ON A SUMSET CONJECTURE OF ERDŐS

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Abstract. Erdős conjectured that for any set $A \subseteq \mathbb{N}$ with positive lower asymptotic density, there are infinite sets $B, C \subseteq \mathbb{N}$ such that $B + C \subseteq A$. We verify Erdős' conjecture in the case that A has *Banach* density exceeding $\frac{1}{2}$. As a consequence, we prove that, for $A \subseteq \mathbb{N}$ with positive Banach density (a much weaker assumption than positive lower density), we can find infinite $B, C \subseteq \mathbb{N}$ such that $B + C$ is contained in the union of A and a translate of A. Both of the aforementioned results are generalized to arbitrary countable amenable groups. We also provide a positive solution to Erdős' conjecture for subsets of the natural numbers that are *pseudorandom*.

1. Introduction

For $A \subseteq \mathbb{N}$, the *lower (asymptotic) density of* A is defined to be

$$\underline{d}(A) := \liminf_{n\to\infty} \frac{|A \cap [1, n]|}{n}.$$

Here, and throughout this paper, for $a, b \in \mathbb{N}$, $[a, b]$ denotes

$$\{c \in \mathbb{N} : \ a \leq c \leq b\}.$$

Moreover if A and B are subsets of $\mathbb{N}$, then $A + B$ denotes the sumset $\{a + b : \ a \in A \text{ and } b \in B\}$. In [5] and [6] Erdős conjectured the following generalization of Hindman's theorem on sumsets (see [10]): If A is a set of natural numbers of positive lower density, then there is an infinite subset A' of A such that $A' + A'$ is contained in a translate of A. This density version of Hindman's theorem was inspired by the celebrated Szemerédi theorem on arithmetic progressions (see [17]), which can be regarded as a density version of van der Waerden's theorem from [18]. Later, Straus provided a counterexample to this conjecture of Erdős, as reported in [7] on page 105. The conjecture was thus modified (cf. [15] and page 85 of [8]) as follows.

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Conjecture (Erdős). *If $A \subseteq \mathbb{N}$ has $\underline{d}(A) > 0$, then there are two infinite sets $B, C \subset \mathbb{N}$ such that $B + C \subset A$.*

We will refer to this as "Erdős' $B + C$ conjecture". Partial results on this conjecture have been obtained by Nathanson in [15], where he proved in particular that one can find an infinite set B and an arbitrarily large finite set F such that $B + F \subset A$.

In this paper, we make progress on the $B + C$ conjecture by proving the following "one-shift" version for sets of positive *Banach density*, where, for $A \subseteq \mathbb{N}$, the (upper) Banach density of A is defined to be

$$\mathrm{BD}(A) := \lim_{n\to\infty} \sup_{m\in\mathbb{N}} \frac{|A \cap [m, m+n]|}{n}.$$

Theorem 1.1. *If $\mathrm{BD}(A) > 0$, then there are infinite $B, C \subseteq \mathbb{N}$ and $k \in \mathbb{N}$ such that $B + C \subseteq A \cup (A + k)$.*

Observe that

$$\underline{d}\,(A) \leq \mathrm{BD}\,(A)\,,$$

whence the hypothesis of positive Banach density is weaker than the hypothesis of positive lower density.

We also settle Erdős' conjecture for sets of large Banach density.

Theorem 1.2. *If $\mathrm{BD}(A) > \frac{1}{2}$, then there are infinite $B, C \subseteq \mathbb{N}$ such that $B + C \subseteq A$.*

We derive Theorem 1.1 from Theorem 1.2 by showing that every subset of the natural numbers of positive Banach density has finitely many translates whose union has Banach density at least $\frac{1}{2}$ and then use Ramsey's theorem to obtain our shifts.

In the proof of Theorem 1.1, we will see that whether $b_i + c_j$ is in A or $A + k$ depends only on whether or not $i < j$ holds, where $B = (b_i)$ and $C = (c_j)$ are increasing enumerations of B and C respectively.

We generalize both of the aforementioned results to the case of arbitrary countable amenable groups. However, we present proofs for the two contexts separately as the proofs for subsets of the natural numbers are easier and/or require less technical machinery.

In the final section, we prove the $B + C$ conjecture for sets A that are *pseudorandom* in a precise technical sense. Here we remain in the setting of sets of natural numbers as we do not know how to generalize one of the key ingredients (Fact 5.4) to the setting of amenable groups.

We use *nonstandard analysis* to derive our results and we assume that the reader is familiar with elementary nonstandard analysis. For those not familiar with the subject, the survey article [12] contains a light introduction to nonstandard methods with combinatorial number theoretic aims in mind. The specific technical results from nonstandard analysis that we will need are found in Section 2, where we review the Loeb measure. In Section 2, we also recall the basic facts from the theory of amenable groups that we need.

In Sections 3 and 4, we prove Theorems 1.2 and 1.1 respectively (as well as their amenable counterparts). In Section 5, we prove Erdős' conjecture for pseudorandom sets.

Throughout the paper, we do not include 0 in the set $\mathbb{N}$ of natural numbers. Also, if B, C are subsets of a group G, then BC denote the set of products

$$\{bc : \ b \in B \text{ and } c \in C\}.$$

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2. Preliminaries

2.1. **Loeb measure.** Throughout this paper, we always work in a countably saturated nonstandard universe.

We recall the definition of Loeb measure, which is defined relative to a fixed hyperfinite set X. For every internal $A \subseteq X$, the measure of A is defined to be $\mu(A) := \mathrm{st}(\frac{|A|}{|X|})$. This defines a finitely additive measure μ on the algebra of internal subsets of X, which canonically extends to a countably additive probability measure μ_L on the σ-algebra of *Loeb measurable* sets of X.

