Weight Complexes for arithmetic varieties

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Question of Serre

Theorem 1 (G.& Soulé,Crelle v. 478, 1996)**.** *Let* $\mathbf{V}$ *be the category of varieties over a field* k *of characteristic zero. Let* $\mathbf{M}$ *be the category of Chow motives for any adequate equivalence relation.*

Then any variety X *has a class* $[X] \in K_0(\mathbf{M})$ *such that:*

1. *If* X *is a smooth variety over* k*, then* $[X]$ *is the class of the motive* $(X, 1_X)$*.*

2. *If* $Y \subset X$ *is a closed subvariety, then*

$$[X] = [X] + [X - Y] \, .$$

Weight Complexes
Let $\mathrm{Hot}(\mathbf{M})$ be the homotopy category of bounded complexes of Chow motives over k.

Theorem 2. *There is a contravariant functor from the category of* proper *morphisms between varieties in* $\mathbf{V}$ *to* $\mathrm{Hot}(\mathbf{M})$, $X \mapsto W(X)$, *such that:*

(i) Any open immersion $i : U \to X$ *induces a map* $i_* : W(U) \to W(X)$, *compatible with composition.*

(ii) If $Y \subset X$ *is a closed subvariety, there is a canonical triangle:*

$$W(X \setminus Y) \to W(X) \to W(Y) \to W(X \setminus Y)[1]$$

(iii) If X *is a smooth projective variety, then* $W(X) = (X, 1_X)$.

Proof of theorem 1: $[X] := \chi(W(X))$.
Corollary 3. *The weight filtration is defined integrally.*

Main Idea of Proof of theorem 2

Definition 4. Envelope $= f : X \to Y$, *proper s.t. for all fields* $X(F) \twoheadrightarrow Y(F)$

Key points:

1. For all X, $\exists f : \tilde{X} \to X$, nonsingular resolution by envelope.

2. Envelopes are universal descent morphisms for K-theory and the homology of Gersten complexes.

3. Two proper hyper-envelopes $p_i : \widetilde{X}_{\cdot i} \to X$ determine *homotopy equivalent* complexes of motives.

Getting rid of compact Supports:

Guillen & Navarro-Aznar in Pub. I.H.E.S. vol. ?, used cubical methods to also construct a functor which is contravariant for *all* maps.

Arithmetic Analog

The context:

$\mathbf{V}$ = category of varieties (= schemes of finite type) over a base S for which De Jong's theorem holds:

E.g.:

Spec($\mathcal{O}_K$) K =number field, or Spec(k) k = field.

$CH^*(X) := CH^*(X)_{\mathbb{Q}}$

Note that $CH^*(X)$ is invariant under purely inseparable extensions – so we will ignore them.

Motives will have rational coefficients.

The Main Theorem

Theorem 5. *There is a contravariant functor from the category of proper morphisms between Deligne-Mumford stacks over S to the category of homotopy classes of maps between bounded complexes of motives over S, $W : \mathcal{S}tack_S \to \mathrm{Hot}(\mathbf{M}_S)$ such that:*

(i) Any open immersion $i : \mathfrak{U} \to \mathfrak{X}$ induces a map $i_* : W(\mathfrak{U}) \to W(\mathfrak{X})$, compatible with composition.

(ii) If $\mathfrak{Y} \subset \mathfrak{X}$ is a closed substack, there is a canonical triangle:

$$W(\mathfrak{X} \setminus \mathfrak{Y}) \to W(\mathfrak{X}) \to W(\mathfrak{Y}) \to W(\mathfrak{X} \setminus \mathfrak{Y})[1]$$

(iii) If X is a regular scheme, proper over S, then $W(X)$ is the usual motive of X.

(iv) If V is a regular scheme, projective over S, and G is a finite group acting on V, then $W([V/G]) = W(V)^G$

Bivariant Chow groups(Fulton)

Definition 6. *If* $f : X \to Y$*, a bivariant class* $\alpha \in \mathsf{CH}^i(X \to Y)$*, consists of:*

$$(T \to Y) \mapsto \left(\alpha_T^* : \mathsf{CH}_p(T) \to \mathsf{CH}_{p-i}(X \times_Y T)\right)$$

If Y is regular, then $\mathsf{CH}^*(X \to Y) \simeq \mathsf{CH}_*(X)$.

