Monad measure spaces and combinatorial number theory

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Densities

Definition

Suppose that $A \subseteq \mathbb{Z}$.

1. The *upper density of A* is

$$\bar{d}(A) := \limsup_{n\to\infty} \frac{|A \cap [-n, n]|}{2n+1}.$$

2. The *Banach density of A* is

$$\mathrm{BD}(A) := \lim_{n\to\infty} \sup_{x\in\mathbb{Z}} \frac{|A \cap [x-n, x+n])|}{2n+1}.$$

Positive density implies structure

A common theme in combinatorial number theory is to prove that a set of positive density must possess some structure. Perhaps the most famous example of such a theorem is:

Theorem (Szemeredi, 1975)

If $A \subseteq \mathbb{Z}$ is such that $\mathrm{BD}(A) > 0$, then A contains arbitrarily long arithmetic progressions.

Structure notions of largeness

Definition

Suppose that $A \subseteq \mathbb{Z}$. We say that A is:

1. *thick* if, for every $n \in \mathbb{N}$, there is $x \in \mathbb{Z}$ such that $x + [-n, n] \subseteq A$ (equivalently, $\mathrm{BD}(A) = 1$);
2. *syndetic* if there is $n \in \mathbb{N}$ such that $A + [-n, n] = \mathbb{Z}$;
3. *piecewise syndetic* if there is $m \in \mathbb{N}$ such that $A + [-m, m]$ is thick.

Our motivating theorem:

Theorem (Jin, 2002)

If $A, B \subseteq \mathbb{Z}$ are such that $\mathrm{BD}(A), \mathrm{BD}(B) > 0$, then $A + B$ is piecewise syndetic.

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Inspiration from Real Analysis

Fact

Suppose that $A, B \subseteq [-1, 1]$ are such that $\lambda(A), \lambda(B) > 0$. Then $A + B$ contains an interval.

Proof.

Let a and b be Lebesgue points for A and B respectively. Choose $r > 0$ sufficiently small such that

$$\frac{\lambda(A \cap (a-r, a+r))}{2r}, \frac{\lambda(-B \cap (-b-r, -b+r))}{2r} \approx 1.$$

For $x \in (-\frac{r}{2}, \frac{r}{2})$, we have

$$\frac{\lambda((-B+x) \cap (-b-r, -b+r))}{2r} \approx \frac{1}{2}.$$

It follows that $(A-a) \cap (-B+b+x) \neq \emptyset$, so $a+b+(\frac{r}{2}, \frac{r}{2}) \subseteq A+B$. □

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Shifting from continuous to discrete

Question

How do we make the passage discrete to continuous? How do we make the passage from densities to measures?

Answer

Furstenberg's Correspondence Principle, which is a meta-principle that roughly turns combinatorial questions about densities into questions in ergodic theory. *Nonstandard analysis* provides a very slick way to present this idea.

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An axiomatic approach to $\mathbb{R}^*$

We will work in a *nonstandard universe* $\mathbb{R}^*$ that has the following properties:

1. $(\mathbb{R}; +, \cdot, 0, 1, <)$ is an *ordered subfield* of $(\mathbb{R}^*; +, \cdot, 0, 1, <)$.
2. $\mathbb{R}^*$ has a *positive infinitesimal* element, that is, there is $\delta \in \mathbb{R}^*$ such that $\delta > 0$ but $\delta < r$ for every $r \in \mathbb{R}^{>0}$.
3. For every $n \in \mathbb{N}$ and every function $f : \mathbb{R}^n \to \mathbb{R}$, there is a "natural extension" $f : (\mathbb{R}^*)^n \to \mathbb{R}^*$. The natural extensions of the field operations $+, \cdot : \mathbb{R}^2 \to \mathbb{R}$ coincide with the field operations in $\mathbb{R}^*$. Similarly, for every $A \subseteq \mathbb{R}^n$, there is a subset $A^* \subseteq (\mathbb{R}^*)^n$ such that $A^* \cap \mathbb{R}^n = A$.
4. $\mathbb{R}^*$, equipped with the above assignment of extensions of functions and subsets, "behaves logically" like $\mathbb{R}$.

Standard parts

- Say that $x \in \mathbb{R}^*$ is *finite* if $|x| \leq n$ for some $n \in \mathbb{N}$.
- For example, for any $r \in \mathbb{R}$ and any (positive or negative) infinitesimal δ, $r + \delta$ is finite.
- Conversely:

Fact

If $x \in \mathbb{R}^*$ is finite, then there is a unique $r \in \mathbb{R}^{>0}$ such that $x - r$ is infinitesimal. We call r *the standard part of* x and denote it by $\mathrm{st}(x)$.