2.2. **Amenable Groups.** Suppose that G is a group. A *(left) Følner sequence* for G is a sequence $(F_n)_{n\in\mathbb{N}}$ of of finite subsets of G such that, for every $g \in G$, we have

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0.$$

Observe that if (F_n) is a Følner sequence for G and (x_n) is any sequence in G, then $(F_n x_n)$ is also a Følner sequence for G. Observe also that, if $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$, then $\frac{|gF_\nu \triangle F_\nu|}{|F_\nu|} \approx 0$ for every $g \in G$. (In the terminology of [4], F_ν is a *Følner approximation for G*.)

A countable group G is said to be *amenable* if there is a Følner sequence for G. For example, if $G = \mathbb{Z}$, then G is amenable, where one can take as (F_n) any sequence of intervals whose length approaches infinity. The class of amenable groups is very rich, including all solvable-by-finite groups, and is closed under subgroups, quotients, and extensions.

In an amenable group, one can define a notion of (upper) Banach density. In the rest of this subsection, fix a countable amenable group G. For $A \subseteq G$, the *Banach density of A*, denoted $\mathrm{BD}(A)$, is defined to be

$$\mathrm{BD}(A) := \sup\{\limsup_{n\to\infty} \frac{|A \cap F_n|}{|F_n|} \ : \ (F_n) \text{ a Følner sequence for } G\}.$$

It can be shown that this supremum is actually attained in the sense that, for any $A \subseteq G$, there is a Følner sequence (F_n) for G such that $\lim_{n\to\infty} \frac{|A\cap F_n|}{|F_n|} = \mathrm{BD}(A)$.

It is evident from the definition that $\mathrm{BD}(A) = \mathrm{BD}(gA) = \mathrm{BD}(Ag)$ for all $g \in G$ and $A \subseteq G$. However, it is not a priori immediate that this agrees with the usual notion of Banach density in the case that $G = \mathbb{Z}$ as here one allows arbitrary Følner sequences rather than just sequences of intervals. Nevertheless, it is shown in [1, Remark 1.1] that if G is a countable amenable group and (F_n) is *any* Følner sequence for G, then there is a sequence (g_n) from G such that $\mathrm{BD}(A) = \limsup_{n\to\infty} \frac{|A\cap F_n g_n|}{|F_n|}$, whence we see immediately that the two notions of Banach density agree in the case of the integers.

For finite $H \subseteq G$ and $\epsilon > 0$, we say that a finite set $F \subseteq G$ is (H, ϵ)-*invariant* if, for every $h \in H$, we have

$$\frac{|hF \triangle F|}{|F|} < \epsilon.$$

One can equivalently define a countable group to be amenable if, for every finite $H \subseteq G$ and $\epsilon > 0$, there is a finite subset of G that is (H, ϵ)-invariant. (This definition has the advantage that it extends to groups of arbitrary cardinality.) In this language, we have that $\mathrm{BD}(A)$ is the supremum of those γ for which, given any finite $H \subseteq G$ and any $\epsilon > 0$, there is a finite $F \subseteq G$ that is (H, ϵ)-invariant and satisfying $\frac{|A\cap F|}{|F|} \geq \gamma$.

Finally, we will need a version of the pointwise ergodic theorem for countable amenable groups due to E. Lindenstrauss [14]. First, we say that a Følner sequence (F_n) is *tempered* if there is a constant $C > 0$ such that, for every $n \in \mathbb{N}$, we have

$$|\bigcup_{k<n} F_k^{-1} F_n| \leq C|F_n|.$$

For example, if $G = \mathbb{Z}$ and our F_n are simply disjoint intervals with length and endpoints going to infinity, a tempered subsequence can always be obtained by insisting that the length of the n^{th} interval in the subsequence is at least as large as the right endpoint of the $(n-1)^{\text{st}}$ interval.

Fortunately, there is an abundance of tempered Følner sequences for any countable abelian group.

Fact 2.1 (Lindenstrauss [14]). *Suppose that G is a countable amenable group. Then every Følner sequence for G has a tempered subsequence. In particular, for $A \subseteq G$, there is a tempered Følner sequence (F_n) for G such that* $\mathrm{BD}(A) = \lim_{n\to\infty} \frac{|A\cap F_n|}{|F_n|}$.

Here is the pointwise ergodic theorem for countable amenable groups:

Fact 2.2 (Lindenstrauss [14]). *Suppose that G is a countable amenable group acting on a probability space $(X, \mathcal{B}, \mu)$ by measure preserving transformations* *and* (F_n) *is a tempered Følner sequence for* G*. If* $f \in L^1(\mu)$ *and*

$$A\left(F_n, f\right)(x) := \frac{1}{|F_n|}\sum_{g\in F_n} f(gx)$$

for every $n \in \mathbb{N}$*, then the sequence*

$$\left(A\left(F_n, f\right)\right)_{n\in\mathbb{N}}$$

converges almost everywhere to a G*-invariant* $\bar{f} \in L^1(\mu)$*. Consequently, by the Lebesgue dominated convergence theorem,* $A\left(F_n, f\right)$ *converges to* $\bar{f}$ *in* $L^1(\mu)$ *and, in particular,*

$$\int f d\mu = \int \bar{f} d\mu.$$

2.3. **A result of Bergelson.** Throughout our paper, we will make use of the following result of Bergelson, which is Theorem 1.1 in [2]:

Fact 2.3. *Suppose that* $(X, \mathcal{B}, \mu)$ *is a probability space and* (A_n) *is a sequence of measurable sets for which there is* $a \in \mathbb{R}^{>0}$ *such that* $\mu(A_n) \geq a$ *for each* n*. Then there is infinite* $P \subseteq \mathbb{N}$ *such that, for every finite* $F \subseteq P$*, we have* $\mu(\bigcap_{n\in F} A_n) > 0$*.*

3. The high density case

The main result of this section is the following:

Theorem 3.1. *Suppose that* G *is a countable amenable group and* $A \subseteq G$ *is such that* $\mathrm{BD}(A) > \frac{1}{2}$*. Then there are injective sequences* $(b_n)_{n\in\mathbb{N}}$ *and* $(c_n)_{n\in\mathbb{N}}$ *in* G *such that:*

- $c_n \in A$ *for all* $n \in \mathbb{N}$*;*
- $b_i c_j \in A$ *for* $i \leq j$*;*
- $c_i b_j \in A$ *for* $i < j$*.*

In the first subsection, we prove the analogous fact for subsets of the natural numbers as in this case we can avoid using Fact 2.2 and instead resort to more elementary methods. We prove the case of a general amenable group in the second subsection.

