lemma 3.4: For all (t, x), and for all i and j we have

(1)$|\partial_{t_i} \sigma(t, x; r)| \leq Cr^2$

(2) If (t_0, x_0) is such that $S(t_0, x_0, y) = 0$ for all y, and $(t_1, x_1) \in B(t_0, x_0; r)$, then $|\partial_{x_j} \sigma(t_1, x_1; r)| \leq (C/r)\sigma(t_0, x_0; r)$

Proof: $(\sigma(t, x; r))^2 = \sum_{0<|\alpha|,|\beta|,|\alpha|+|\beta|\leq d} (\tau_{\alpha,\beta}(t, x)(cr)^{|\alpha|+|\beta|})^2$, so

$$\partial_{t_i} \sigma(t, x; r)\sigma(t, x; r) = \sum_{0<|\alpha|,|\beta|,|\alpha|+|\beta|\leq d} \tau_{\alpha,\beta}(t, x)(cr)^{|\alpha|+|\beta|} \cdot \partial_{t_i} \tau_{\alpha,\beta}(t, x)(cr)^{|\alpha|+|\beta|}$$

So by Cauchy-Schwartz,

$$|\partial_{t_i} \sigma(t, x; r)\sigma(t, x; r)| \leq \sigma(t, x; r)(\sum |\partial_{t_i} \tau_{\alpha,\beta}(t, x)(cr)^{|\alpha|+|\beta|}|^2)^{1/2}$$

Therefore

$$|\partial_{t_i} \sigma(t, x; r)| leq C \sum |\partial_{t_i} \tau_{\alpha,\beta}(t, x)(cr)^{|\alpha|+|\beta|}|$$

Since each $|\alpha| + |\beta| \geq 2$, we conclude that $|\partial_{t_i} \sigma(t, x; r)| < Cr^2$. This completes the proof of (1).

We turn to (2). Suppose $S(t_0, x_0, y) = 0$ for all y. Then $B(t_0, x_0; r) = \{(t_1, x_1) : |t_1 - t_0| \leq \sigma(t_0, x_0; r), |x_1 - x_0| < r\}$. Arguing exactly as above, replacing the t differentiation with an x differentiation, gives us $|\partial_{x_j} \sigma(t_0, x_1; r)| < C \sum |\partial_{x_j} \tau_{\alpha,\beta}(t_0, x_1)|(cr)^{|\alpha|+|\beta|}$ Hence since $|t_1 - t_0| \leq \sigma(t_0, x_0; r)$,

$$|\partial_{x_j} \sigma(t_1, x_1; r)| < C\sigma(t_0, x_0; r) + C \sum |\partial_{x_j} \tau_{\alpha,\beta}(t_0, x_1)(cr)^{|\alpha|+|\beta|}| \tag{3.13}$$

By equation (3.3) and the characterization of the function $E_{\alpha,\beta}$,

$$\partial_{x_j} \tau_{\alpha,\beta}(t_0, x_1) = \partial_{x_j} \partial_x^\alpha \partial_y^\beta S(t_0, x_1, x_1) + F_{\alpha,\beta}(t_0, x_0, x_1)$$

here $|F_{\alpha,\beta}(t_0,x_0,x_1)| \leq C\sum_{0\leq|\alpha'|+|\beta'|\leq|\alpha|+|\beta|}|\partial_x^{\alpha'}\partial_y^{\beta'}S(t_0,x_1,x_1)|$ Hence by (3.4) and (3.9), $|\partial_{x_j}\tau_{\alpha,\beta}(t_0,x_1)|(cr)^{|\alpha|+|\beta|+1} \leq C\sigma(t_0,x_0;r)$. We conclude therefore that $\sum|\partial_{x_j}\tau_{\alpha,\beta}(t_0,x_1)|(cr)^{|\alpha|+|\beta|} \leq (C'/r)\sigma(t_0,x_0;r)$. So by (3.13)

$$|\partial_{x_j}\sigma(t_1,x_1;r)| \leq (C'/r)\sigma(t_0,x_0;r) + C\sigma(t_0,x_0;r) < (C''/r)\sigma(t_0,x_0;r)$$

Thus we have conclusion (2) of this Lemma and the proof is complete.

4: Almost-orthogonality

Lemma 4.1: There exists C such that $||T_{i,j_1}T^*_{i,j_2}|| < C2^i2^{-|j_1-j_2|/2}$

Proof: Since $||T_{i,j_1}T^*_{i,j_2}|| = ||(T_{i,j_1}T^*_{i,j_2})^*|| = ||T_{i,j_2}T^*_{i,j_1}||$, without loss of generality we may assume that $j_2 \geq j_1$. The kernel of $T_{i,j_1}T^*_{i,j_2}$ is $P(t_1,x_1,t_2,x_2) =$

$$\int 2^{2i}\sigma(t_2,x_2,2^{-j_2})^{-1}\sigma(t_1,x_1,2^{-j_1})^{-1}\gamma(2^i\sigma(t_2,x_2,2^{-j_2})^{-1}(t_2+S(t_2,x_2,y)-s))$$

$$\gamma(2^i\sigma(t_1,x_1,2^{-j_1})^{-1}(s-t_1+S(t_1,x_1,y))K_{j_1}(t_1,x_1,y)K_{j_2}(t_2,x_2,y)ds\,dy \quad (4.1)$$

(when $i=0$ replace γ by γ_0). We will show that $\int|P(t_1,x_1,t_2,x_2)|dt_1dx_1 < C2^{2i}2^{j_1-j_2}$. The estimate $\int|P(t_1,x_1,t_2,x_2)|dt_2dx_2 < C$ is easy (integrate with respect to t_2, s, x_2, then y) and the Lemma then follows from Lemma 2.1.

First we will do a coordinate change to insure that $S(t_2,x_2,y)=0$ for all y. Note that $\int|P(t_1,x_1,t_2,x_2)|dt_1\,dx_1 = \int|P(t_1+S(t_2,x_2,x_1),x_1,t_2,x_2)|dt_1\,dx_1$ and thus it suffices to prove estimates on $\int|P(t_1+S(t_2,x_2,x_1),x_1,t_2,x_2)|dt_1\,dx_1$. Changing variables in (4.1) from s to $s'=s-S(t_2,x_2,y)$, we get $P(t_1+S(t_2,x_2,x_1),x_1,t_2,x_2) =$

$$\int 2^{2i}\sigma(t_2,x_2;2^{-j_2})^{-1}\sigma(t_1+S(t_2,x_2,x_1),x_1;2^{-j_1})^{-1}K_{j_1}(t_1+S(t_2,x_2,x_1),x_1,y)$$

$$K_{j_2}(t_2,x_2,y)\gamma(2^i\sigma(t_2,x_2;2^{-j_2})^{-1}(t_2-s))\gamma(2^i\sigma(t_1+S(t_2,x_2,x_1),x_1;2^{-j_1})^{-1}$$

$$(s+S(t_2,x_2,y)-S(t_1+S(t_2,x_2,x_1),x_1,y)-t_1-S(t_2,x_2,x_1)))dy$$

$$= \int 2^{2i}\sigma(t_2,x_2;2^{-j_2})^{-1}\sigma(t_1,x_1;2^{-j_1})^{-1}\gamma(2^i\sigma(t_2,x_2;2^{-j_2})^{-1}(t_2+S_1(t_2,x_2,y)-s)$$

$$\gamma(2^i\sigma(t_1,x_1;2^{-j_1})^{-1}(s-t_1-S_1(t_1,x_1,y)))K'_{j_1}(t_1,x_1,y)K_{j_2}(t_2,x_2,y)dy \quad (4.2)$$