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Extending sequences

- Recall that every function $f : \mathbb{R} \to \mathbb{R}$ has a nonstandard extension $f : \mathbb{R}^* \to \mathbb{R}^*$.
- Partial functions $f : A \to \mathbb{R}$ have nonstandard extensions $f : A^* \to \mathbb{R}^*$ as well.
- In particular, if $(a_n \ : \ n \in \mathbb{N})$ is a sequence of reals, viewing (a_n) as the function $a : \mathbb{N} \to \mathbb{R}$, we get a nonstandard extension $a : \mathbb{N}^* \to \mathbb{R}^*$. We also write this in sequence notation $(a_n \ : \ n \in \mathbb{N}^*)$ and refer to a_N for $N \in \mathbb{N}^* \setminus \mathbb{N}$ as an *extended term of the sequence*.

Subsequential limits

Lemma

If (a_n) is a sequence and $L \in \mathbb{R}$, then L is a subsequential limit of (a_n) if and only if there is an extended term a_N with $\text{st}(a_N) = L$.

Corollary

If (a_n) is a bounded sequence, then

$$\limsup a_n = \max\{\text{st}(a_N) \ : \ N \in \mathbb{N}^* \setminus \mathbb{N}\}.$$

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Nonstandard characterization of densities

- For $A \subseteq \mathbb{Z}$, we have

$$\bar{d}(A) = \max \left\{ \operatorname{st}\left(\frac{|A^* \cap [-N, N]|}{2N+1} \right) \; : \; N \in \mathbb{N}^* \setminus \mathbb{N} \right\}.$$

- In a similar manner, for every $N > \mathbb{N}$, there is $x \in \mathbb{Z}^*$ such that

$$\operatorname{BD}(A) = \operatorname{st}\left(\frac{|A^* \cap [x - N, x + N]|}{2N+1} \right).$$

- The point of passing to the nonstandard framework is that the quantities $\operatorname{st}\left(\frac{|A^* \cap [-N,N]|}{2N+1}\right)$ are actually certain measures on A^*, called *Loeb measures*. To define Loeb measure, we first need the concept of internal sets and hyperfinite sets.

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Internal sets

- *Internal* subsets of $\mathbb{R}^*$ are the "definable" subsets of $\mathbb{R}^*$ in some precise way that we won't define. They "logically behave" like ordinary subsets of $\mathbb{R}$. For example, $A^* \cap [-N, N]$ is an internal set.
- The set of all infinitesimals is not internal. Indeed, nonempty internal subsets of $\mathbb{R}^*$ bounded above have a sup. But what would the sup of the infinitesimals be?
- An internal set is *hyperfinite* if there is an internal bijection between it and an interval of the form $[1, M]$ from $\mathbb{N}^*$. Internal subsets of hyperfinite sets are hyperfinite, so, e.g., $A^* \cap [-N, N]$ is hyperfinite.

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Saturation

Definition

The nonstandard extension is said to be *countably saturated* if, whenever $(A_n : n \in \mathbb{N})$ is a family of internal sets with the finite intersection property, then $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$.

Any reasonable construction of the nonstandard extension is countably saturated and we assume throughout that our model is countably saturated.

Loeb measure

- Suppose that $E \subseteq \mathbb{R}^*$ is hyperfinite. Then there is a unique $M \in \mathbb{N}^*$ such that there is an internal bijection $E \to [1, M]$; we call M the *internal cardinality of E* and denote it by $|E|$.
- Fix a hyperfinite set E and define a function $\mu : \mathcal{P}_{\text{int}}(E) \to [0, 1]$ by $\mu(A) := \text{st}\left(\frac{|A|}{|E|}\right)$. ($\mathcal{P}_{\text{int}}$ is the *internal powerset*.)
- Then μ is a finitely additive measure. By countable saturation, μ (trivially!) satisfies the conditions of the Caratheodory extension theorem, so extends to a countably additive measure on a certain σ-algebra containing the internal subsets of E; this measure is called the *Loeb measure*.
- Motivating fact: Consider the function $f : [1, M] \to [0, 1]$ given by $f(k) := \text{st}(\frac{k}{M})$. Then the measure on $[0, 1]$ induced by the Loeb measure on $[1, M]$ is the usual Lebesgue measure.