3.1. **The case of the integers.** The main goal of this subsection is the following theorem.

Theorem 3.2. *Suppose that* $A \subseteq \mathbb{N}$ *is such that* $\mathrm{BD}(A) > \frac{1}{2}$*. Then there are infinite* $B, C \subseteq \mathbb{N}$ *with* $C \subseteq A$ *such that* $B + C \subseteq A$*.*

We first need a lemma.

Lemma 3.3. *Suppose that* $A \subseteq \mathbb{N}$ *has* $\mathrm{BD}(A) = \alpha > 0$*. Suppose that* (I_n) *is a sequence of intervals with* $|I_n| \to \infty$ *and for which* $\lim_{n\to\infty} \frac{|A\cap I_n|}{|I_n|} = \alpha$*. Then there is* $L \subseteq \mathbb{N}$ *satisfying:*

- $\limsup_{n\to\infty} \frac{|L\cap I_n|}{|I_n|} \geq \alpha$*;*

- *for every finite $F \subseteq L$, we have $A \cap \bigcap_{x \in F}(A - x)$ is infinite.*

Proof. It suffices to find $L \subseteq \mathbb{N}$ and $x_0 \in {}^*A \setminus A$ for which $\limsup_{n\to\infty} \frac{|L\cap I_n|}{|I_n|} \geq \alpha$ and $x_0 + L \subseteq {}^*A$. Indeed, if we can find such L and x_0, then given any finite $F \subseteq L$ and any finite $K \subseteq A$, the statement "there exists $x_0 \in {}^*A$ such that $x_0 + F \subseteq {}^*A$ and $x_0 \notin K$" is true in the nonstandard extension, whence we can conclude that $A \cap \bigcap_{x\in F}(A - x)$ is infinite.

For each n, let b_n denote the right endpoint of I_n. By passing to a subsequence of (I_n) if necessary, we may assume that the sequences (b_n) and $(|I_n|)$ are strictly increasing. Fix $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ and note that $\frac{|{}^*A\cap I_H|}{|I_H|} \approx \alpha$.

In what follows, we let μ denote the Loeb measure on I_H. Also, for any $m \in {}^*\mathbb{N}$ (standard or nonstandard) and for any hyperfinite $X \subseteq {}^*\mathbb{N}$, we set $\delta_m(X) := \frac{|X|}{|I_m|}$.

We fix $K \in {}^*\mathbb{N}\setminus\mathbb{N}$ for which $2b_K \cdot \delta_H(I_K) \approx 0$ and consider $M \in {}^*\mathbb{N}\setminus\mathbb{N}$ with $M \leq K$. We claim that, for μ-almost all $x \in I_H$, we have $\delta_M({}^*A \cap (x + I_M)) \approx \alpha$. Indeed, since $\mathrm{BD}(A) = \alpha$, we can conclude that, for all $x \in I_H$, we have $\mathrm{st}(\delta_M({}^*A \cap (x + I_M))) \leq \alpha$. We now compute

$$\frac{1}{|I_H|} \sum_{x\in I_H} \delta_M({}^*A \cap (x + I_M)) = \frac{1}{|I_M|} \sum_{y\in I_M} \frac{1}{|I_H|} \sum_{x\in I_H} \chi_{{}^*A}(x + y).$$

By the choice of K, it follows that

$$\frac{1}{|I_H|} \sum_{x\in I_H} \delta_M({}^*A \cap (x + I_M)) \approx \frac{1}{|I_M|} \sum_{y\in I_M} \delta_H({}^*A \cap I_H) \approx \alpha.$$

Coupled with our earlier observation, this proves the claim.

We now fix a standard positive real number $\epsilon < \frac{1}{2}$. Inductively assume that we have chosen natural numbers $n_1 < n_2 < \cdots < n_{i-1}$ and internal subsets $X_1, X_2, \ldots, X_{i-1} \subseteq I_H$ such that, for each $j = 1, 2, \ldots, i - 1$ and each $x \in X_j$, we have

$$\mu(X_j) > 1 - \epsilon^j \text{ and } \delta_{n_j}({}^*A \cap (x + I_{n_j})) \geq \alpha - \frac{1}{j}.$$

Consider the internal set

$$Z := \{M \in {}^*\mathbb{N} \ : \ n_{i-1} < M \leq K \text{ and}$$
$$\delta_H\left(\left\{x \in I_H \ : \ \delta_M({}^*A \cap (x + I_M)) \geq \alpha - \frac{1}{i}\right\}\right) > 1 - \epsilon^i\}.$$

Since Z is internal and contains every nonstandard element of ${}^*\mathbb{N}$ below K, it follows that there is $n_i \in Z \cap \mathbb{N}$. For this n_i, we set

$$X_i := \{x \in I_H \ : \ \delta_{n_i}({}^*A \cap (x + I_{n_i})) \geq \alpha - \frac{1}{i}\}.$$

