Ends of Finitely Generated Groups from a Nonstandard Perspective

Isaac Goldbring

University of Illinois at Urbana Champaign

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Nonstandard Extensions

- We form a many-sorted structure by taking as sorts sets relevant to our work and we name all elements of these basic sorts.
- For example, we will have a sort for $\mathbb{N}$ and a sort for $\mathbb{R}$ and a sort for our metric space X and a sort for our group G, as well as sorts for $\mathcal{P}(\mathbb{N})$, $\mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathbb{N} \times G)$, $\mathcal{P}(\mathbb{N} \times \mathbb{N} \times G)$, etc...
- Then relevant objects in the story are elements of these sorts. For example, a sequence $a_0, \ldots, a_n$ from G is the set $\{(i, a_i) \mid 0 \leq i \leq n\} \in \mathcal{P}(\mathbb{N} \times G)$.
- Whenever Y and $\mathcal{P}(Y)$ are among our basic sets, we include the membership relation between elements of Y and elements of $\mathcal{P}(Y)$ as a basic relation in our language.

Nonstandard Extensions (cont'd)

- We now go to a big elementary extension of our many-sorted structure and we decorate the sorts in this extension with $*$, i.e. $\mathbb{N}^*$, $\mathbb{R}^*$, etc...
- Suppose X and $\mathcal{P}(X)$ are basic sorts. Since the original structure satisfies the sentence

  $$\forall y, z \in \mathcal{P}(X)(y = z \leftrightarrow \forall a \in X(a \in y \leftrightarrow a \in z)),$$

  we see that an element $y \in \mathcal{P}(X)^*$ is determined uniquely by the set of $a \in X^*$ with $a \in y$.
- We may thus replace each $y \in \mathcal{P}(X)^*$ by the set $\{a \in X^* \mid a \in^* y\}$, thus identifying $\mathcal{P}(X)^*$ with a subset of $\mathcal{P}(X^*)$. The elements of $\mathcal{P}(X)^*$ are called **internal** subsets of X^*. Subsets of X^* that are not internal are called **external**.

Saturation

We assume that our nonstandard extension is as saturated as necessary, so whenever $(A_i \mid i < \kappa)$ is a family of internal sets with the finite intersection property, it follows that their intersection is nonempty.

A useful special case of this is **overflow**, which states that if $A \subseteq \mathbb{N}^*$ is internal and contains arbitrarily large standard natural numbers, then it contains an infinite natural number.

Nonstandard Extensions of Metric Spaces

Suppose (X, d) is a metric space. Then its nonstandard extension (X^*, d) is almost a metric space; the only difference is that the"metric" takes values in $\mathbb{R}^*$. There are always some important *external* subsets of X^* to consider:

$$X_{\text{ns}} := \{x \in X^* \mid \exists x' \in X \text{ s.t. } d(x, x') \text{ is infinitesimal}\};$$

$$X_{\text{fin}} := \{x \in X^* \mid \exists x' \in X \text{ s.t. } d(x, x') \text{ is finite}\};$$

$$X_{\text{inf}} := X^* \setminus X_{\text{fin}}.$$

Proper Metric Spaces and Proper Functions

Definition

1. A metric space is **proper** if every closed ball is compact.
2. A function $f : X \to Y$ between metric spaces is **proper** if $f^{-1}(K)$ is compact for every compact $K \subseteq Y$.

Facts

1. A metric space X is proper if and only if $X_{\text{ns}} = X_{\text{fin}}$.
2. A continuous map $f : X \to Y$ between proper metric spaces is proper if and only if $f(X_{\text{inf}}) \subseteq Y_{\text{inf}}$.

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The Notions $\propto_R$ and $\propto$

From now on, (X, x_0) is a pointed proper *unbounded* metric space. For $R \in \mathbb{R}^*$, $B(x_0, R)$ denotes the set of $x \in X^*$ such that $d(x, x_0) \leq R$.

Definition

Let $x, y \in X^*$.

1. For $R \in (\mathbb{R}^{>0})^*$, say $x \propto_R y$ if there is $\alpha \in C([0,1], X)^*$ such that $\alpha(0) = x$, $\alpha(1) = y$, and $\alpha(t) \in X^* \setminus B(x_0, R)$ for all $t \in [0,1]^*$.
2. Say $x \propto y$ if there is $\alpha \in C([0,1], X)^*$ such that $\alpha(0) = x$, $\alpha(1) = y$, and $\alpha(t) \in X_{\text{inf}}$ for all $t \in [0,1]^*$.