Here $S_1(t,x,y) = S(t_2,x_2,x) + S(t + S(t_2,x_2,x),x,y) - S(t_2,x_2,y)$, σ now corresponds to S_1 instead of S, and where K'_{j_1} is the function $K_{j_1}(t_1{+}S(t_2,x_2,x_1),x_1,y)$ Note that now $S_1(t_2,x_2,y) = 0$ for all y. That we can switch σ's like this follows from the transformation properties of the $\tau_{\alpha,\beta}$ under this type of coordinate change. (See the discussion preceeding the proof of Lemma (3.1)).

As a result we may assume we are dealing with S_1 instead of S and it suffices to prove $\int |P_0(t_1,x_1,t_2,x_2)|dt_1\,dx_1 < C2^{3i}2^{j_1-j_2}$, where P_0 is P with S replaced by S_1 and K_{j_1} replaced by K'_{j_1}.

Since γ is supported on $|x| < 1$, the integrand in (4.2) is nonzero only if s satisfies

$$|s - t_2 - S_1(t_2,x_2,y)| < 2^{-i}\sigma(t_2,x_2;2^{-j_2}) < \sigma(t_2,x_2;2^{-j_2}) \tag{4.3a}$$

$$|s - t_1 - S_1(t_1,x_1,y)| < 2^{-i}\sigma(t_1,x_1;2^{-j_1}) < \sigma(t_1,x_1;2^{-j_1}) \tag{4.3b}$$

$(4.3a)$ means that $(s,y) \in B(t_2,x_2;2^{-j_2})$. $(4.3a)$ and $(4.3b)$ taken together imply that $B(t_2,x_2;2^{-j_2}) \cap B(t_1,x_1;2^{-j_1}) \neq 0$. Thus $B(t_2,x_2;2^{-j_1}) \cap B(t_1,x_1;2^{-j_1}) \neq 0$, so by Lemma 3.2, $\sigma(t_1,x_1;2^{-j_1}) \sim \sigma(t_2,x_2;2^{-j_1})$, and by Fact 2 we have $(t_1,x_1) \in B(t_2,x_2;C2^{-j_1})$ whenever $P(t_1,x_1,t_2,x_2)$ is nonzero.

Note now that if in (4.2) we replace (s,y) by (t_2,x_2) in the expression $\gamma(2^i\sigma(t_1,x_1;2^{-j_1})^{-1}(s-t_1-S_1(t_1,x_1,y)))K'_{j_1}(t_1,x_1,y)$, we obtain the kernel $P'(t_1,x_1,t_2,x_2) =$

$$\int 2^{2i}\sigma(t_2,x_2;2^{-j_2})^{-1}\sigma(t_1,x_1;2^{-j_1})^{-1}\gamma(2^i\sigma(t_2,x_2;^{-j_2})^{-1}(t_2 + S_1(t_2,x_2,y) - s))$$

$$\gamma(2^i\sigma(t_1,x_1;2^{-j_1})^{-1}(t_2 - t_1 + S_1(t_1,x_1,x_2))K'_{j_1}(t_1,x_1,x_2)K_{j_2}(t_2,x_2,y)ds\,dy$$

If we first integrate in s and then in y the cancellation condition on K gives us that $|P'| <$

$$2^{2i}2^{-j_2}\sigma(t_1,x_1;2^{-j_1})^{-1}\gamma(2^i\sigma(t_1,x_1;2^{-j_1})^{-1}(t_2 - t_1 + S_1(t_1,x_1,x_2))K'_{j_1}(t_1,x_1,x_2)$$

Then, integrating with respect to first t_1 then x_1 we have $|P'|dt_1\,dx_2 < C2^{-j_2}$; for the t_1 integration observe that since $\sigma(t_1,x_1;2^{-j_1}) \sim \sigma(t_2,x_2;2^{-j_1})$, the $\gamma((2^i\sigma(t_1,$

$x_1; 2^{-j_1})^{-1}(t_2 - t_1 + S_1(t_1, x_1, x_2))$ factor is supported on a set of t-diameter $<$ $C\sigma(t_2, x_2; 2^{-j_1})$.

As a result, if we denote the function $\gamma(2^i\sigma(t_1, x_1; 2^{-j_1})^{-1}(s - t_1 - S_1(t_1, x_1, y)))K_{j_1}(t_1, x_1, y)$ by $F_{(t_1,x_1,t_2,x_2)}(s, y)$, then $\int |P(t_1, x_1, t_2, x_2)|dt_1\, dx_1 <$ $C2^{-j_2}+$

$$\int_{(s,y)\in B(t_2,x_2;2^{-j_2}),(t_1,x_1)\in B(t_2,x_2;C2^{-j_1})} 2^{2i}\sigma(t_2, x_2; 2^{-j_2})^{-1}\sigma(t_1, x_1; 2^{-j_1})^{-1}$$

$$\gamma(2^i\sigma(t_2, x_2; 2^{-j_2})^{-1}(t_2 + S_1(t_2, x_2, y) - s))|F_{(t_1,x_1,t_2,x_2)}(s, y) - F_{(t_1,x_1,t_2,x_2)}(t_2, s_2)|$$

$$|K_{j_2}(t_2, x_2, y)|dydsdx_1dt_1 \tag{4.4}$$

We use the mean-value theorem to estimate the difference $|F_{(t_1,x_1,t_2,x_2)}(s, y) - F_{(t_1,x_1,t_2,x_2)}(t_2, s_2)|$. Because $S_1(t_2, x_2, y)$ is always 0, the ball $B(t_2, x_2, 2^{-j_2})$ is the convex box $\{(t, x) : |t - t_2| < \sigma(t_2, x_2; 2^{-j_2}), |x - x_2| < 2^{-j_2}\}$ and as a result $|F_{(t_1,x_1,t_2,x_2)}(s, y) - F_{(t_1,x_1,t_2,x_2)}(t_2, s_2)| <$

$$\sigma(t_2, x_2; 2^{-j_2}) \sup_{(s,y)\in B(t_2,x_2,2^{-j_2})} |\partial_s F_{(t_1,x_1,t_2,x_2)}(s, y)|$$

$$+2^{-j_2} \sup_{(s,y)\in B(t_2,x_2,2^{-j_2})} |\nabla_y F_{(t_1,x_1,t_2,x_2)}(s, y)|$$