Correspondences over S

Definition 7. *Suppose that X and Y are proper,*
$\mathbf{Corr}^*_S(X,Y) := CH^*(X \times_S Y \to X)$

If $\alpha \in \mathsf{Corr}^{-d}(X,Y)$, get $\alpha_* : \mathsf{CH}^i(X) \to \mathsf{CH}^{i-d}(Y)$.

Definition 8. *Given X, Y, and Z,*
$\alpha \in \mathsf{Corr}^{-d}(X,Y) = \mathsf{CH}^{-d}(X \times_S Y \to X)$, *and*
$\beta \in \mathsf{Corr}^{-e}(Y,Z) = \mathsf{CH}^{-e}(Y \times_S Z \to Y)$, *their* composition *is defined by:*
β *induces* $\beta_{X \times_S Y} \in \mathsf{CH}^{-e}(X \times_S Y \times_S Z \to X \times_S Y)$
hence $\beta \cdot \alpha_Z \in A^{-d-e}(\pi_Z \cdot \pi'_Y : X \times_S Y \times_S Z \to X)$,
which we push forward by $X \times_S Y \times_S Z \to X \times_S Z$ *to get* $\beta \cdot \alpha$.

Proposition 9. *Composition of correspondences is a bilinear pairing, and is associative.*

Definition 10. *We write* $\mathbf{Corr}_S$ *for the graded* $\mathbb{Q}$*-linear category with Hom-sets equal to the graded vector space of correspondences.*

Note that there is a *covariant* functor Γ from the category of proper morphisms between varieties to $\mathbf{Corr}_S^0$:

$$(g : X \to Y) \to \Gamma(g) = (\Gamma_g)_* : \mathsf{CH}_*(X) \to \mathsf{CH}^*(X \times_S Y)$$

Γ extends to a functor from simplicial objects to Chain complexes in $\mathbf{Corr}_S^*$

Exact Sequence

Definition 11. *A (homological) complex* X_* *in* $\mathbf{Corr}_S$ *is said to be* acyclic *if for all* regular T, i, $\mathbf{Corr}^i_S(T, X_*)$ *is exact.*

A map of complexes $f_* : X_* \to Y_*$ *is said to be a quasi-isomorphism if for all* T, $\mathbf{Corr}^i_S(T, X_*) \to \mathbf{Corr}^i_S(T, Y_*)$ *is a quasi-iso.*

Triviality: If X_* is a bounded below acyclic complex, and T_* is a complex of regular varieties, then $\mathbf{Corr}_S(T_*, X_*)$ is acyclic.

Proper Hypercovers

Theorem 12. $f. : X. \to Y.$ *a proper hypercover* $\Rightarrow \Gamma(f)$ *a quasi-iso.*

Proof. For all regular T, $\mathrm{CH}_*(X. \times_S T) \to \mathrm{CH}_*(Y. \times_S T)$, is a quasi-iso.

Key point: Proper morphisms satisfy universal homological descent for CH_*. □

Remark 13. *This was not in the Crelle paper.*

Complexes associated to stacks

Let $\mathfrak{X}$ be a Deligne-Mumford stack over S.

Chow's lemma $\Rightarrow \exists\ f : X \to \mathfrak{X}$ proper & surjective, X a variety, and hence a proper hypercover $f : X. \to \mathfrak{X}$.

Proposition 14. *For all $k \geq 0$, we have a quasi-isomorphism:*

$$(i \mapsto \mathsf{CH}_k(X_i)) \simeq CH_k(\mathfrak{X})$$

Proof. Reduce to points, then homology of finite groups. □

De Jong $\Rightarrow \exists\ f : X. \to \mathfrak{X}$ proper hypercover, X_i regular $\forall i$.

Definition of W, step 1

Let $\mathbf{P}$ be the category of varieties proper over S, and let $\mathrm{Ar}(s.\mathbf{P})$ be the category of morphisms between simplicial objects in $\mathbf{P}$.