Set $X := \bigcap_{i=1}^{\infty} X_i$ and observe that $\mu(X) > 0$. Fix $y_0 \in X$ and observe that, for all $i \in \mathbb{N}$, we have

$$\delta_{n_i}({}^*A \cap (y_0 + I_{n_i})) > \alpha - \frac{1}{i}.$$

Set x_0 to be the minimum element of ${}^*A \cap [y_0, b_H]$ and set

$$L := ({}^*A \cap (x_0 + \mathbb{N})) - x_0.$$

Note that $x_0 - y_0 \in \mathbb{N}$ and $x_0 + L \subseteq {}^*A$. Since $x_0 - y_0$ is finite, it follows that

$$\lim_{i\to\infty} \delta_{n_i}(L \cap I_{n_i}) = \lim_{i\to\infty} \delta_{n_i}({}^*A \cap (x_0 + I_{n_i})) = \lim_{i\to\infty} \delta_{n_i}({}^*A \cap (y_0 + I_{n_i})) = \alpha.$$

□

Proof of Theorem 3.2. Fix a sequence (I_n) of intervals such that $|I_n| \to \infty$ and

$$\lim_{n\to\infty} \frac{|A \cap I_n|}{|I_n|} = \alpha.$$

Fix L as in the conclusion of Lemma 3.3. Let $L = (l_n)$ be an increasing enumeration of L. Recursively define an increasing sequence $D := (d_n)_{n\in\mathbb{N}}$ from A such that $l_i + d_n \in A$ for $i \le n$. Fix $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $\mathrm{st}(\frac{|{}^*L \cap I_\nu|}{|I_\nu|}) \ge \alpha$. Recalling that $\alpha > \frac{1}{2}$, it follows that, for every $n \in \mathbb{N}$, we have

$$\mathrm{st}\left(\frac{|{}^*L \cap ({}^*A - d_n) \cap I_\nu|}{|I_\nu|}\right) \ge 2\alpha - 1 > 0.$$

By Fact 2.3, we may, after passing to a subsequence of (d_n), assume that, for every $n \in \mathbb{N}$, we have

$$\mathrm{st}\left(\frac{|{}^*L \cap \bigcap_{i\le n}({}^*A - d_i) \cap I_\nu|}{|I_\nu|}\right) > 0.$$

In particular, this implies that, for every $n \in \mathbb{N}$, we have $L \cap \bigcap_{i\le n}(A - d_i)$ is infinite. Take $b_1 \in L$ arbitrary and take $c_1 \in D$ such that $b_1 + c_1 \in A$. Fix $b_2 \in (L \cap (A - c_1)) \setminus \{b_1\}$ and take $c_2 \in D$ such that $\{b_1 + c_2, b_2 + c_2\} \subseteq A$. Take $b_3 \in (L \cap (A - c_1) \cap (A - c_2)) \setminus \{b_1, b_2\}$ and take $c_3 \in D$ such that $\{b_1 + c_3, b_2 + c_3, b_3 + c_3\} \subseteq A$. Continue in this way to construct the desired B and C. □

3.2. **The case of an arbitrary countable amenable group.** In this section, we assume that G is a countable amenable group and prove Theorem 3.1.

Before proving Theorem 3.1, we need a lemma analogous to Lemma 3.3.

Lemma 3.4. *Suppose that (F_n) is a tempered Følner sequence. If $A \subseteq G$ is such that* $\limsup_{n\to\infty} \frac{|A\cap F_n|}{|F_n|} = \alpha$*, then there is $L \subseteq G$ satisfying:*

- $\liminf_{n\to\infty} \frac{|L\cap F_n|}{|F_n|} \ge \alpha$*;*
- *for every finite $F \subseteq L$, we have $A \cap \bigcap_{x\in F} x^{-1}A$ is infinite.*

Proof. Fix $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $\frac{|{}^*A\cap F_\nu|}{|F_\nu|} \approx \alpha$. Notice that, for all $g \in G$, we have $\frac{|gF_\nu \triangle F_\nu|}{|F_\nu|} \approx 0$. Since G is countable, there is a full measure (with respect to the Loeb measure on F_ν) subset E of F_ν for which the map

$(g,x) \mapsto gx : G \times E \to E$ defines a measure preserving action of G on E. For $\xi \in E$, we define

$$f_n(\xi) := A\left(F_n, \chi_{^*A\cap E}\right)(\xi) = \frac{1}{|F_n|}\sum_{g\in F_n}\chi_{^*A\cap E}(g\xi)$$

where $\chi_{^*A\cap E}$ denotes the characteristic function of $^*A\cap E$. Observe that

$$f_n(\xi) = \frac{|F_n \cap ({}^*A\cap E)\xi^{-1}|}{|F_n|} \leq \frac{|F_n\cap {}^*A\xi^{-1}|}{|F_n|}. \quad (\dagger)$$

By Fact 2.2, there is $\bar{f} \in L^1(\mu)$ such that (f_n) converges to $\bar{f}$ almost everywhere and in $L^1(\mu)$, whence $\int \bar{f}d\mu = \alpha$. (Here, μ denotes the restriction of the Loeb measure on F_ν to E.)

We next claim that $\bar{f}$ is almost everywhere bounded above by α. If this is not the case, then there is $k\in\mathbb{N}$ such that the set of $\xi\in E$ for which $\bar{f}(\xi)\geq \alpha+\frac{1}{k}$ has positive measure. Since f_n converges to $\bar{f}$ almost everywhere, there is $\xi\in E$ such that $\lim_{n\to\infty} f_n(\xi)\geq\alpha+\frac{1}{k}$, whence, by (†), we have

$$\liminf_{n\to\infty}\frac{|F_n\xi\cap {}^*A|}{|F_n|}\geq\alpha+\frac{1}{k}.$$

By transfer, for each $n\in\mathbb{N}$ there is $x_n\in G$ such that

$$|F_nx_n\cap A| = |F_n\xi\cap {}^*A|.$$

Since (F_nx_n) is also a Følner sequence for G this implies

$$\mathrm{BD}\left(A\right)\geq\limsup_n\frac{|F_nx_n\cap A|}{|F_n|}\geq\alpha+\frac{1}{k}.$$

This contradicts the fact that $\mathrm{BD}(A)=\alpha$.

