LOCALLY COMPACT CONTRACTIVE LOCAL GROUPS

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Abstract. We study locally compact contractive local groups, that is, locally compact local groups with a contractive pseudo-automorphism. We prove that if such an object is locally connected, then it is locally isomorphic to a Lie group. We also prove a related structure theorem for locally compact contractive local groups which are not necessarily locally connected. These results are local analogues of theorems for locally compact contractive groups.

1. Introduction

Throughout G is a local group as defined in [1]. We let 1 be its identity, $\Lambda \subseteq G$ be the domain of its inversion map, and $\Omega \subseteq G \times G$ be the domain of its product map. All local groups in this paper are assumed to be hausdorff, and likewise, "topological group" means "hausdorff topological group".

An automorphism φ of a topological group H is said to be *contractive* if

$$\lim_{n\to\infty} \varphi^n(x) = 1 \text{ for all } x \in H,$$

and we call a topological group *contractive*[1] if it has a contractive automorphism. In [5] it is shown that locally compact connected contractive topological groups are (finite-dimensional, real) Lie groups. In response to a question by Svetlana Selivanova we prove here a local analogue of this result. To formulate this analogue precisely, we define a *contractive pseudo-automorphism* of G to be a morphism $\varphi\colon G \to G$ of local groups such that for some open neighborhood U of 1 in G the map $\phi|U : U \to G$ is injective and open, and $\lim_{n\to\infty} \varphi^n(x) = 1$ for all $x \in U$. Call G *contractive* if G has a contractive pseudo-automorphism.

Theorem 1.1. *If G is locally compact, locally connected, and contractive, then G is locally isomorphic to a contractive Lie group.*

The recent solution [1] of a local version of Hilbert's 5th problem is of no help here, and we use instead an old result due to Mal'cev [2] to the effect that local groups satisfying a certain generalized associative law embed into topological groups. In Section 2 we prove Mal'cev's theorem. In Section 3 we show that if G is contractive in a strong way, then G obeys the generalized associative law that makes Mal'cev's theorem applicable. In Section 4 we use this to derive Theorem 1.1 from the corresponding global result in [5].

[1] *Contractible* in [5], but this term has another meaning in topology.

We also prove a related structure theorem for locally compact contractive local groups that are not necessarily locally connected.

See [1] for the definition of $G|U$ for an open neighborhood U of 1 in G, and of "morphism of local groups" (also called "local group morphism" below). Recall also from [1] that two local groups are said to be locally isomorphic if they have isomorphic restrictions to open neighborhoods of their identity. Here are definitions of some auxiliary notions. Let $X \subseteq G$. We call X *symmetric* if $X \subseteq \Lambda$ and $X^{-1} = X$; in particular, G is symmetric iff $\Lambda = G$. The largest symmetric subset of X is its *symmetrization* X_s:

$$X_s := \{x \in X \cap \Lambda : \ x^{-1} \in X \cap \Lambda\} \qquad (\text{so } G_s = \Lambda \cap \Lambda^{-1}).$$

If U is an open neighborhood of 1, then so is U_s. If $\varphi : G \to G$ is a contractive pseudo-automorphism of G, then $\varphi(G_s) \subseteq G_s$, and the restriction of φ to a map $G_s \to G_s$ is a contractive pseudo-automorphism of $G|G_s$. We call G *neat* if $\Lambda = G$ and $(xy, y^{-1}) \in \Omega$ for all $(x, y) \in \Omega$. Note that $G|U$ is neat for any symmetric open neighborhood U of 1 with $U \times U \subseteq \Omega$.

2. Mal'cev's theorem

Theorem 2.1 below provides a necessary and sufficient condition for a neat local group to admit an injective local group morphism into a topological group. Because some of its byproducts are useful in the next section we repeat Mal'cev's construction [2], and include details omitted in Mal'cev's proof. Throughout we let m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$.