Functor $T : \mathrm{Ar}(s.\mathbf{P}) \to \mathrm{Hot}(\mathbf{Corr})$:

Given $f : Y. \to Z.$, choose a regular proper hypercover:

$$\begin{array}{ccc} \tilde{Y}. & \xrightarrow{\tilde{f}} & \tilde{Z}. \\ \downarrow & & \downarrow \\ Y. & \xrightarrow{f} & Z. \end{array}$$

$T(f) := \mathrm{Cone}(\Gamma(\tilde{f}))[-1]$

Definition of W, step 2

Suppose that g is a morphism in Ar($s.\mathbf{P}$):

$$\begin{array}{ccc} Y_{\cdot 1} & \xrightarrow{f_1} & Z_{\cdot 1} \\ {\scriptstyle g_Y}\downarrow & & \downarrow{\scriptstyle g_Z} \\ Y_{\cdot 2} & \xrightarrow{f_2} & Z_{\cdot 2} \end{array}$$

Then there is a canonical map $T(g) : T(f_1) \to T(f_2)$ in Hot($\mathbf{Corr}$)

Proof: regular=projective.

Definition of W, step 3

We say that g is a morphism in $\mathrm{Ar}(s.\mathbf{P})$ is a *Gersten Equivalence* if for all q:

$$R_{q,*}(Y._1) \to R_{q,*}(Y._2) \oplus R_{q,*}(Z._1) \to R_{q,*}(Z_2.)$$

Lemma 15. *If the $Y._i$ and $Z._i$ are regular, and g is a Gersten equivalence, then $T(g)$ is a quasi-isomorphism.*

Definition of W, step 3
Let $\mathfrak{X}$ be a stack. There exists:
$p : X. \to \mathfrak{X}$, a proper hypercover , and
$i : X. \hookrightarrow \bar{X}.$, a compactification over S
such that $Y. = \bar{X}. \setminus X.$ is a closed subsimplicial scheme.

If $\bar{X}.$ above is regular, and $f : \tilde{Y}. \to Y.$ is a regular hypercover, we call

$$\tilde{Y}. \to \bar{X}. \hookleftarrow X. \to \mathfrak{X}$$

a *resolution* of $\mathfrak{X}$.
Theorem 16. *Given two resolutions f_1 and f_2 of $\mathfrak{X}$, there is a canonical quasi-isomorphism $T(f_1) \to T(f_2)$.*

More generally, given a morphism $g : \mathfrak{X}_1 \to \mathfrak{X}_2$ of stacks, and resolutions f_1, f_2, there is a canonical $T(g) : T(f_1) \to T(f_2)$.

Proof: dense compactifications form a directed poset.

Definition of W

Definition 17. *The category $\mathbf{M}_S$ of motives over S is defined by taking adding projectors to $\mathbf{Corr}_S$ and reversing arrows. W is the associated functor.*

Boundedness

Every stack $\mathfrak{X}$ has a dense open set of the form $[X/G]$ for some finite group G.

Pick equivariant compactification $[\bar{X}/G]$

Apply De Jong to $\bar{X}$ with G acting on it:

There exists $\pi : Y \to X$ with a group H acting on Y, and a homomorphism $H \to G$ so that the map is equivariant, Y is regular, and there is an equivariant dense open $U \subset X$ such that $[\pi^{-1}(U)/H] \simeq [U/G]$.

Then use M-V sequence for W plus noetherian induction.

No compact supports

Theorem 18. *There is a contravariant functor from the category of* ALL *morphisms between varieties in* $\mathbf{V}$ *to* $\mathrm{Hot}(\mathbb{M})$, $X \mapsto \widehat{W}(X)$, *such that:*

(i) If $Y \subset X$ *is a closed subvariety, there is a canonical triangle:*

$$\widehat{W}(X) \to \widehat{W}(X \setminus Y) \to \widehat{W}^Y(X) \to \widehat{W}(X \setminus Y)[1]$$

(ii) If X *is a smooth projective variety, then* $\widehat{W}(X) = (X, 1_X)$.

(iii) If X *is regular, proper over* S, *and* $Y \subset X$ *is closed, then* $\widehat{W}^Y(X) \simeq W(Y)$.

Idea of Proof

Let X be a variety over S. Choose a compactification $X \subset \bar{X}$, with complement Y. Given a regular hypercover $\tilde{X}. \to \bar{X}$, with $Y. \subset \tilde{X}. = \tilde{X}. \setminus X.$ we want to define $W^{Y.}(\tilde{X}.)$, and then set

$$W(X) := \text{Cone}(W^{Y.}(\tilde{X}.) \to W(\tilde{X}.))[\pm 1]$$

Use operational Chow groups, together with "projective" resolutions of the ΓY_i.