This is $< 2^iC\sigma(t_2, x_2; 2^{-j_2})\sigma(t_1, x_1; 2^{-j_1})^{-1}|x_1 - y|^{-n} + C2^{j_1-j_2}|x_1 - y|^{-n-1}$: For the second term, estimate $\partial_y S_1(t_1, x, y)$ by expanding in x and y about $x = y = x_1$ and using (3.9). By Lemma 3.1 and the fact that $|x_1 - y| \sim 2^{-j_1}$, this quantity is at most $C'2^i2^{j_1-j_2}|x_1 - y|^{-n}$. As a result (4.4) $<$

$$\int_{(t_1,x_1)\in B(t_2,x_2;C2^{-j_1}),|x_1-y|\sim 2^{-j_1},|x_2-y|\sim 2^{-j_2}} 2^{3i}2^{j_1-j_2}$$

$$\sigma(t_2, x_2; 2^{-j_2})^{-1}\sigma(t_1, x_1; 2^{-j_1})^{-1}\gamma(2^i\sigma(t_2, x_2; 2^{-j_2})^{-1}(t_2 + S_1(t_2, x_2, y) - s))$$

$$|x_1 - y|^{-n}|x_2 - y|^{-n}dsdt_1dx_1dy \tag{4.5}$$

We now integrate with respect to s, t_1, x_1, then y, keeping in mind that $|s - t_2| <$ $2^{-i}\sigma(t_2, x_2; 2^{-j_2})$ for $(s, y) \in B(t_2, x_2; 2^{-j_2})$ and $|t_1 - t_2| < C'\sigma(t_2, x_2; 2^{-j_1})$ in the domain of integration. The result is that (4.5) $< 2^{3i}2^{j_1-j_2}$. This completes the proof of Lemma 4.1.

Lemma 4.2: $\exists C$ such that $||T^*_{i,j_1}T_{i,j_2}|| < C2^{\frac{3}{2}i}2^{-|j_1-j_2|/2}$

Proof: The kernel of $T^*_{i,j_1}T_{i,j_2}$, call it $Q(s_1,y_1,s_2,y_2)$, is given by

$$\int 2^{2i}\sigma(t,x;2^{-j_2})^{-1}\sigma(t,x;2^{-j_1})^{-1}\gamma(2^i\sigma(t,x;2^{-j_2})^{-1}(t+S(t,x,y_2)-s_2))$$

$$\gamma(2^i\sigma(t,x;2^{-j_1})^{-1}(t+S(t,x,y_1)-s_1))K_{j_1}(t,x,y_1)K_{j_2}(t,x,y_2)dx\,dt \tag{4.6}$$

(When $i=0$ replace γ by γ_0). Again we might as well assume $j_1 \leq j_2$. Analogous to before, our objective will be to show $\int |Q(s_1,y_1,s_2,y_2)|ds_1dy_1 < C2^{3i}2^{j_1-j_2}$. Once more $\int |Q(s_1,y_1,s_2,y_2)|ds_2dy_2 < C$ is a simple matter, and Lemma 2.1 will then complete the proof of this lemma.

As before we perform a convenient coordinate change, this time to insure $S(s_2,y_2,y)=0$ for all y. To this end, we consider $Q(s_1+S(s_2,y_2,y_1),y_1,s_2,y_2)$ and change variables in (4.6) from t to $t'=t-S(s_2,y_2,x)$. Like in the previous proof, we have replaced S by $S_1(t,x,y)=S(s_2,y_2,x)+S(t+S(s_2,y_2,x),x,y)-S(s_2,y_2,y)$, the σ's for S have been replaced by the σ's for S_1, and we have replaced the kernels $K_{j_1}(t,x,y)$ and $K_{j_2}(t,x,y)$ by $K'_{j_1}(t,x,y)=K_{j_1}(t+S(s_2,y_2,x),x,y)$ and $K'_{j_2}(t,x,y)=K_{j_2}(t+S(s_2,y_2,x),x,y)$ respectively. Thus in the following we may replace $Q(s_1,y_1,s_2,y_2)$ by $Q'(s_1,y_1,s_2,y_2)=Q(s_1+S(s_2,y_2,y_1),y_1,s_2,y_2)$, $S(t,x,y)$ by $S_1(t,x,y)$, $K_{j_1}(t,x,y)$ by $K'_{j_1}(t,x,y)$, and $K_{j_2}(t,x,y)$ by $K'_{j_2}(t,x,y)$.

In analogy to (4.3a) and (4.3b), note that when the integrand in (4.6) is nonzero, $(s_1,y_1)\in B(t,x;2^{-j_1})$ and $(s_2,y_2)\in B(t,x;2^{-j_2})$. So by Fact 2, $(t,x)\in B(s_1,y_1;C2^{-j_1})\cap B(s_2,y_2;C2^{-j_2})$, and then by Fact 2 again we have $(s_1,y_1)\in B(s_2,y_2;C^22^{-j_1})$. So since $S_1(s_2,y_2,y)=0$ for all y, we have $|t-s_2|< C'\sigma(s_2,y_2;2^{-j_2})$ and $|s_1-s_2|<C'\sigma(s_2,y_2;2^{-j_1})$.

Our strategy here is not all that different from that of Lemma 4.1. We would like to "fix" (t,x) at (s_2,y_2) in (4.6). Correspondingly, we define the function $G_{(s_1,y_1,s_2,y_2,t,x)}(t',x')=$

$$\sigma(t',x;2^{-j_2})^{-1}\sigma(t',x';2^{-j_1})^{-1}\gamma(2^i\sigma(t',x;2^{-j_2})^{-1}(t+S(t,x,y_2)-s_2))$$

$$\gamma(2^i\sigma(t',x';2^{-j_1})^{-1}(t'+S(t',x',y_1)-s_1))K'_{j_1}(t',x',y_1)K'_{j_2}(t',x,y_2)$$

We then write $Q = Q' + Q''$, where $Q'(s_1, y_1, s_2, y_2) =$

$$\int 2^{2i}\sigma(s_2,x;2^{-j_2})^{-1}\sigma(s_2,y_2;2^{-j_1})^{-1}\gamma(2^i\sigma(s_2,x;2^{-j_2})^{-1}(t+S(t,x,y_2)-s_2))$$

$$\gamma(2^i\sigma(s_2,y_2;2^{-j_1})^{-1}(s_2+S(s_2,y_2,y_1)-s_1))K'_{j_1}(s_2,y_2,y_1)K'_{j_2}(s_2,x,y_2)dx\,dt \tag{4.7}$$

And $|Q''(s_1, y_1, s_2, y_2| \leq$

$$\int 2^{2i}|G_{(s_1,y_1,s_2,y_2,t,x)}(t,x) - G_{(s_1,y_1,s_2,y_2,t,x)}(s_2,y_2)|dx\,dt \tag{4.8}$$