We call G **globalizable** if there is a topological group H and an open neighborhood U of the identity in H such that $G = H|U$. Note that if G is globalizable and symmetric, then G is neat.

Let $a_1, \dots, a_n, b \in G$. We define the notion $(a_1, \dots, a_n) \rightsquigarrow b$, by induction on n as follows:

- If $n = 0$, then $(a_1, \dots, a_n) \rightsquigarrow b$ iff $b = 1$;
- $(a_1) \rightsquigarrow b$ iff $a_1 = b$;
- If $n > 1$, then $(a_1, \dots, a_n) \rightsquigarrow b$ iff for some $i \in \{1, \dots, n-1\}$, there exist $b', b'' \in G$ such that $(a_1, \dots, a_i) \rightsquigarrow b'$, $(a_{i+1}, \dots, a_n) \rightsquigarrow b''$, $(b', b'') \in \Omega$ and $b' \cdot b'' = b$.

Informally, $(a_1, \dots, a_n) \rightsquigarrow b$ if for some way of introducing parentheses into the sequence $(a_1, \dots, a_n)$ all intermediate products are defined and the resulting product equals b. A priori, there may be distinct $b, c \in G$ such that $(a_1, \dots, a_n) \rightsquigarrow b$ and $(a_1, \dots, a_n) \rightsquigarrow c$.

We call G **globally associative** if for all $a_1, \dots, a_n, b, c \in G$ such that $(a_1, \dots, a_n) \rightsquigarrow b$ and $(a_1, \dots, a_n) \rightsquigarrow c$ we have $b = c$. If G is globally asociative, so is its restriction $G|U$ to any open neighborhood U of 1. If there is an injective local group morphism from G into a topological group, then G is globally associative. For neat G the converse holds:

Theorem 2.1. *Suppose G is neat and globally associative. Then there is an injective local group morphism $\iota\colon G \to H$ into a topological group H such that if $\phi\colon G \to L$ is any local group morphism into a topological group L, then there is a unique continuous group morphism $\tilde{\phi}\colon H \to L$ with $\tilde{\phi}\circ\iota = \phi$.*

Proof. Let $G^* := \bigcup_n G^{\times n}$ be the set of words on G. Consider a word $x = (x_1, \ldots, x_m) \in G^{\times m}$. If $(x_i, x_{i+1}) \in \Omega$, $1 \le i < m$, then we call the word

$$(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_m) \in G^{\times(m-1)}$$

a *contraction of x of type I.* If also $x_{i+1} = x_i^{-1}$, then we call

$$(x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_m) \in G^{\times(m-2)}$$

a *contraction of x of type II.* If $(a, b) \in \Omega$ and $x_i = ab$, $1 \le i \le m$ then

$$(x_1, \ldots, x_{i-1}, a, b, x_{i+1}, \ldots, x_m) \in G^{\times(m+1)}$$

is an *expansion of x of type I.* Finally, for $a \in G$ and $0 \le i \le m$ we call

$$(x_1, \ldots, x_i, a, a^{-1}, x_{i+1}, \ldots, x_m) \in G^{\times(m+2)}$$

an *expansion of x of type II.* Define an *admissible* sequence to be a finite sequence $w_1, \ldots, w_N$ of words $w_i \in G^*$ with $N \ge 1$ such that w_{i+1} is a contraction or expansion of w_i, for all i with $1 \le i < N$. This gives an equivalence relation $\sim$ on G^* by: $x \sim y$ iff there is an admissible sequence $w_1, \ldots, w_N$ such that $w_1 = x$ and $w_N = y$. Let H be the set of equivalence classes $[x]$ of elements $x = (x_1, \ldots, x_m) \in G^*$. It is easy to check that we have a binary operation and a unary operation on H given by

$$[(x_1, \ldots, x_m)] \cdot [(y_1, \ldots, y_n)] := [(x_1, \ldots, x_m, y_1, \ldots, y_n)]$$

and

$$[(x_1, \ldots, x_m)]^{-1} := [(x_m^{-1}, \ldots, x_1^{-1})].$$

Endowed with these operations, H is a group with identity element $1_H = [\emptyset]$, the equivalence class of the empty sequence. Note that also $1_H = [(1)]$.