By our claim and the fact that $\int\bar{f}d\mu=\alpha$, we see that $\bar{f}$ is almost everywhere equal to α. In particular, there is $\xi\in {}^*A\cap E$ such that $\lim_{n\to\infty}f_n(\xi)=\alpha$. Since $G\cap E$ has measure 0, whence we can further insist that $\xi\in {}^*A\setminus G$. Fix such ξ and set $L := {}^*A\xi^{-1}\cap G$. By (†) and the choice of ξ, we have $\liminf_{n\to\infty}\frac{|L\cap F_n|}{|F_n|}\geq\alpha$.

It remains to show that $A\cap\bigcap_{x\in F}x^{-1}A$ is infinite for every finite subset F of L. Fix such an F. For each $x\in F$, we have $x\xi\in {}^*A$. Since $\xi\notin G$, for any finite $K\subseteq G$, the statement "there exists $h\in {}^*A$ such that $h\notin K$ and, for every $x\in F$, we have $xh\in {}^*A$" holds in the nonstandard extension. Thus, by transfer, for any given finite subset K of G, there is $h\in A$ such that $h\notin K$ and $xh\in A$ for each $x\in F$. □

The proof of Theorem 3.1 from Lemma 3.4 is almost the same as the proof of Theorem 3.2 from Lemma 3.3, but we include the proof for the sake of the reader.

Proof of Theorem 3.1. Fix $A \subseteq G$ such that $\alpha := \mathrm{BD}(A) > \frac{1}{2}$. Fix a tempered Følner sequence (F_n) for G such that

$$\lim_{n\to\infty} \frac{|A \cap F_n|}{|F_n|} = \alpha.$$

Fix L as in the conclusion of Lemma 3.4. Fix an injective enumeration $L = (l_n)$ of L. Recursively define an injective sequence $D = (d_n)_{n\in\mathbb{N}}$ from A such that $l_i d_n \in A$ for $i \leq n$. Fix $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$. For any $g \in G$, we have

$$\mathrm{st}\left(\frac{|g^*A \cap F_\nu|}{|F_\nu|}\right) = \mathrm{st}\left(\frac{|g^*A \cap gF_\nu|}{|F_\nu|}\right) = \mathrm{st}\left(\frac{|^*A \cap F_\nu|}{|F_\nu|}\right) = \alpha;$$

since we also have $\mathrm{st}(\frac{|^*L \cap F_\nu|}{|F_\nu|}) \geq \alpha$, it follows that, for every $n \in \mathbb{N}$, we have

$$\mathrm{st}\left(\frac{|^*L \cap d_n^{-1}{}^*A \cap F_\nu|}{|F_\nu|}\right) \geq 2\alpha - 1 > 0.$$

By Fact 2.3, we may, after passing to a subsequence of (d_n), assume that, for every $n \in \mathbb{N}$, we have

$$\mathrm{st}\left(\frac{|^*L \cap \bigcap_{i\leq n} d_i^{-1}{}^*A \cap F_\nu|}{|F_\nu|}\right) > 0.$$

In particular, this implies that, for every $n \in \mathbb{N}$, we have $L \cap \bigcap_{i \leq n} d_i^{-1} A$ is infinite. Take $b_1 \in L$ arbitrary and take $c_1 \in D$ such that $b_1 c_1 \in A$. Fix $b_2 \in (L \cap c_1^{-1}A) \setminus \{b_1\}$ and take $c_2 \in D$ such that $\{b_1c_2, b_2c_2\} \subseteq A$. Take $b_3 \in (L \cap c_1^{-1}A \cap c_2^{-1}A) \setminus \{b_1, b_2\}$ and take $c_3 \in D$ such that $\{b_1c_3, b_2c_3, b_3c_3\} \subseteq A$. Continue in this way to construct the desired B and C. □

We say that (F_n) is a *two-sided Følner sequence* for G if, for all $g \in G$, we have

$$\lim_{n\to\infty} \frac{|(gF_n \triangle F_n)| + |(F_n g \triangle F_n)|}{|F_n|} = 0.$$

Of course, if G is abelian, then every Følner sequence is two-sided. If G is amenable, then two-sided Følner sequences for G exist. However, it is unclear, given $A \subseteq G$ with positive Banach density, whether or not there is a two-sided Følner sequence for G witnessing the Banach density of A.

If we repeat the previous proof with Ad_n^{-1} instead of $d_n^{-1}A$, we get the following result.

Theorem 3.5. *Suppose that (F_n) is a two-sided Følner sequence for G and $A \subseteq G$ is such that $\lim_{n\to\infty} \frac{|A\cap F_n|}{|F_n|} = \mathrm{BD}(A) > \frac{1}{2}$. Then there are infinite $B, C \subseteq G$ with $C \subseteq A$ such that $BC \subseteq A$.*

Let us end this section by showing how to derive Theorem 3.2 from Theorem 3.1 directly. Suppose that $A \subseteq \mathbb{N}$ has Banach density exceeding $\frac{1}{2}$. Then $\mathrm{BD}(A) > \frac{1}{2}$ *when viewed as a subset of* $\mathbb{Z}$. By Theorem 3.1, there are infinite sequences $B, C \subseteq \mathbb{Z}$ such that $C \subseteq A$ and $B + C \subseteq A$. Since

$C \subseteq A \subseteq \mathbb{N}$, this forces all but finitely many elements of B to belong to $\mathbb{N}$; replacing B with $B \cap \mathbb{N}$ yields the desired result.