Like before we may use the cancellation condition to estimate Q'. In the dt integration of (4.7) do a change of variables $t' = t + S(t, x, y_2) - s_2$. Since $S(t, x, x) = 0$, $\partial_t S(t, x, x) = 0$ and this change of variables has Jacobian $1 + 0(|x - y_2|)$. Furthermore, $K'_{j_1}(s_2, y_2, y_1)K'_{j_2}(s_2, x, y_2) = K_{j_1}(s_2 + S(s_2, y_2, x), y_2, y_1)K_{j_2}(s_2 + S(s_2, y_2, x), x, y_2) = K_{j_1}(s_2, y_2, y_1)K_{j_2}(s_2, x, y_2) + O(|x-y_2|^{-n+1})$ using $(K1)$. As a result Q' may be written as the sum of 2 terms. The first is $Q'_1(s_1, y_1, s_2, y_2) =$

$$\int 2^{2i}\sigma(s_2,x;2^{-j_2})^{-1}\sigma(s_2,y_2;2^{-j_1})^{-1}\gamma(2^i\sigma(s_2,x;2^{-j_2})^{-1}t')$$

$$\gamma(2^i\sigma(s_2,y_2;2^{-j_1})^{-1}(s_2+S(s_2,y_2,y_1)-s_1))K_{j_1}(s_2,y_2,y_1)K_{j_2}(s_2,x,y_2)dx\,dt'$$

Integrating with respect to t' then x the cancellation condition $(K2)$ gives us $|Q'_1(s_1, y_1, s_2, y_2)| < C2^i\times$

$$2^{-j_2}\sigma(s_2,y_2;2^{-j_1})^{-1}\gamma(2^i\sigma(s_2,y_2;2^{-j_1})^{-1}(s_2+S(s_2,y_2,y_1)-s_1))K_{j_1}(s_2,y_2,y_1)$$

Integrating with repect to s_1 then y_1 then shows $\int |Q'_1|ds_1\,dy_1 < C2^{-j_2}$, better than the estimate we need.

Our second term Q'_2 is bounded in absolute value by

$$C\int_{|y_2-y_1|\sim 2^{-j_1},|y_2-x|\sim 2^{-j_2}} \sigma(s_2,x;2^{-j_2})^{-1}\sigma(s_2,y_2;2^{-j_1})^{-1}\gamma(2^i\sigma(s_2,x;2^{-j_2})^{-1}t')$$

$$2^{2i}|x-y_2|\gamma(2^i\sigma(s_2,y_2;2^{-j_1})^{-1}(s_2+S(s_2,y_2,y_1)-s_1))|y_2-y_1|^{-n}||x-y_2|^{-n}dx\,dt'$$

However $|x - y_2| \sim 2^{-j_2}$, so when this expression is integrated with respect to s_1 and y_1 we again get something $< C2^{-j_2}$.

We conclude that $\int |Q'(s_1, y_1, s_2, y_2)| ds_2\, dy_2 < C2^{-j_2}$. We thus may direct our attention to Q''. Analogous to the proof of Lemma 4.1, we observe that since $B(s_2, y_2; C2^{-j_2}) = \{(s, y) : |s - s_2| < \sigma(s_2, y_2; C2^{-j_2}), |y - y_2| < C2^{-j_2}\}$, we have that $|G_{(s_1,y_1,s_2,y_2,t,x)}(t, x) - G_{(s_1,y_1,s_2,y_2,t,x)}(s_2, y_2)| <$

$$C\sigma(s_2, y_2; 2^{-j_2}) \sup_{(t',x')\in B(s_2,y_2;2^{-j_2})} |\partial_{t'} G_{(s_1,y_1,s_2,y_2,t',x')}(t', x')|$$

$$+C2^{-j_2} \sup_{(t',x')\in B(s_2,y_2;2^{-j_2})} |\nabla_{x'} G_{(s_1,y_1,s_2,y_2,t',x')}(t', x')|$$

To estimate this expression we examine the derivative on a term by term basis. The details, not hard, are omitted. The facts about $(t', x') \in B(s_2, y_2; 2^{-j_2})$ being used are

(1) For $r = 2^{-j_1}$ and 2^{-j_2}, $\sigma(t', x'; r) \sim \sigma(s_2, y_2; r)$

(2) $\sigma(t', x'; 2^{-j_2}) \leq 2^{j_1-j_2}\sigma(t', x'; 2^{-j_1})$

(3) For $r = 2^{-j_1}$, 2^{-j_2}, $\partial_t\sigma(t', x'; r) < Cr^2$ and $\nabla_x\sigma(t', x'; r) < (C/r)\sigma(s_2, y_2; r)$

(4) $\gamma(t)$ and $\gamma'(t)$ are supported on $|t| < 1$.

(5) The estimates $(K1)$.

The conclusion is that $|G_{(s_1,y_1,s_2,y_2,t,x)}(t, x) - G_{(s_1,y_1,s_2,y_2,t,x)}(s_2, y_2)| <$

$$C2^i2^{j_1-j_2}\sigma(s_2, y_2; 2^{-j_1})^{-1}\sigma(s_2, y_2; 2^{-j_2})^{-1}|x - y_1|^{-n}|x - y_2|^{-n}$$

As a result, $|Q''(s_1, y_1, s_2, y_2)|ds_1\, dy_1 <$

$$\int_{|t-s_2|<C\sigma(s_2,y_2;2^{-j_2}),|x-y_2|\sim 2^{-j_2},|x-y_1|\sim 2^{-j_1},|s_1-s_2|<C\sigma(s_2,y_2;2^{-j_1})}$$

$$2^{2i}2^i2^{j_1-j_2}\sigma(s_2, y_2; 2^{-j_1})^{-1}\sigma(s_2, y_2; 2^{-j_2})^{-1}|x - y_1|^{-n}|x - y_2|^{-n}dx\, dt\, ds_1\, dy_1$$

$< C2^{2i}2^i2^{j_1-j_2}$ This estimate for $\int |Q''(s_1, y_1, s_2, y_2)|dy_1\, dx_1$ completes the proof of Lemma 4.2.

5: Estimates on Individual $T_{i,j}$'s

Theorem 5.1: $||T_{i,j}|| < C2^{-\delta i}$ for all i and j.

Proof: Recall $T_{i,j}T_{i,j}^*$ has kernel $P(t_1, x_1, t_2, x_2) =$

$$\int 2^{2i}\sigma(t_2, x_2; 2^{-j})^{-1}\sigma(t_1, x_1; 2^{-j})^{-1}\gamma(2^i\sigma(t_2, x_2; 2^{-j})^{-1}(t_2 + S(t_2, x_2, y) - s))$$

$$\gamma(2^i\sigma(t_1, x_1; 2^{-j})^{-1}(s - t_1 - S(t_1, x_1, y))K_j(t_1, x_1, y)K_j(t_2, x_2, y)dy\, ds \qquad (5.1)$$

We will prove that $\int |P(t_1, x_1, t_2, x_2)|dt_2dx_2 < C2^{-\delta i}$, by symmetry we will then also have $\int |P(t_1, x_1, t_2, x_2)|dt_1dx_1 < C2^{-\delta i}$.