Define $\iota\colon G \to H$ by $\iota g := [(g)]$. Clearly, ιG generates the group H. We now show that ι is injective. (This is the part asserted without proof by Mal'cev [2].) The key to doing this is the following.

Claim 1: Suppose that $x, y, z \in G^*$ and x contracts to y and y expands to z. Then one can also go from x to z by first expanding once or twice and then contracting once.

There are some obvious cases where the relevant contraction and expansion operations "commute" and can just be interchanged. (This includes the case where y is a contraction of x of type II or z is an expansion of y of type II.) So we can assume that y is a contraction of $x := (x_1, \ldots, x_m)$ of type I,

$$y = (x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_m), \quad 1 \le i < m,\ (x_i, x_{i+1}) \in \Omega,$$

and z is an expansion of y of type I of the form

$$z = (x_1, \ldots, x_{i-1}, a, b, x_{i+2}, \ldots, x_m), \qquad (a, b) \in \Omega,\ ab = x_i x_{i+1}.$$

Now G is neat, so $(ab, x_{i+1}^{-1}) \in \Omega$ and $x_i = (ab)x_{i+1}^{-1}$. Define

$$u := (x_1, \ldots, x_{i-1}, ab, x_{i+1}^{-1}, x_{i+1}, \ldots, x_n),$$
$$v := (x_1, \ldots, x_{i-1}, a, b, x_{i+1}^{-1}, x_{i+1}, \ldots, x_n).$$

Then u is an expansion of x of type I, v is an expansion of u of type I, and z is a contraction of v of type II. This proves the claim.

Define a *special* sequence to be an admissible sequence $w_1, \ldots, w_N$ such that for some $M \in \{1, \ldots, N\}$, w_{i+1} is an expansion of w_i for $1 \le i < M$, and w_{i+1} is a contraction of w_i for $M \le i < N$.

Claim 2: Let $x, y \in G^*$ and $x \sim y$. Then there is a special sequence $w_1, \ldots, w_N$ such that $w_1 = x$ and $w_N = y$.

To prove this, let $w_1, \ldots, w_n$ be any admissible sequence (typically, part of an admissible sequence connecting x to y), and suppose it is not special. Then $n \ge 3$ and we have a largest $m \in \{2, \ldots, n-1\}$ such that w_{m-1} contracts to w_m and w_m expands to w_{m+1}. Apply Claim 1 to w_{m-1}, w_m, w_{m+1} in the role of x, y, z, so w_m gets replaced by one or two words. If the resulting admissible sequence is not yet special, apply the same procedure to it. We have to show that after a finite number of such steps we end up with a special sequence. The critical case is when $m \in \{2, \ldots, n-1\}$ is such that w_i contracts to w_{i+1} for $1 \le i < m$, and w_i expands to w_{i+1} for $m \le i < n$. Then the reader can easily check that after at most

$$(n-m) + 2(n-m) + \cdots + 2^{m-1}(n-m) = (2^m - 1)(n-m)$$

such steps (applications of Claim 1) we obtain a special sequence. This concludes the proof of Claim 2.

Now suppose that $a, b \in G$ and $\iota(a) = \iota(b)$, that is, $(a) \sim (b)$. By Claim 2, we can take $x \in G^*$ such that x is obtained from (a) by a finite succession of expansions (hence $x \rightsquigarrow a$), and (b) is obtained from x by a finite succession of contractions (hence $x \rightsquigarrow b$). Hence $a = b$ by global associativity, so ι is injective. Note:

$$\iota(1) = 1_H, \ \iota(a^{-1}) = \iota(a)^{-1} \text{ for } a \in G, \ \iota(ab) = \iota(a)\iota(b) \text{ for } (a,b) \in \Omega.$$

Let $\mathcal{B}$ be the set of open neighborhoods of 1 in G, and $\iota\mathcal{B} := \{\iota U \mid U \in \mathcal{B}\}$. We verify the conditions (i)-(v) below that make $\iota\mathcal{B}$ a neighborhood base at 1_H for a (necessarily unique) group topology on H, which by convention includes here the requirement of being hausdorff.