4. A one-shift result for sets of positive Banach density

The main result of this section is the following.

Theorem 4.1. *If $A \subseteq G$ has positive Banach density, then there are injective sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ in G and $h, h' \in G$ such that:*

- *$c_n \in A$ for each n;*
- *$b_i c_j \in hA$ for $i \leq j$;*
- *$c_j b_i \in h'A$ for $i < j$.*

The proof proceeds in two steps. First, we show that we can "fatten" A to a set QA, where $Q \subseteq G$ is finite, for which $\mathrm{BD}(QA) > \frac{1}{2}$. We then apply Theorem 3.1 to QA and apply Ramsey's theorem to obtain the desired result. The first step was done in [11] in the case of the natural numbers, so we cover this case separately for those readers who are primarily interested in the case of subsets of the natural numbers.

4.1. The case of the integers.

Definition 4.2. For $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, we define $A_{[n]} \subseteq \mathbb{N}$ by declaring $k \in A_{[n]}$ iff $[kn, kn+n-1] \cap A \neq \varnothing$. In other words, if the natural numbers are partitioned into equal sized blocks of length n, then $A_{[n]}$ is the sequence of the "block numbers" that intersect A.

The following is Theorem 3.8 in [11].

Fact 4.3. *For any A with $\mathrm{BD}(A) > 0$ and any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\mathrm{BD}(A_{[n]}) \geq 1 - \epsilon$.*

We are now ready to prove the one-shift result in the case of subsets of the natural numbers.

Theorem 4.4. *If $A \subseteq \mathbb{N}$ is such that $\mathrm{BD}(A) > 0$, then there exist infinite sets $B, C \subseteq \mathbb{N}$ and $k \in \mathbb{Z}$ such that $B + C \subset A \cup (A + k)$.*

Proof. By the previous lemma, there exists $n \in \mathbb{N}$ such that $\mathrm{BD}(A_{[n]}) > \frac{1}{2}$. Applying Theorem 3.1 to $A_{[n]}$, we obtain sets $B_{[n]} = (b_i), C_{[n]} = (c_j)$ such that $B_{[n]} + C_{[n]} \subset A_{[n]}$. In other words, every $[nb_i + nc_j, nb_i + nc_j + n - 1]$ intersects A. Using n^2 colors we may code every pair of natural numbers $\{i, j\}$ with $i < j$ based on which $\nu \in [0, n-1]$ is such that $nb_i + nc_j + \nu$ is the first element of A in $[nb_i + nc_j, nb_i + nc_j + n - 1]$, and which $\xi \in [0, n-1]$ is such that $nc_i + nb_j + \xi$ is the first element of A in $[nc_i + nb_j, nc_i + nb_j + n - 1]$. By Ramsey's theorem, there exists an infinite $J \subseteq \mathbb{N}$ monochromatic for this coloring.

We now replace $B_{[n]}$ and $C_{[n]}$ by infinite subsequences whose indices come from J. In particular, there is a fixed pair ν and ξ such that, for any $i < j$, $nb_i + nc_j + \nu \in A$ while $nc_i + nb_j + \xi \in A$. If we now let $k = \nu - \xi$,

$B = \{nb_i + \nu : i \text{ is odd}\}$, and $C = \{nc_j : j \text{ is even}\}$, we see that $B + C \subset A \cup (A + k)$, with the translate of A for a given element of $B+C$ determined by whether $i < j$ or $i > j$. It is important to note that by taking only the odd indices from one set and the even indices from the other set we avoid the case in which the indices are the same, something that was not determined by the use of Ramsey's Theorem. □

4.2. **The case of an arbitrary amenable group.** In this subsection, we once again assume that G is a countable amenable group.

In order to prove the analog of Fact 4.3 in the case of an arbitrary amenable group, we will need the following fact, which is a particular case of Theorem 4.5 in [16]. (There one assumes that the amenable group is unimodular, which is immediate in our case since our groups are discrete.)

Fact 4.5. *Suppose that* $\varepsilon \in \left(0, \frac{1}{10}\right)$. *Define* $N(\varepsilon) = \left\lceil \frac{\log(\varepsilon)}{\log(1-\varepsilon)} \right\rceil$. *For every finite subset* H *of* G *and every* $\delta \in (0, \varepsilon)$ *there are* (H, δ)*-invariant finite sets*

$$\{1_G\} \subset T_1 \subset T_2 \subset \ldots \subset T_{N(\varepsilon)},$$

a finite subset K *of* G *containing* H, *and a positive real number* $\eta < \delta$ *such that, for every finite subset* F *of* G *which is* (K, η)*-invariant, there are finite sets* $C_i \subset G$ *and* $T_i^{(c)} \subset T_i$ *for* $i = 1, 2, \ldots, N(\varepsilon)$ *such that:*

- $\left\{ T_i^{(c)} c \mid i \leq N(\varepsilon),\, c \in C_i \right\}$ *is a family of pairwise disjoint sets;*
- $\bigcup_{i=1}^{N(\varepsilon)} T_i^{(c)} C_i \subset F$;
- $\left| \bigcup_{i=1}^{N(\varepsilon)} T_i^{(c)} C_i \right| > (1 - 2\varepsilon) \cdot |F|$.

Lemma 4.6. *For any* $A \subseteq G$ *with* $\mathrm{BD}(A) > 0$ *and for every* $\rho > 0$, *there is a finite subset* Q *of* G *such that* $\mathrm{BD}(QA) > 1 - \rho$.