The definition (5.1) for $P(t_1, x_1, t_2, x_2)$ can be rewritten

$$\int \sigma(t_2, x_2; 2^{-j})^{-1}\sigma(t_1, x_1; 2^{-j})^{-1}K_j(t_1, x_1, y)K_j(t_2, x_2, y)\gamma(2^i\sigma(t_1, x_1; 2^{-j})^{-1}s)$$

$$2^{2i}\gamma(2^i\sigma(t_2, x_2; 2^{-j})^{-1}(t_2 + S(t_2, x_2, y) - t_1 - S(t_1, x_1, y) - s))dy\, ds \qquad (5.1')$$

$P(t_1, x_1, t_2, x_2)$ is clearly nonzero only if there is an s with $|s| < 2^{-i}\sigma(t_1, x_1; 2^{-j})$ and $|t_2 + S(t_2, x_2, y) - t_1 - S(t_1, x_1, y) - s| < 2^{-i}\sigma(t_2, x_2; 2^{-j})$ and thus only if $|t_2 + S(t_2, x_2, y) - t_1 - S(t_1, x_1, y)| < 2^{-i}\sigma(t_2, x_2; 2^{-j}) + 2^{-i}\sigma(t_1, x_1; 2^{-j}) < C_1 2^{-i}\sigma(t_1, x_1; 2^{-j})$. (The last inequality follows from the fact that $\sigma(t_1, x_1; 2^{-j}) \sim \sigma(t_2, x_2; 2^{-j})$ which we saw subsequent to eqs (4.3)). Recall that $t^*(t_1, x_1, x_2, y)$ satisfies $t^*(t_1, x_1, x_2, y) + S(t^*(t_1, x_1, x_2, y), x_2, y) - t_1 - S(t_1, x_1, y) = 0$ and that $|\partial_t S| < 1/2$. As a result $|t_2 + S(t_2, x_2, y) - t_1 - S(t_1, x_1, y)| < C_1 2^{-i}\sigma(t_1, x_1, 2^{-j})$ only if $|t_2 - t^*(t_1, x_1, x_2, y)| < 2C_1 2^{-i}\sigma(t_1, x_1, 2^{-j})$, and hence the integrand in (5.1) is nonzero only if $|t_2 - t^*(t_1, x_1, x_2, y)| < 2C_1 2^{-i}\sigma(t_1, x_1, 2^{-j})$.

However, expanding in t_2 about $t^*(t_1, x_1, x_2, y)$, we get

$$\gamma(2^i\sigma(t_2, x_2; 2^{-j})^{-1}(t_2 + S(t_2, x_2, y) - t_1 - S(t_1, x_1, y) - s)) =$$

$$\gamma(2^i\sigma(t_2, x_2; 2^{-j})^{-1}L(t_1, x_1, x_2, y)(t_2 - t^*(t_1, x_1, x_2, y) - s))$$
$$+O(2^i\sigma(t_2, x_2; 2^{-j})^{-1}(t_2 - t^*(t_1, x_1, x_2, y))^2)$$

Here $L(t_1, x_1, x_2, y)$ is a function of absolute value at least $1/2$. But

$$\int_{|t_2 - t^*(t_1, x_1, x_2, y)| < 2C_1 2^{-i}\sigma(t_1, x_1, 2^{-j})} | \int 2^{2i}\sigma(t_2, x_2; 2^{-j})^{-1}\sigma(t_1, x_1; 2^{-j})^{-1}$$

$$K_j(t_1,x_1,y)K_j(t_2,x_2,y)O(2^i\sigma(t_2,x_2;2^{-j})^{-1}(t_2-t^*(t_1,x_1,x_2,y))^2)$$

$$\gamma(2^i\sigma(t_1,x_1;2^{-j})^{-1}s)dy\,ds|dt_2\,dx_2$$

$< C2^{-i}\sigma(t_2,x_2;2^{-j})$ (Integrate successively with respect to s, t_2, x_2, y.) Hence if we let $P'(t_1,x_1,t_2,x_2) =$

$$\int 2^{2i}\sigma(t_2,x_2;2^{-j})^{-1}\sigma(t_1,x_1;2^{-j})^{-1}\gamma(2^i\sigma(t_2,x_2;2^{-j})^{-1}L(t_2-t^*(t_1,x_1,x_2,y)-s))$$

$$\gamma(2^i\sigma(t_1,x_1;2^{-j})^{-1}s)K_j(t_1,x_1,y)K_j(t_2,x_2,y)dy\,ds \tag{5.2}$$

it suffices to prove $|P'(t_1,x_1,t_2,x_2)|dt_2dx_2 < C2^{-\delta i}$, which is the estimate we will prove.

Let $\delta_0 > 0$ and let α', β' be multiindices such that $|\tau_{\alpha',\beta'}(t_1,x_1)|(c2^{-j})^{|\alpha'|+|\beta'|} =$

$$|\partial_{x_2}^{\alpha'}\partial_y^{\beta'}t^*(t_1,x_1,x_1,x_1)_{x_2=x_1=y}|(c2^{-j})^{|\alpha'|+|\beta'|} > \delta_0\sigma(t_1,x_1;2^{-j})$$

Since the multiindices of order $d-1$ are spanned by the multiindices of the form ∂_z^{d-1}(i.e differentiation in the same direction $d-1$ times), we can find a y_k and a $\partial_z = \sum_l \alpha_l\partial_{x_l} + \sum_1 \beta_l\partial_{y_l}$ with $\sum|\alpha_l|^2+|\beta_l|^2 = 1$ such that

$$|\partial_z^{|\alpha'|+|\beta'|-1}\partial_{y_k}t^*(t_1,x_1,x_2,y)_{x_2=y=x_1}|(c2^{-j})^{|\alpha'|+|\beta'|} > \delta_1\sigma(t_1,x_1;2^{-j})$$

If we expand $\partial_z^{|\alpha'|+|\beta'|-1}\partial_{y_k}t^*(t_1,x_1,x_2,y)$ in x_2 and y about $x_2 = y = x_1$, for $|x_2-x_1|,|y-x_1| < 22^{-j}$ we have

$$|\partial_z^{|\alpha'|+|\beta'|-1}\partial_{y_k}t^*(t_1,x_1,x_2,y) - \partial_z^{|\alpha'|+|\beta'|-1}\partial_{y_k}t^*(t_1,x_1,x_1,x_1)|$$

$$\leq C\sum_{|\alpha'|+|\beta'|<|\alpha|+|\beta|\leq d}\partial_{x_2}^{\alpha}\partial_y^{\beta}t^*(t_1,x_1,x_1,x_1)2^{-j(|\alpha|+|\beta|-|\alpha'|-|\beta'|)}$$

(The higher order terms are negligible since at least one of the derivatives of order at most d is of absolute value at least ϵ.) Denoting $|\alpha'|+|\beta'|$ by l, we thus have

$$|\partial_z^{l-1}\partial_{y_k}t^*(t_1,x_1,x_2,y)(c2^{-j})^l - \partial_z^{l-1}\partial_{y_k}t^*(t_1,x_1,x_1,x_1)(c2^{-j})^l|$$

$$\leq (1/c)C\sum_{l<|\alpha|+|\beta|\leq d+1}\partial_{x_2}^{\alpha}\partial_y^{\beta}t^*(t_1,x_1,x_1,x_1)(c2^{-j})^{|\alpha|+|\beta|} \leq (1/c)C'\sigma(t_1,x_1;2^{-j})$$

(Again, the higher order terms are negligible.) Therefore if c has been chosen large enough, $|\partial_z^{l-1}\partial_{y_k}t^*(t_1,x_1,x_2,y)(c2^{-j})^l| > \delta_2\sigma(t_1,x_1;2^{-j})$ for all $|x_2-x_1|, |y-x_1| < 22^{-j}$.