(i) Let $U, V \in \mathcal{B}$; we need $W \in \mathcal{B}$ such that $\iota W \subseteq \iota U \cap \iota V$. Since ι is injective, we can take $W = U \cap V$.

(ii) Let $U \in \mathcal{B}$; we need $V \in \mathcal{B}$ such that $\iota V \cdot \iota V \subseteq \iota U$. Choose $V \in \mathcal{B}$ such that $V \times V \subseteq \Omega$ and $V^2 \subseteq U$. Then for $g, g' \in V$, we have

$$\iota(g) \cdot \iota(g') = \iota(gg') \in \iota U.$$

(iii) Let $U \in \mathcal{B}$; we need $V \in \mathcal{B}$ such that $(\iota V)^{-1} \subseteq \iota U$. Choose $V \in \mathcal{B}$ such that $V^{-1} \subseteq U$, for example $V = U \cap U^{-1}$. Then clearly $(\iota V)^{-1} \subseteq \iota U$.

(iv) Let $h \in H$ and $U \in \mathcal{B}$; we need $V \in \mathcal{B}$ such that $h(\iota V)h^{-1} \subseteq \iota U$. Since H is generated by ιG we can reduce to the case $h = \iota g$, $g \in G$. Choose $V \in \mathcal{B}$ such that $\{g\} \times V \subseteq \Omega$, $(gV) \times \{g^{-1}\} \subseteq \Omega$, and $(gV)g^{-1} \subseteq U$.

(v) (hausdorff requirement) $\bigcap\{\iota U \mid U \in \mathcal{B}\} = \{1_H\}$. This holds because G is hausdorff.

With H now being a topological group, ι is clearly continuous at 1. Then the local homogeneity lemma 2.16 of [1] yields that ι is continuous at each $a \in G$, and thus ι is a local group morphism.

Let L be any topological group and $\phi\colon G \to L$ a morphism of local groups.

Claim 3: Suppose $x_1, \ldots, x_m, y_1, \ldots, y_n \in G$ and $(x_1, \ldots, x_m) \sim (y_1, \ldots, y_n)$. Then $\phi(x_1)\cdots\phi(x_m) = \phi(y_1)\cdots\phi(y_n)$.

It is routine to verify the claim when $y = (y_1, \ldots, y_n)$ is a contraction or expansion of $x = (x_1, \ldots, x_m)$, and the general case then follows.

By Claim 3 we can define a group morphism $\tilde{\phi}\colon H \to L$ by

$$\tilde{\phi}([g_1, \ldots, g_n]) := \phi(g_1)\cdots\phi(g_n), \qquad (g_1, \ldots, g_n \in G),$$

so $\tilde{\phi} \circ \iota = \phi$. To check continuity of $\tilde{\phi}$, let V be an open neighborhood of the identity in L. Then $U := \phi^{-1}(V)$ is an open neighborhood of 1 in G and $\iota U \subseteq \tilde{\phi}^{-1}(V)$, so $\tilde{\phi}^{-1}(V)$ is a neighborhood of 1_H in H. Thus $\tilde{\phi}$ is continuous. $\square$

Let G be neat and globally associative. The universal property of ι, H in Theorem 2.1 determines ι, H up to unique isomorphism over G, and so, without claiming that G is globalizable, we may call H the **globalization** of G. The construction in the proof of the theorem and the local homogeneity lemma 2.16 of [1] show that $\iota : G \to H$ is not just continuous but also open. In particular ιG is open in H and ι is a homeomorphism onto ιG. Accordingly, we identify G with $\iota G \subseteq H$ via ι. Note that G generates H. The following properties of H are also evident from its construction.