Proof. Set $\alpha := \mathrm{BD}(A) > 0$. Pick $\varepsilon > 0$ such that

$$\frac{\alpha - 3\varepsilon}{\alpha + \varepsilon} > 1 - \rho.$$

Since $\alpha + \epsilon > \mathrm{BD}(A)$, there is finite $H \subseteq G$ and $\delta \in (0, \varepsilon)$ such that, for every (H, δ)-invariant set F, we have

$$\frac{|F \cap A|}{|F|} < \alpha + \varepsilon.$$

Fix $K \subseteq G$ finite, $\eta > 0$, and

$$\{1_G\} \subset T_1 \subset T_2 \subset \ldots \subset T_{N(\varepsilon)}$$

obtained from ε, δ, and H as in Fact 4.5. Define

$$Q = \bigcup_{i=1}^{N(\varepsilon)} T_i T_i^{-1}$$

and

$$B = QA.$$

We claim that $\mathrm{BD}(B) > 1-\rho$. Towards this end, fix a Følner sequence $(F_n)_{n\in\mathbb{N}}$ of G such that

$$\limsup_n \frac{|A\cap F_n|}{|F_n|} = \alpha.$$

We claim that

$$\limsup_n \frac{|B\cap F_n|}{|F_n|} > 1-\rho.$$

Fix $n_0 \in \mathbb{N}$ and pick $n \geq n_0$ such that F_n is (K,η)-invariant and

$$\frac{|F_n\cap A|}{|F_n|} > \alpha - \varepsilon. \quad (\dagger)$$

Fix sets C_i and $T_i^{(c)} \subset T_i$ for $i \leq 1, 2, \ldots, N(\varepsilon)$ obtained from F_n as in Fact 4.5. Define

$$\mathcal{T} = \left\{ T_i^{(c)} c \mid i \leq N(\varepsilon), c \in C_i \right\}$$

and observe that $\mathcal{T}$ is a finite family of pairwise disjoint (H,δ)-invariant finite sets such that

$$\frac{|\bigcup \mathcal{T}|}{|F_n|} > 1 - 2\varepsilon. \quad (\dagger\dagger)$$

Define

$$\mathcal{T}_0 = \{T \in \mathcal{T} \mid T \cap A \neq \varnothing\}.$$

We have

$$\begin{aligned}
(\alpha-\varepsilon)|F_n| &< |A\cap F_n| \\
&\leq \left|A \cap \bigcup \mathcal{T}\right| + 2\varepsilon |F_n| \\
&= \left|A \cap \bigcup \mathcal{T}_0\right| + 2\varepsilon |F_n| \\
&= \sum_{T\in\mathcal{T}_0} |A\cap T| + 2\varepsilon |F_n| \\
&\leq \sum_{T\in\mathcal{T}_0} |T|(\alpha+\varepsilon) + 2\varepsilon |F_n| \\
&= \left|\bigcup \mathcal{T}_0\right|(\alpha+\varepsilon) + 2\varepsilon |F_n|.
\end{aligned}$$

In the above string of equalities and inequalities, the first line follows from $(\dagger)$, the second line follows from $(\dagger\dagger)$, the third line follows from the definition of $\mathcal{T}_0$, the fourth line follows from the fact that the members of $\mathcal{T}_0$ are pairwise disjoint, and the fifth line follows from the fact that the elements of $\mathcal{T}_0$ are (H,δ)-invariant and the choice of H and δ.

It follows that

$$\frac{|\bigcup \mathcal{T}_0|}{|F_n|} \geq \frac{\alpha - 3\varepsilon}{\alpha+\varepsilon}.$$

Observe that $B \supset \bigcup \mathcal{T}_0$ and therefore

$$\frac{|B \cap F_n|}{|F_n|} \geq \frac{|\bigcup \mathcal{T}_0|}{|F_n|} \geq \frac{\alpha - 3\varepsilon}{\alpha + \varepsilon} > 1 - \rho.$$

□

Theorem 4.1 now follows from Lemma 4.6 in the same way that Theorem 4.4 followed from Lemma 4.3.

We leave it to the reader to verify that Theorem 4.4 also follows from the special case of Theorem 4.1 for $G = \mathbb{Z}$.

5. The pseudorandom case

In this section, we prove that the $B + C$ conjecture holds for A that are *pseudorandom* in a sense to be described below. We start by recalling some preliminary facts and definitions.

Suppose that H is a Hilbert space and $U : H \to H$ is a unitary operator. We say that $x \in H$ is *weakly mixing (for U)* if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |\langle U^i x, x \rangle| = 0.$$

We will need the following result; see [13, Theorem 3.4] for a proof.

Fact 5.1. *$x \in H$ is weakly mixing if and only if* $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\langle U^i x, y \rangle| = 0$ *for every $y \in H$.*

We will also need the following easy fact.

Fact 5.2. *Suppose that (r_n) is a sequence of nonnegative real numbers. Then* $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} r_n = 0$ *if and only if, for every $\epsilon > 0$, we have*

$$\underline{d}(\{n \in \mathbb{N} \ : \ r_n \leq \epsilon\}) = 1.$$

In what follows, we will need the notion of *upper (asymptotic) density.* For $A \subseteq \mathbb{N}$, the upper density of A, denoted $\overline{d}\,(A)$, is defined to be

$$\overline{d}\,(A) := \limsup_{n \to \infty} \frac{|A \cap [1, n]|}{n}.$$

For $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, we set $A_N := {}^*A \cap [1, N]$ and write μ_N for the Loeb measure on $[1, N]$. We always consider $[1, N]$ to be equipped with its Loeb measure μ_N.