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Nonstandard characterization of densities again

- For $N \in \mathbb{N}^* \setminus \mathbb{N}$, let μ_N be the Loeb measure on $[-N, N]$.
- For internal $E \subseteq \mathbb{Z}^*$, we simply write $\mu_N(E)$ instead of $\mu_N(E \cap [-N, N])$.
- We then have

$$\overline{d}(A) = \max \{\mu_N(A^*) \ : \ N \in \mathbb{N}^* \setminus \mathbb{N}\}.$$

- Likewise, for every $N > \mathbb{N}$, there is $x \in \mathbb{Z}^*$ such that

$$\mathrm{BD}(A) = \mu_N((A^* - x)).$$

- Densities have become measures!

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We need a new space

- Notice that the Lebesgue Density Theorem cannot possibly hold for Loeb measure spaces.
- For example, on any interval, the set of even elements of $\mathbb{Z}^*$ have measure $\frac{1}{2}$.
- However, we noticed that the usual Lebesgue measure on $[0, 1]$ is the quotient measure space associated to the Loeb measure space on $[-N, N]$ when two elements of $[-N, N]$ are identified if they differ by an amount infinitely smaller than N.
- So to get a Lebesgue Density Theorem to hold, we need to go to quotient spaces.

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Monad measure space

- Given $x, y \in \mathbb{Z}^*$, we say that x and y are equivalent, denoted $x \sim y$, if $|x - y| \in \mathbb{N}$.
- Equivalence classes are called *monads* and are simply $\mathbb{Z}$-chains.
- Given $N > \mathbb{N}$, let $\mathcal{H}_N := \{[x] \ : \ x \in [-N, N]\}$.
- We call $X \subseteq \mathcal{H}_N$ *measurable* if $\bigcup X$ is Loeb measurable in $[-N, N]$ and then we declare $\mathfrak{m}(X) := \mu_N(\bigcup X)$. We refer to this quotient measure space as a *monad measure space*.
- For example, if $E \subseteq [-N, N]$ is internal, then $\{[x] \ : \ x \in E\}$ is measurable as its unionset is $E + \mathbb{Z} = \bigcup_m (E + [-m, m])$, a countable union of internal sets, whence Loeb measurable.

Lebesgue density theorem for monad measure spaces

Definition

Suppose that $E \subseteq \mathbb{Z}^*$ is internal and $x \in \mathbb{Z}^*$. We say that x is a *density point of E* if there is $M > \mathbb{N}$ such that, for all $\mathbb{N} < N < M$, we have $\mu_N(E - x + \mathbb{Z}) = 1$.

Theorem (LDT for Monad Measure Spaces-DGJLLM, 2013)

Suppose that $N > \mathbb{N}$ and $E \subseteq [-N, N]$ is internal. Then μ_N-almost all points of E are density points of E.

Points of syndeticity

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Suppose that $E \subseteq \mathbb{Z}^*$ is internal and $x \in \mathbb{Z}^*$. We say that x is a *point of syndeticity of E* if there is $m \in \mathbb{N}$ such that $x + \mathbb{Z} \subseteq E + [-m, m]$.

Lemma

Suppose that $A \subseteq \mathbb{Z}$ is such that A^* has a point of syndeticity. Then A is piecewise syndetic.

Proof.

Fix $x \in \mathbb{Z}^*$ and $m \in \mathbb{N}$ such that $x + \mathbb{Z} \subseteq A^* + [-m, m]$. Then, for every $k \in \mathbb{N}$, the nonstandard model believes the statement "there is $x \in \mathbb{Z}^*$ such that $x + [-k, k] \subseteq A^* + [-m, m]$." Apply transfer. □

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Fix $x \in \mathbb{Z}^*$ and $m \in \mathbb{N}$ such that $x + \mathbb{Z} \subseteq A^* + [-m, m]$. Then, for every $k \in \mathbb{N}$, the nonstandard model believes the statement "there is $x \in \mathbb{Z}^*$ such that $x + [-k, k] \subseteq A^* + [-m, m]$." Apply transfer. □

Nonstandard Jin's Theorem

Theorem

Suppose that $N > \mathbb{N}$ and that $E, F \subseteq [-N, N]$ are internal and have positive Loeb measure. Take $x \in E$ and $y \in F$ density points for E and F. Then $x + y$ is a syndetic point for $E + F$.

Proof.

- Take $M > \mathbb{N}$ small enough so that $\mu_M(E - x + \mathbb{Z}) = \mu_M(-F + y + \mathbb{Z}) = 1$.
- Arguing as before, this gives us that $x + y + [-\frac{M}{2}, \frac{M}{2}] \subseteq E + F + \mathbb{Z} = \bigcup_{m \in \mathbb{N}}(E + F + [-m, m])$.
- Countable saturation tells us that there is $m \in \mathbb{N}$ such that $x + y + [-\frac{M}{2}, \frac{M}{2}] \subseteq E + F + [-m, m]$.
- Since $M > \mathbb{N}$, we get $x + y + \mathbb{Z} \subseteq E + F + [-m, m]$.