Lemma 2.2.

(1) *For any symmetric open neighborhood U of 1 in G with $U \times U \subseteq \Omega$, we have $G|U = H|U$ (and so $G|U$ is globalizable).*
(2) *If G is connected, then H is connected.*
(3) *G is locally compact if and only if H is locally compact.*

Remark. Olver [4] has another variant of Mal'cev's theorem, where G is a local Lie group, G, Ω, Λ are connected, and instead of neatness, it is assumed that for any $x \in G$ and neighborhood U of 1, there are $x_1, \ldots, x_n \in U$ such that $(x_1, \ldots, x_n) \rightsquigarrow x$.

3. Contractive injective endomorphisms

In this section $\varphi : G \to G$ is an injective morphism of local groups such that $\lim_{n\to\infty} \varphi^n(x) = 1$ for all $x \in G$. (If φ is also open, then G is contractive.)

Lemma 3.1. *Suppose that $a_1, \ldots, a_n, a \in G$ and $(a_1, \ldots, a_n) \rightsquigarrow a$. Then also $(\varphi(a_1), \ldots, \varphi(a_n)) \rightsquigarrow \varphi(a)$.*

Proof. We proceed by induction on n. The conclusion of the lemma is obvious when $n = 0$ or 1. Suppose that $n > 1$. Choose $i \in \{1, \ldots, n-1\}$ and $b', b'' \in G$ with $(b', b'') \in \Omega$ such that

$$(a_1, \ldots, a_i) \rightsquigarrow b', \qquad (a_{i+1}, \ldots, a_n) \rightsquigarrow b'', \qquad b' \cdot b'' = a.$$

By the induction hypothesis we have

$$(\varphi(a_1), \ldots, \varphi(a_i)) \rightsquigarrow \varphi(b'), \qquad (\varphi(a_{i+1}), \ldots, \varphi(a_n)) \rightsquigarrow \varphi(b'').$$

Also $(\varphi(b'), \varphi(b'')) \in \Omega$ and $\varphi(b')\varphi(b'') = \varphi(b'b'') = \varphi(a)$, and therefore $(\varphi(a_1), \cdots, \varphi(a_n)) \rightsquigarrow \varphi(a)$. □

The following is taken from [1]. By recursion on n we define the relation $(a_1, \ldots, a_n) \to b$ for $a_1, \ldots, a_n, b \in G$ as follows:

- If $n = 0$, then $(a_1, \ldots, a_n) \to b$ iff $b = 1$;
- $(a_1) \to b$ iff $a_1 = b$;
- If $n > 1$, then $(a_1, \ldots, a_n) \to b$ iff for all $i \in \{1, \ldots, n-1\}$ there exist $b', b'' \in G$ such that $(a_1, \ldots, a_i) \to b'$, $(a_{i+1}, \ldots, a_n) \to b''$, $(b', b'') \in \Omega$ and $b' \cdot b'' = b$.

An easy induction on n shows that for $a_1, \ldots, a_n, b, c \in G$, if

$$(a_1, \ldots, a_n) \to b, \quad (a_1, \ldots, a_n) \rightsquigarrow c,$$

then $b = c$. By Lemma 2.5 of [1] there is for each $n > 0$ a neighborhood $\mathcal{U}_n$ of 1 such that for all $a_1, \ldots, a_n \in \mathcal{U}_n$ there is $b \in G$ with $(a_1, \cdots, a_n) \to b$.