Suppose that $A \subseteq \mathbb{N}$ is such that $\overline{d}(A) = \alpha > 0$ and $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ is such that $\frac{|A_N|}{N} \approx \alpha$. Notice that $\chi_{A_N} \in L^2(\mu_N)$. We have a measure preserving transformation $T : [1, N] \to [1, N]$ defined by

$$T(x) := x + 1 \ (\text{mod} N).$$

The transformation T gives rise to the unitary operator $U_T : L^2(\mu_N) \to L^2(\mu_N)$ given by $U_T(f) := f \circ T$.

We are now ready to define our notion of pseudorandom.

Definition 5.3. Suppose that $A \subseteq \mathbb{N}$ is such that $\overline{d}(A) = \alpha > 0$. We say that A is *pseudorandom* if there is $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that, in the notation preceding the definition, we have that $\chi_{A_N} - \alpha$ is weakly mixing (for U_T).

Equivalently A is pseudorandom if and only if there is $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ as above such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left|\mu_N(A_N \cap (A-i)_N) - \alpha^2\right| = 0.$$

It appears to be a little awkward to give a standard reformulation of the aforementioned notion of pseudorandom. Certainly, if there is an increasing sequence (b_k) of natural numbers such that:

- $\lim_k b_k = \infty$,
- $\lim_k \frac{|A\cap[1,b_k]|}{b_k} = \alpha$, and
- $\lim_n \frac{1}{n}\sum_{i=1}^{n} \lim_k \left|\frac{|A\cap(A-i)\cap[1,b_k]|}{b_k} - \alpha^2\right| = 0$,

then A is pseudorandom (just take $N = b_K$ for any $K \in {}^*\mathbb{N} \setminus \mathbb{N}$).

In order to prove that pseudorandom sets satisfy the $B+C$ conjecture, we will need one last fact whose proof is nearly identical to the proof of Theorem 4.6 in [3] (just replace arbitrary hyperfinite intervals by hyperfinite intervals of the form $[1, N]$).

Fact 5.4. *If $A \subseteq \mathbb{N}$ is such that $\overline{d}(A) > 0$, then there is $L \subseteq \mathbb{N}$ such that $\underline{d}(L) = \overline{d}(A)$ and*

$$\overline{d}\left(\bigcap_{l\in F}(A-l)\right) > 0$$

for every finite $F \subseteq L$.

We are now ready to prove the main result of this section.

Theorem 5.5. *If $A \subseteq \mathbb{N}$ is pseudorandom, then there are infinite $B, C \subseteq \mathbb{N}$ such that $B + C \subseteq A$.*

Proof. Set $\alpha := \overline{d}(A)$ and take N as above witnessing that A is pseudorandom. For ease of notation, we set $\mu := \mu_N$. By Fact 5.4, we may fix $L = (l_n)$ with $\underline{d}(L) = \alpha$ and such that

$$\overline{d}\left(\bigcap_{l\in F}(A-l)\right) > 0$$

for every finite $F \subseteq L$. Set $\beta := \mu(L_N) \geq \alpha$. Observe that $U_T^i(\chi_{A_N}) = \chi_{(A-i)_N}$. Since $\chi_{A_N} - \alpha$ is weak mixing, by Fact 5.1, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\mu((A-i)_N \cap L_N) - \alpha\beta| = 0.$$

By Fact 5.2, for every $\epsilon > 0$, we have that

$$R_\epsilon := \{n \in \mathbb{N} \ : \ |\mu((A-n)_N \cap L_N) - \alpha\beta| < \epsilon\}$$

has lower density equal to 1. In particular, for any $\epsilon > 0$ and any finite $F \subseteq L$, we have that

$$\overline{d}\left(\bigcap_{l \in F}(A-l) \cap R_\epsilon\right) > 0.$$

Setting $\eta := \frac{\alpha^2}{2}$, this allows us to inductively define a sequence (d_n) such that, for each $n \in \mathbb{N}$, we have $d_n \in \bigcap_{i \leq n}(A-l_i) \cap R_\eta$. In particular, we have $\mu((A-d_n)_N \cap L_N) > \eta$ for each $n \in \mathbb{N}$. We now apply Fact 2.3 to the family $((A-d_n)_N \cap L_N)$ to get a subsequence (e_n) of (d_n) such that

$$\mu(\bigcap_{i \leq n}(A-e_i)_N \cap L_N) > 0$$

for each $n \in \mathbb{N}$. Finally, as in the proof of Theorem 3.1, this allows us to define subsequences $B = (b_n)$ and $C = (c_n)$ of (l_n) and (e_n), respectively, for which $B + C \subseteq A$. □

We end this section with a question. First, for H a Hilbert space and $U : H \to H$ a unitary operator, we say that $x \in H$ is *almost periodic (for U)* if $\{U^n x : n \in \mathbb{Z}\}$ is relatively compact (in the norm topology). Using the notation of Definition 5.5, we say that A is *almost periodic* if χ_{A_N} is an almost periodic element of $L^2([0,N])$ (for U_T).

Question 5.6. *If A is almost periodic, does A satisfy the conclusion of the $B+C$ conjecture?*

This distinction between weakly mixing and almost periodic subsets of $\mathbb{N}$ is reminiscent of Furstenberg's proof of Szemeredi's Theorem (see [9]), where it is shown how to prove Szemeredi's Theorem by first establishing it for the weakly mixing and compact cases and then showing how to derive it for the general case by "Furstenberg towers" that are "built from" both of these cases. It thus makes sense to ask:

Question 5.7. *If the previous question has an affirmative answer, is there a way to decompose an arbitrary $A \subseteq \mathbb{N}$ of positive lower density into a "tower" built from weakly mixing and almost periodic parts in a way that allows one to prove the $B+C$ conjecture?*

It is unclear to us whether there are many concrete examples of pseudo-random subsets of the natural numbers, but we believe the value of Theorem 5.5 is that it may be a first step in proving the $B+C$ conjecture via the route outlined in Question 5.7.

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