In what follows, for notational convenience, we write $\sigma_1 = \sigma(t_1,x_1;2^{-j})$ and $\sigma_2 = \sigma(t_2,x_2;2^{-j})$. We split (5.2) into 2 parts, $P_1(t_1,x_1,x_2,y) =$

$$\int 2^{2i}\sigma_2^{-1}\sigma_1^{-1}\gamma_0(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*)\gamma(2^i\sigma_2^{-1}L(t_2-t^*-s))$$
$$\gamma(2^i\sigma_1^{-1}s)K_j(t_1,x_1,y)K_j(t_2,x_2,y)dy\,ds$$

and where $P_2(t_1,x_1,t_2,x_2)$ is the same except the bump function γ_0 is replaced by $1-\gamma_0$. We now apply a Van der Corput-like argument, and for P_1 we need the following simple lemmas concerning the growth of functions subintervals of the real line with nonvanishing qth derivative.

Lemma 5.1: If $I \subset R$ is an interval, and $f : I \to R$ is a function with $|f^{(q)}| \geq 1$ then the set $\{t : |f(t)| \leq 1\}$ has measure $\leq C_q$.

Proof: We consider the first statement first. The proof is similar in spirit to that of Van der Corput's lemma. By downwards induction on $r \leq q$, we show that there is a collection F_r of at most $2^{q-r+1}-1$ subintervals of I of length ≤ 2 such that $|f^r(t)| > 1$ outside the union of the intervals in F_r. For $r = q$ this is obvious. Suppose we are at step r of the induction. By induction hypothesis, $I-$ the union of the intervals in F_r is the union of at most 2^{q-r+1} intervals, on each of which we have $|f^{(r+1)}| \geq 1$. Suppose J is one of these intervals. Then there is a subinterval J' of J of length at most 2 such that $|f^{(r)}| \geq 1$ outside this interval. We define F_r to be the union of F_{r+1} and the J' corresponding to a J. Thus induction follows.

Lemma 5.2: If $I \subset R$ is an interval, and $f : I \to \mathbf{R}$ is a function with $|f^{(q)}| \geq \epsilon_1$ then the set $\{t : |f(t)| \leq \epsilon_2\}$ has measure $\leq C_q(\epsilon_2/\epsilon_1)^{1/q}$.

Proof: Just use $1/\epsilon_2\ f((\epsilon_2/\epsilon_1)^{1/q}t)$ in Lemma (5.1).

Proceeding to the analysis of P_1, integrating with respect to t_2 then s we get $\int |P_1(t_1,x_1,t_2,x_2)|dt_2dx_2 <$

$$C\int \gamma_0(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*(t_1,x_1,x_2,y))|K_j(t_1,x_1,y)K_j(t_2,x_2,y)|dydx_2$$

$$< C'2^{2nj}\int_{|y-x_1|,|x_2-x_1|<22^{-j}} \gamma_0(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*(t_1,x_1,x_2,y)dydx_2 \qquad (5.3)$$

But letting $q = |\alpha'| + |\beta'| - 1$, we have $|\partial_z^q\partial_{y_k}t^*| > \delta_2\sigma_1(c2^{-j})^{-q-1}$. Thus by Lemma (5.2), along any ray in direction z we have $\gamma_0(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*) \neq 0$ on a set of measure at most $C2^{-i/4q}2^{-j}$. Thus first integrating (5.3) in the z-direction, and then in the other $2n-1$ directions, we get (5.3) $< C2^{-i/4q}$. We are thus done with the analysis of P_1.

We now proceed to the analysis of P_2. We are going to integrate by parts in the y_k direction. To this end, we write $P_2 = P_2' + P_2''$, where $P_2'(t_1,x_1,t_2,x_2) =$

$$-\int 2^{2i}\sigma_2^{-1}\sigma_1^{-1}\frac{(\partial_{y_k}L)(t_2 - t^*(t_1,x_1,x_2,y)-s)}{L\partial_{y_k}t^*(t_1,x_1,x_2,y)}(1-\gamma_0)(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*)$$

$$\gamma(2^i\sigma_2^{-1}L_{t_1,x_1,x_2,y}(t_2-t^*(t_1,x_1,x_2,y)-s))\gamma(2^i\sigma_1^{-1}s)K_j(t_1,x_1,y)K_j(t_2,x_2,y)dy\,ds$$

and where $P_2''(t_1,x_1,t_2,x_2) =$

$$\int 2^{2i}\sigma_2^{-1}\sigma_1^{-1}\frac{\partial_{y_k}[L(t_2 - t^*(t_1,x_1,x_2,y)-s)]}{L\partial_{y_k}t^*(t_1,x_1,x_2,y)}(1-\gamma_0)(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*)$$

$$\gamma(2^i\sigma_2^{-1}(t_2-t^*(t_1,x_1,x_2,y)-s))\gamma(2^i\sigma_1^{-1}s)K_j(t_1,x_1,y)K_j(t_2,x_2,y)dy\,ds$$

But $\int|P_2'(t_1,x_1,t_2,x_2)|dt_2dx_2$ is easy to estimate; we utilize the facts that $|(t_2 - t^*(t_1,x_1,x_2,y)-s)| < 2^{-i}\sigma_1$ and $|(1/\partial_{y_k}t^*(t_1,x_1,x_2,y))| < 2^{i/4}2^{-j}\sigma_1^{-1}$. We get

$$\int|P_2'(t_1,x_1,t_2,x_2)|dt_2dx_2 < C2^{-\frac{3}{4}i}2^{-j}\times$$

$$\int 2^{2i}\sigma_2^{-1}\sigma_1^{-1}\gamma(2^i\sigma_2^{-1}L(t_2-t^*-s))\gamma(2^i\sigma_1^{-1}s)|K_j(t_1,x_1,y)K_j(t_2,x_2,y)|dy\,ds\,dt_2dx_2$$

which is $< C2^{-\frac{3}{4}i}2^{-j} < C2^{-\frac{3}{4}i}$ after an integration successively with respect to t_2, s, x_2, y.