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□

Proof of Jin's Theorem

- Suppose that $A, B \subseteq \mathbb{Z}$ are such that $\mathrm{BD}(A), \mathrm{BD}(B) > 0$.
- Fix $N > \mathbb{N}$ and take $c, d \in \ \mathbb{Z}^*$ such that

$$\mu_N(A^* - c) = \mathrm{BD}(A),\ \mu_N(B^* - d) = \mathrm{BD}(B).$$

- Set $E := (A^* - c) \cap [-N, N]$ and $F := (B^* - d) \cap [-N, N]$.
- Then $E + F$ has a point of syndeticity z, so there is $m \in \mathbb{N}$ such that

$$z + \mathbb{Z} \subseteq E + F + [-m, m].$$

- It follows that $c + d + z + \mathbb{Z} \subseteq A^* + B^* + [-m, m]$, so $A^* + B^* = (A + B)^*$ has a point of syndeticity.

Quantitative version

- Notice that we only used the existence of a single density point for each of A^* and B^*, even though the Lebesgue Density Theorem guarantees us many density points. We can use this to obtain strengthenings of Jin's theorem. For example:

Definition

We say that $A \subseteq \mathbb{Z}$ is *upper syndetic of level* α if there is $m \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$, we have

$$\bar{d}(\{x \in \mathbb{Z} \ : \ x + [-k, k] \subseteq A + [-m, m]\}) \geq \alpha.$$

Theorem (DGJLLM, 2013)

If $A, B \subseteq \mathbb{Z}$ are such that $\bar{d}(A) = \alpha > 0$ and $\mathrm{BD}(B) > 0$, then $A + B$ is upper syndetic of level α.

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Measure of syndeticity points

Suppose that $E \subseteq \mathbb{Z}^*$ is internal and $N > \mathbb{N}$. We set

$$\mathcal{S}_{E,N} := \{z \in [-N, N] \ : \ z \text{ is a syndeticity point for } E\}.$$

Lemma

$\mathcal{S}_{E,N}$ is μ_N-measurable. If $\mu_N(\mathcal{S}_{E,N}) = \alpha > 0$, then for all (standard) $\epsilon > 0$, there is $m \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$, we have

$$\mu_N(\{z \in [-N, N] \ : \ z + [-k, k] \subseteq E + [-m, m]\} \geq \alpha - \epsilon.$$

Proof.

$\mathcal{S}_{E,N} = \bigcup_{i=1}^{\infty} \mathcal{S}^i_{E,N}$, where $\mathcal{S}^i_{E,N} = (\bigcap_{x \in \mathbb{Z}}(E + [-i, i] + x)) \cap [-N, N]$. □

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Proof of Weak Quantitative Jin

- Take $N > \mathbb{N}$ such that $\mu_N(A^*) = \alpha$.
- Then almost all points of $A^* \cap [-N, N]$ are points of density of A^*.
- One can actually show that we can find a point of density b of B^* whose absolute value is infinitely smaller than N.
- Therefore, almost all points of $(A^* + b) \cap [-N, N]$ are points of syndeticity of $A^* + B^*$, so $\mu_N(\mathcal{S}_{(A+B)^*,N}) \geq \alpha$.
- Therefore, for any $\epsilon > 0$, we have $m \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$, we have

$$\mu_N(\{x \in [-N, N] \ : \ x + [-k, k] \subseteq (A + B)^* + [-m, m]\} \geq \alpha - \epsilon.$$

- By the nonstandard characterization of upper density, this says that $A + B$ is upper syndetic of level $\alpha - \epsilon$.

Logarithmic density

Definition

For $A \subseteq \mathbb{N}$, the *logarithmic Banach density of A* is

$$\ell\mathrm{BD}(A) = \lim_{n\to\infty} \sup_{k\geq 1} \frac{1}{\ln n} \sum_{x\in A\cap[k,nk]} \frac{1}{x}.$$

Facts

1. $\ell\mathrm{BD}(\mathbb{N}) = 1$;
2. $\ell\mathrm{BD}(A) \leq \mathrm{BD}(A)$;
3. If $\ell\mathrm{BD}(A) = \alpha$, then there is $N \in \mathbb{N}^* \setminus \mathbb{N}$ and $k \in \mathbb{N}^*$ such that $\ell\mathrm{BD}(A) \approx \frac{1}{\ln N} \sum_{x\in A^*\cap[k,Nk]} \frac{1}{x}$.