Corollary 3.2. *G is globally associative.*

Proof. Let $a_1, \ldots, a_n, b, c \in G$ be such that

$$(a_1, \ldots, a_n) \rightsquigarrow b \text{ and } (a_1, \ldots, a_n) \rightsquigarrow c.$$

It is enough to derive $b = c$. By Lemma 3.1 we have

$$(\varphi^m(a_1), \ldots, \varphi^m(a_n)) \rightsquigarrow \varphi^m(b) \text{ and } (\varphi^m(a_1), \ldots, \varphi^m(a_n)) \rightsquigarrow \varphi^m(c),$$

for all $m > 0$. Choose m so large that $\varphi^m(a_1), \ldots, \varphi^m(a_n) \in \mathcal{U}_n$. It follows that $\varphi^m(b) = \varphi^m(c)$, and thus $b = c$. □

For the remainder of this section L denotes a topological group.

A **near-automorphism** of L is an injective, continuous, open group morphism $L \to L$. We call a near-automorphism $\tau\colon L \to L$ **contractive** if $\lim_{n\to\infty} \tau^n(x) = 1$ for all $x \in L$.

For example, $x \mapsto px : \mathbb{Z}_p \to \mathbb{Z}_p$ is a contractive near-automorphism of the compact additive group $\mathbb{Z}_p$ of p-adic integers, and is *not* an automorphism. Thus non-trivial compact groups may admit contractive near-automorphisms, but do not admit contractive automorphisms; see [5], 1.8(b).

Remark 3.3. If $\tau : L \to L$ is a contractive near-automorphism of L, then τ is a contractive pseudo-automorphism of L viewed as a local group.

Lemma 3.4. *Suppose $\tau : L \to L$ is a near-automorphism. Let L_1 be the connected component of 1 in L. Then $\tau(L_1) = L_1$ and so $\tau|L_1$ is an automorphism of L_1. If L has only finitely many connected components, then τ is an automorphism of L.*

Proof. Since τ is continuous and open, $\tau(L_1)$ is a connected open subgroup of L, and hence also closed in L, and thus $\tau(L_1) = L_1$. The set L/L_1 of cosets is the set of connected components of L. Suppose L/L_1 is finite. Since $\tau(L_1) = L_1$, the function

$$xL_1 \mapsto \tau(x)L_1 : L/L_1 \to L/L_1$$

is injective, hence bijective. It follows that τ is an automorphism of L. $\square$

Lemma 3.5. *Suppose G is neat and φ is open. Let H be the globalization of G and let $\tilde{\varphi} : H \to H$ be the unique extension of φ to an endomorphism of H. Then the map $\tilde{\varphi}$ is open, and for $D := \bigcup_n \ker(\tilde{\varphi}^n)$ we have:*

(1) *D is a discrete normal subgroup of H and $\tilde{\varphi}^{-1}(D) = D$;*
(2) *$\tilde{\varphi}$ descends to a contractive near-automorphism*

$$\varphi_D : H/D \to H/D, \quad \varphi_D(xD) := \tilde{\varphi}(x)D;$$

(3) *for any symmetric open neighborhood $U \subseteq G$ of 1 with $U \times U \subseteq \Omega$, the image $\pi(U)$ of U in H/D is open, and the map*

$$x \mapsto xD \; : \; U \to H/D$$

is an isomorphism $G|U \to (H/D)|\pi(U)$ of local groups.

Proof. The openness of φ gives the openness of $\tilde{\varphi}$. It is easy to check that D is a normal subgroup of H and $\tilde{\varphi}^{-1}(D) = D$. Each φ^n is injective, so $D \cap G = \{1\}$, which gives (1), and so $\tilde{\varphi}$ descends to a near-automorphism φ_D of H/D. To show that φ_D is contractive, let $x \in H$ be given. Then $x = x_1 \cdots x_m$, with $x_1, \ldots, x_m \in U$, so $\tilde{\varphi}^n(x) = \varphi^n(x_1) \cdots \varphi^n(x_m) \to 1$ as $n \to \infty$. Item (3) is straightforward. $\square$

4. The structure of locally compact contractive local groups

In this section H is a topological group, and H_1 is the connected component of its identity. Our aim here is to prove local analogues of the following two structure theorems for locally compact contractive groups.