In $P_2''(t_1,x_1,t_2,x_2)$ we integrate by parts. We integrate the $\partial_{y_k}[L(t_2 - t^*(t_1,x_1,x_2,y)-s)]\gamma(2^i\sigma_2^{-1}L(t_2-t^*(t_1,x_1,x_2,y)-s))$ which for some function δ is equal to $2^{-i}\sigma_2\partial_{y_k}\delta(2^i\sigma_2^{-1}L(t_2-t^*(t_1,x_1,x_2,y)-s))$ because $\int\gamma = 0$. After our integration by parts, we have that $\int|P_2''(t_1,x_1,t_2,x_2)|dt_2dx_2 <$

$$\int 2^{2i}\sigma_2^{-1}\sigma_1^{-1}2^{-i}\sigma_2\delta(2^i\sigma_2^{-1}L(t_2-t^*(t_1,x_1,x_2,y)-s))\gamma(2^i\sigma_1^{-1}s)$$

$$|\partial_{y_k} \frac{(1-\gamma_0)(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*(t_1,x_1,x_2,y))K_j(t_1,x_1,y)K_j(t_2,x_2,y)}{L\partial_{y_k}t^*(t_1,x_1,x_2,y)}|dy\,ds\,dx_2\,dt_2 \tag{5.4}$$

We examine the y_k derivative on a term by term basis and bound the resulting terms. The first term is

$$|(\partial_{y_k}L^{-1})\frac{(1-\gamma_0)(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*(t_1,x_1,x_2,y))K_j(t_1,x_1,y)K_j(t_2,x_2,y)}{\partial_{y_k}t^*(t_1,x_1,x_2,y)}|$$

$$\begin{aligned}
&< C\frac{(1-\gamma_0)(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*(t_1,x_1,x_2,y))|K_j(t_1,x_1,y)K_j(t_2,x_2,y)|}{\partial_{y_k}t^*(t_1,x_1,x_2,y)}\\
&< C2^{i/4}2^{-j}\sigma_1^{-1}|K_j(t_1,x_1,y)K_j(t_2,x_2,y)|\\
&< C2^{i/4}2^{-j}\sigma_1^{-1}2^{2nj}
\end{aligned}$$

The next term is

$$|L(\partial_{y_k}\frac{1}{\partial_{y_k}t^*})(1-\gamma_0)(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*)K_j(t_1,x_1,y)K_j(t_2,x_2,y)|$$

$$\begin{aligned}
&< C|\frac{\partial^2_{y_ky_k}t^*}{(\partial_{y_k}t^*)^2}(1-\gamma_0)(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*)K_j(t_1,x_1,y)K_j(t_2,x_2,y)|\\
&< C'2^{i/2}2^{-2j}\sigma_1^{-2}|\partial^2_{y_ky_k}t^*||K_j(t_1,x_1,y)K_j(t_2,x_2,y)|\\
&< C''2^{i/2}2^{-2j}\sigma_1^{-2}2^{-2nj}|\partial^2_{y_ky_k}t^*|
\end{aligned}$$

However, expanding $\partial^2_{y_ky_k}t^*(t_1,x_1,x_2,y)$ in x_2 and y about $x_2=y=x_1$ in the by now usual fashion gives us $|\partial^2_{y_ky_k}t^*(t_1,x_1,x_2,y)| < C''2^{2j}\sigma_1$, and the above is $< C''''2^{i/2}\sigma_1^{-1}2^{2nj}$

Our next term is

$$|L(\partial_{y_k}t^*)^{-1}\partial_{y_k}[(1-\gamma_0)(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*)]K_j(t_1,x_1,y)K_j(t_2,x_2,y)|$$

$$\begin{aligned}
&< |L(\partial_{y_k}t^*)^{-1}2^{i/4}2^{-j}\sigma_1^{-1}\partial^2_{y_ky_k}t^*\gamma_0'(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*)K_j(t_1,x_1,y)K_j(t_2,x_2,y)|\\
&< C|\partial^2_{y_ky_k}t^*(\partial_{y_k}t^*)^{-2}\gamma_0'(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*)K_j(t_1,x_1,y)K_j(t_2,x_2,y)|
\end{aligned}$$

and so exactly as above we have that this term is $< C2^{i/2}\sigma_1^{-1}2^{2nj}$

Our final 2 terms, where the derivative lands on either $K_j(t_1,x_1,y)$ or $K_j(t_2,x_2,y)$, are bounded by $C|(\partial_{y_k}t^*)^{-1}|(1-\gamma_0)(2^{i/4}2^{-j}\sigma_1^{-1}\partial_{y_k}t^*)2^{(2n+1)j}$ $< C'2^{i/4}2^{-j}\sigma_1^{-1}2^{(2n+1)j}$.

We conclude that

$$|\partial_{y_k}[(L\partial_{y_k}t^*)^{-1}(1-\gamma_0)(2^{\frac{i}{4}}2^j\sigma_1^{-1}t^*_{y_k})K_j(t_1,x_1,y)K_j(t_2,x_2,y)]| < C''2^{i/2}\sigma_1^{-1}2^{-2nj}$$

Substituting this in (5.4), we get that (5.4) $< C$ times

$$\int_{|y-x_1|,|y-x_2|<2^{-j}} 2^{\frac{3i}{2}+2nj}(\sigma_1)^{-2}|\delta(2^i\sigma_2^{-1}L(t_2-t^*-s)|\gamma(2^i\sigma_1^{-1}s)dydsdx_2dt_2 \quad (5.5)$$

Integrating with respect to t_2, s, x_2, then y, we get (5.5) $< C2^{-i/2}$. As a result $\int |P_2''(t_1,x_1,t_2,x_2)|dt_2\,dx_2 < C2^{-i/2}$ and we have completed the proof of Theorem 5.1.

6: L^p Estimates

In section 5 we saw that $||T_if||_2 \leq C2^{-\delta i}||f||_2$. Here we apply the Calderon-Zygmund theory to the non-Euclidean balls introduced in section 3 to prove that for $1<p<\infty$ $||T_if||_p \leq Ci||f||_p$ (with C depending on p now as well as the other parameters M, n, etc). Then interpolation between 2 and $(p-1)/2$ if $p<2$, and between 2 and $2p$ if $p>2$, gives us $||T_i^*f||_p \leq C'2^{-\delta' i}||f||_p$. Adding these up over i gives us $||Tf||_p \leq C'||f||_p$, and we have the Main Theorem.

Recall that $T_if = \sum_j T_{i,j}f$, where $T_{i,j}f =$

$$\int 2^i\sigma(t,x,2^{-j})^{-1}\gamma(2^i\sigma(t,x,2^{-j})^{-1}(t+S(t,x,y)-s))K_j(t,x,y)f(s,y)ds\,dy$$

If $L_{i,j}(t,x,s,y) = 2^i\sigma(t,x,2^{-j})^{-1}\gamma(2^i\sigma(t,x,2^{-j})^{-1}(t+S(t,x,y)-s))K_j(t,x,y)$, T_i has kernel $\sum_j L_{i,j}(t,x,s,y)$.

Lemma 6.1: $\exists k>1$ such that

If $(t',x') \in B(t,x;r)$, then

$$\int_{(s,y)\notin B(t,x,2^kr)} |L_i(t,x,s,y)-L_i(t',x',s,y)|ds\,dy \leq Ci \quad (6.1a)$$

If $(s',y') \in B(s,y;r)$, then

$$\int_{(t,x)\notin B(s,y,2^kr)} |L_i(t,x,s,y)-L_i(t,x,s',y')|dt\,dx \leq Ci \quad (6.1b)$$

Proof: The proofs of $(6.1a)$ and $(6.1b)$ are similar, so we will only prove the slightly more difficult equation $(6.1a)$ here.