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Loeb space

Definition

For internal $A \subseteq [k, Nk]$, we set $\nu(A) := \operatorname{st}(\frac{1}{\ln N} \sum_{x \in A} \frac{1}{x})$. As before, we get a Loeb measure.

Example

For all $k \leq a \leq b \leq Nk$, we have $\nu([a, b]) = \operatorname{st}(\frac{\ln b - \ln a}{\ln N})$. In particular, $\nu([k, Nk]) = 1$ and $\nu([ac, bc]) = \nu_L([a, b])$.

Example

If $c > 1$, then $\nu(c \cdot [a, b]) = \operatorname{st}(\frac{1}{c \ln N} \sum_{x \in [a,b]} \frac{1}{x}) \neq \nu([a, b])$.

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Multiplicative monad spaces

- Given $x \leq y \in \mathbb{N}^*$, we now say $x \sim y$ if $\lfloor \frac{y}{x} \rfloor \in \mathbb{N}$.
- This is an equivalence relation; quotient map $\varphi : [k, Nk] \to \mathcal{H}_{k,N}$.
- $\mathcal{H}_{k,N}$ inherits a (dense) linear order and multiplication.
- We equip $\mathcal{H}_{k,N}$ with the quotient measure $\mathfrak{m}$.
- Set $V_N := \bigcap_{k \geq 1} [1, N^{1/k})$.

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Approximate geometric progressions

Definition

Suppose $n \in \mathbb{N}$ and $X, Y \subseteq \mathbb{N}^*$. We say that X is an *n-approximate subset of Y* if, for every $x \in X$, there is $y \in Y$ such that $\frac{y}{n} \leq x \leq ny$.

Theorem (DGJLLM, 2014)

Suppose that $\ell\mathrm{BD}(A) > 0$ and $k \in \mathbb{N}$. Then there is $n \in \mathbb{N}$ such that, for any $m \in \mathbb{N}$, there is a geometric progression

$$G = \{ar^i \ : \ i = 0, 1, \ldots, k-1\}$$

such that G is an n-approximate subset of A and $a, r > m$.

The theorem is false if one replaces $\ell\mathrm{BD}$ by BD and is also false if one requires genuine geometric progressions rather than approximate ones (e.g. square-free numbers).

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Furstenberg's Recurrence Theorem

Theorem (Furstenberg Recurrence)

Suppose that $T : X \to X$ is a measure preserving transformation, $\mu(A) > 0$, and $k \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that

$$\mu(A \cap T^{-n}(A) \cap T^{-2n}(A) \cap \cdots \cap T^{-(k-1)n}(A)) > 0.$$

- Furstenberg deduced Szemeredi's Theorem from his recurrence theorem via what is now referred to as the Furstenberg Correspondence Principle. Here is the nonstandard presentation.
- Fix $A \subseteq \mathbb{Z}$ with $\mathrm{BD}(A) > 0$. Fix an infinite interval I such that $\mathrm{BD}(A) = \mu_I(A^* \cap I)$.
- Apply Furstenberg's Recurrence Theorem to the measure preservation transformation $x \mapsto x + 1 (\mathrm{mod}\ I)$ on I and use transfer.

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The proof

- Take $k, N \in \mathbb{N}^*$ with $N > \mathbb{N}$ such that $\ell\mathrm{BD}(A) = \nu(A^* \cap [k, Nk])$.
- Let $E = \varphi(A)$ so $\mathfrak{m}(E) \geq \ell\mathrm{BD}(A)$.
- Take $M \in \mathbb{N}^* \setminus \mathbb{N}$, $M < V_N$. Set $x := [M]$.
- Multiplication by x is a measure preserving transformation, so Fursternberg's Recurrence Theorem gives us a geometric progression $\{cq^i \ : \ i = 1, \ldots k\}$ in E, where $q = x^l$ for some $l \in \mathbb{N}$.
- Let $r = M^l$ and take $a \in \varphi^{-1}(cq)$. Then $\varphi(ar^{i-1}) = cq^i$, so each ar^{i-1} is multiplicatively within $t_i \in \mathbb{N}$ from A^*.
- Let $n = \max(t_1, \ldots, t_k)$. So each of $a, ar, ar^2, \ldots, ar^{k-1}$ is multiplicatively within n of an element of A^*.
- Since $a, r > \mathbb{N}$, we can apply transfer.

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