Fact 4.1 ([3], (1.10) and [5], Lemma 1.4)**.** Each locally compact connected contractive group is a Lie group.

Fact 4.2 ([5], Proposition 4.2)**.** If H is locally compact and contractive, then H is isomorphic as topological group to a product $H_1 \times D$, where D is a closed, totally disconnected, normal subgroup of H, and H_1 and D are both contractive. (So H_1 is a Lie group.)

Obviously, if a local group is locally isomorphic to a Lie group, then it is locally compact and locally connected. A strong converse of this implication holds for contractive local groups: Theorem 1.1 from the Introduction, which is our local analogue of Fact 4.1. To prove this converse we need the next lemma whose proof is close to [5], Lemma 1.4, and whose purpose is to reduce to a situation where the results of the previous section are applicable.

Lemma 4.3. *Suppose G is locally compact, φ is a contractive pseudo-automorphism of G, and V is a neighborhood of 1 in G. Then there is an open symmetric neighborhood U of 1 in G such that $U \subseteq V$, $U \times U \subseteq \Omega$, $\varphi(U) \subseteq U$, $\varphi|U : U \to G$ is open and injective, and $\lim_{n\to\infty} \varphi^n(x) = 1$ for all $x \in U$.*

Proof. By restricting G as indicated at the end of the Introduction we can assume that G is symmetric. By shrinking V we may assume in addition: V is compact, symmetric, $V \times V \subseteq \Omega$, and V is contained in an open neighborhood W of 1 in G for which $\varphi|W : W \to G$ is open and injective, and $\lim_{n\to\infty} \varphi^n(x) = 1$ for all $x \in W$. For $X \subseteq G$, we set

$$\varphi^{-n}(X) := \{x \in G \mid \varphi^n(x) \in X\},$$

while $\varphi^n(X)$ has the usual meaning as a direct image; note that then for all $k \in \mathbb{Z}$ we have $\varphi(\varphi^k(X)) \subseteq \varphi^{k+1}(X)$. For $l \in \mathbb{Z}$, set $V_l := \bigcap_{k \le l} \varphi^k(V)$. We claim that then the family (V_l) has the following properties:

1. V_l is symmetric, $V_l \supseteq V_{l+1}$ and $\varphi(V_l) \subseteq V_{l+1}$;
2. $W \subseteq \bigcup_{l\in\mathbb{Z}} V_l$, and $V_l \cap V$ has nonempty interior in G for some l;
3. for every neighborhood X of 1 there exists $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$ we have $\varphi^n(V) \subseteq X$;
4. $(V_l \mid l \in \mathbb{Z})$ is a neighborhood base of 1.

Item (1) is straightforward to check. To prove (2), let $x \in W$. We can take $n_0 \in \mathbb{N}$ such that $\varphi^n(x) \in V$ for all $n \ge n_0$, so $x \in V_{-n_0}$. This proves $W \subseteq \bigcup_{l\in\mathbb{Z}} V_l$. Now each V_l is closed in G, so by Baire's theorem some $V_l \cap V$ has nonempty interior in G, which is (2).

To prove (3), let X be a neighborhood of 1. Take a compact symmetric neighborhood A of 1 with $A \subseteq V$, $VA \times A \subseteq \Omega$, and $A^2 \subseteq X$. By (1) and (2), with A in place of V, we obtain $n_0 \in \mathbb{N}$ such that

$$B := \{x \in A \mid \varphi^n(x) \in A \text{ for all } n \ge n_0\}$$

has nonempty interior in G. Take $b \in \text{interior}(B)$; so $b^{-1} \in A$. Let $x \in V$; then $(xb^{-1}, b) \in \Omega$ and $x = (xb^{-1})b$. By the local homogeneity lemma 2.16 of [1] we can take an open neighborhood $\mathcal{U} = \mathcal{U}_x$ of 1 such that