The left hand side of $(6.1a)$ is at most

$$\sum_j \int_{(s,y)\notin B(t,x,2^k r)} |2^i\sigma(t,x;2^{-j})^{-1}\gamma(2^i\sigma(t,x;2^{-j})^{-1}(t+S(t,x,y)-s))K_j(t,x,y)$$

$$-2^i\sigma(t',x';2^{-j})^{-1}\gamma(2^i\sigma(t',x';2^{-j})^{-1}(t'+S(t',x',y)-s))K_j(t',x',y)|ds\,dy \quad (6.2)$$

Let j_0 be such that $2^{-j_0-1} \leq r \leq 2^{-j_0}$. If the expression $\gamma(2^i\sigma(t,x;2^{-j})^{-1}(t+S(t,x,y)-s))$ appearing in (6.2) is nonzero, we must have $(s,y) \in B(t,x;2^{-j})$. But the integral is being taken only over $(s,y) \notin B(t,x,2^k r)$, so we conclude that if $j > j_0$, $\gamma(2^i\sigma(t,x;2^{-j})^{-1}(t+S(t,x,y)-s)) = 0$ on the domain of integration.

Similarly, if we had $j > j_0$ and $\gamma(2^i\sigma(t',x';2^{-j})^{-1}(t'+S(t',x',y)-s)) \neq 0$, then $(s,y) \in B(t',x';2^{-j})$, so by Fact 2 $(t',x') \in B(s,y;C2^{-j}) \subset B(s,y;C2^{-j_0})$. Since $(t',x') \in B(t,x;2^{-j_0})$, applying Fact 2 again gives us $(s,y) \in B(t,x;C^2 2^{-j_0})$. This is impossible on our domain of integration if k is chosen appropriately large since we are integrating over $(t,x) \notin B(s,y,2^k r)$. So in (6.2) we may assume that $\gamma(2^i\sigma(t',x';2^{-j})^{-1}(t'+S(t',x',y)-s)) \neq 0$ only if $j \leq j_0$ as well. Hence all the terms in (6.2) are zero for $j > j_0$.

We now do the usual coordinate change to make $S(t,x,y) = 0$ for all y: rewrite t' as $t' + S(t,x,x')$, and change variables in s to rewrite s as $s - S(t,x,y)$. As a result $B(t,x,2^k r)$ becomes the rectangle $\{(s,y) : |s-t| < \sigma(t,x,2^k r), |y-x| < 2^k r\}$.

(6.2) is bounded by

$$\sum_{j\leq j_0} \int |2^i\sigma(t,x;2^{-j})^{-1}\gamma(2^i\sigma(t,x;2^{-j})^{-1}(t+S(t,x,y)-s))K_j(t,x,y)$$

$$-2^i\sigma(t',x';2^{-j})^{-1}\gamma(2^i\sigma(t',x';2^{-j})^{-1}(t'+S(t',x',y)-s))K_j(t',x',y)|ds\,dy \quad (6.3)$$

Define $F(t'',x'') = 2^i\sigma(t,x;2^{-j})^{-1}\gamma(2^i\sigma(t,x;2^{-j})^{-1}(t+S(t,x,y)-s))K_j(t,x,y)$. By the Mean Value Theorem and Lemma 3.1, integrand in the jth term of (6.3) is at most C times

$$\sigma(t,x;2^{-j_0})\sup_{(t'',x'')\in B(t,x;2^{-j_0})}|\partial_t F(t'',x'')| + 2^{-j_0}\sup_{(t'',x'')\in B(t,x;2^{-j_0})}|\nabla_x F(t'',x'')|$$

Differentiating the product, then using Lemmas 3.1 and 3.4, equation (3.9) and the fact that $\sigma(t'',x'';2^{-j}) \sim \sigma(t,x;2^{-j})$, we get that for $(t'',x'') \in B(t,x;2^{-j_0})$

$$|\partial_t F(t'',x'')| < C2^{2i}\sigma(t,x;2^{-j})^{-2}2^{jn}, |\nabla_x F(t'',x'')| < C2^{2i}\sigma(t,x;2^{-j})^{-1}2^{j(n+1)}$$

As a result the integrand of the jth term in (6.3) is at most

$$C2^{2i+jn}\frac{\sigma(t,x;2^{-j_0})}{\sigma(t,x;2^{-j})^2} + 2^{2i+jn}2^{j-j_0}\sigma(t,x;2^{-j})^{-1} < C'2^{j-j_0+2i+jn}\sigma(t,x;2^{-j})^{-1}$$

But this integrand is being integrated over an s set of measure $< C2^{-i}\sigma(t,x;2^{-j})$ and a y set of measure $< C2^{-jn}$ and we conclude the jth term of (6.3) is $< C2^i2^{j-j_0}$. However each term in (6.3) is less than C (To see this, integrate with respect to s first). So (6.3) $< Ci + \sum_{j\le j_0-i} 2^i2^{j-j_0} < C'i$.

We are thus finished with the proof of Lemma 6.1.

By the Calderon-Zygmund method applied to our non-Euclidean balls, $(6.1a)$ proves $||T_if||_p \le Ci||f||_p$ for $2 \le p < \infty$ and $(6.1b)$ proves $||T_if||_p \le Ci||f||_p$ for $1 < p \le 2$. By the discussion preceeding the proof of Lemma 6.1, the proof of the Main Theorem is complete.

7: Extensions and Generalizations

There are a few natural directions in which generalizations of the arguments presented here might be attempted. Firstly, there is the issue of extending the arguments here to situations where the submanifolds being integrated over have codimension $m > 1$. The author will prove L^p boundedness for such singular Radon transforms in a subsequent paper, producing a result analogous to the main theorem of [1].

Secondly, there is the question of finding a natural generalization of the conditions $(K1) - (K2)$. It is hoped that an argument like that of the $T(1)$

theorem can replace the arguments of section 4 and thus extend the arguments here to a general class of $K(t, x, y)$; the author thanks E. Stein for pointing this possibility out. Next, it might be worth examining what happens if one weakens the curvature condition. Finally, and more generally, there is the question as to what extent the methods being developed here apply to the study of more general Radon transforms and related oscillatory integral operators. This issue appears to be substantially more difficult than the ones treated here, and should be interesting to explore.

References

[1] M. Christ, A. Nagel, E. Stein, S. Wainger, *Singular and maximal radon transforms.*, Annals of Mathematics, to appear .

[2] S. Cuccagna, *Sobolev estimates for fractional and singular Radon transforms.*, Journal of Functional Analysis **139** (1996), 94-118.

[3] A. Greenleaf, G. Uhlmann, *Estimates for singular Radon transforms and pseudodifferential operators with singular symbols*, Journal of Functional Analysis **89** (1990), 202-232.

[4] D. H. Phong, *Singular integrals and Fourier integral operators*, Essays on Fourier analysis in honor of Elias M. Stein (Princeton,NJ, 1991), 286-320, Princeton Mathematics Series Vol. 42, Princeton University Press, Princeton, NJ, 1995.

[5] D. H. Phong, E. M. Stein, *Hilbert integrals, singular integrals, and Radon transforms I*, Acta Mathematica **157** (1986), 99-157.

[6] F. Ricci, E. Stein, *Harmonic analysis on nilpotent groups and singular integrals I. oscillatory integrals*, Journal of Functional Analysis **73** (1987), 179-194.