$$\{x\} \times \mathcal{U}, \{b\} \times \mathcal{U} \subseteq \Omega, \quad \{xb^{-1}\} \times b\mathcal{U} \subseteq \Omega, \quad b\mathcal{U} \subseteq B,$$

and $x\mathcal{U}$ and $b\mathcal{U}$ are open neighborhoods of x and b respectively. Since V is compact, we have $x_1, \dots, x_m \in V$ such that $V \subseteq x_1\mathcal{U}_{x_1} \cup \dots \cup x_m\mathcal{U}_{x_m}$. Choose $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$ and $\varphi^n(x_1 b^{-1}), \dots, \varphi^n(x_m b^{-1}) \in A$ for all $n \geq n_1$. Since $\varphi^n(B) \subseteq A$ for all $n \geq n_0$, and for $i = 1, \dots, m$ we have

$$x_i\mathcal{U}_{x_i} = (x_i b^{-1})b\mathcal{U}_{x_i}, \qquad b\mathcal{U}_{x_i} \subseteq B,$$

it follows that $\varphi^n(V) \subseteq A^2 \subseteq X$ for all $n \geq n_1$. This proves (3).

Applying (3) to $X = V$ gives $n_1 \in \mathbb{N}$ such that $V \subseteq V_{-n}$ for all $n \geq n_1$, and thus V_{-n} is a neighborhood of 1 for all $n \geq n_1$. Since φ is open near 1, it follows from the last part of (1) that all V_l are neighborhoods of 1. Since $V_n \subseteq \varphi^n(V)$ for all n, this yields (4) as a consequence of (3).

Thus $U :=$ interior(V_0) satisfies the conclusion of the lemma. □

Proof of Theorem 1.1. Let G be locally compact and locally connected, and let $\varphi : G \to G$ be a contractive pseudo-automorphism. Our job is to show that then G is locally isomorphic to a contractive Lie group.

By Lemma 4.3 and a remark at the end of the Introduction we can reduce to the case that G is neat and φ is open and injective, with $\lim_{n\to\infty} \varphi^n(x) = 1$ for all $x \in G$. Then by Lemma 3.5 and with H, D, ϕ_D as in that lemma, G is locally isomorphic to the topological group $L = H/D$, which has ϕ_D as a contractive near-automorphism. Since G is locally connected, G is then also locally isomorphic to L_1, the connected component of the identity of L, and L_1 is a contractive Lie group by Lemma 3.4 and Fact 4.1. □

To obtain the local analogue of Fact 4.2, we need the next lemma, which is essentially Fact 4.2 with a contractive near-automorphism of H instead of a contractive automorphism.

Lemma 4.4. *Suppose H is locally compact and τ is a contractive near-automorphism of H. Then there exists a totally disconnected, closed, normal subgroup P of H such that $(h,p) \mapsto hp\colon H_1 \times P \to H$ is an isomorphism of topological groups, $\tau(P) \subseteq P$, and $\tau|P$ is a contractive near-automorphism of P.*

Proof. By Lemma 3.4, $\tau|H_1$ is a contractive automorphism of H_1, so by Fact 4.1, H_1 is a Lie group. The remainder of the proof is just like that of Proposition 4.2 in [5]. □

Theorem 4.5. *Suppose G is locally compact and contractive. Then G is locally isomorphic to a direct product $L \times P$, where L is a contractive Lie group and P is a totally disconnected locally compact group with a contractive near-automorphism.*

Proof. As in the proof of Theorem 1.1 we reduce to the case that G is neat and we have an injective open morphism $\varphi : G \to G$ of local groups such that $\lim_{n\to\infty} \varphi^n(x) = 1$ for all $x \in G$. Let H, D and φ_D be as in Lemma 3.5. By that lemma, G is locally isomorphic to H/D, and so it remains to apply Lemma 4.4 to H/D and φ_D in the role of H and τ. □

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