Fact

For $x, y \in X_{\text{inf}}$, $x \propto y$ if and only if $x \propto_R y$ for some $R \in \mathbb{R}^+_{\text{inf}}$. (Of course R may depend on x and y.)

Ends(X)

Definition

- A **ray** in X is just a continuous map $r : [0, \infty) \to X$.
- Two proper rays $r_1, r_2 : [0, \infty) \to X$ are said to **converge to the same end** if for every $R \in \mathbb{R}^{>0}$, there exists $N \in \mathbb{N}$ such that $r_1[N, \infty)$ and $r_2[N, \infty)$ lie in the same path component of $X \setminus B(x_0, R)$.
- This defines an equivalence relation on the set of proper rays in X; the equivalence class of the proper ray r will be denoted by end(r).
- The set of equivalence classes will be denoted by Ends(X).

Nonstandard Characterization of Ends

Theorem

Suppose $r_1, r_2 : [0, \infty) \to X$ are proper rays. Then the following are equivalent:

1. $\text{end}(r_1) = \text{end}(r_2)$;
2. *for all* $\sigma, \tau \in \mathbb{R}^+_{\text{inf}}$, $r_1(\sigma) \propto r_2(\tau)$;
3. *for some* $\sigma, \tau \in \mathbb{R}^+_{\text{inf}}$, $r_1(\sigma) \propto r_2(\tau)$.

Proof

$1 \Rightarrow 2$:

- Suppose $\text{end}(r_1) = \text{end}(r_2)$ and fix $\sigma, \tau \in \mathbb{R}^+_{\text{inf}}$.
- Let $A := \{n \in \mathbb{N}^* \mid r_1(\sigma) \propto_n r_2(\tau)\}$.
- Then A is internal and contains $\mathbb{N}$ by our assumption and transfer, so by overflow, there is $\nu \in \mathbb{N}^* \setminus \mathbb{N}$ such that $r_1(\sigma) \propto_\nu r_2(\tau)$.

Nonstandard Characterization of Ends

Theorem

Suppose $r_1, r_2 : [0, \infty) \to X$ are proper rays. Then the following are equivalent:

1. $\mathrm{end}(r_1) = \mathrm{end}(r_2)$;
2. *for all* $\sigma, \tau \in \mathbb{R}^{+}_{\mathrm{inf}}$, $r_1(\sigma) \propto r_2(\tau)$;
3. *for some* $\sigma, \tau \in \mathbb{R}^{+}_{\mathrm{inf}}$, $r_1(\sigma) \propto r_2(\tau)$.

Proof

$2 \Rightarrow 1$:

- Suppose $\mathrm{end}(r_1) \neq \mathrm{end}(r_2)$ and fix $R \in \mathbb{R}^{>0}$ which "separates" $\mathrm{end}(r_1)$ and $\mathrm{end}(r_2)$.
- Let $B_N := \{m \in \mathbb{R}^* \mid m \geq N \text{ and } r_1(m) \not\propto_R r_2(m)\}$. Then each B_N is internal and nonempty by assumption.
- By saturation, have $\sigma \in \bigcap_N B_N$, whence $\sigma \in \mathbb{R}^{+}_{\mathrm{inf}}$ and $r_1(\sigma) \not\propto_R r_2(\sigma)$, whence $r_1(\sigma) \not\propto r_2(\sigma)$.

Geodesic Spaces

Definition

1. A **geodesic segment** in X is an isometric embedding $\alpha : [a, b] \to X$. We say that α **connects** $\alpha(a)$ and $\alpha(b)$.
2. X is a **geodesic metric space** if every two points can be connected by a geodesic segment.

Ends in Proper Geodesic Spaces

Theorem

Suppose X is a proper ***geodesic*** *space and $r_1, r_2 : [0, \infty) \to X$ are proper rays. Then the following are equivalent:*

1. $\text{end}(r_1) = \text{end}(r_2)$;
2. *For all (equiv. for some)* $\sigma, \tau \in \mathbb{R}^+_{\text{inf}}$, $r_1(\sigma) \propto r_2(\tau)$;
3. *For all (equiv. for some)* $\sigma, \tau \in \mathbb{R}^+_{\text{inf}}$ *and every* $\epsilon \in (\mathbb{R}^*)^{>0}$, *there is a hyperfinite sequence* $a_0, \ldots, a_\nu$ *in* X_{inf} *such that* $a_0 = r_1(\sigma)$, $a_\nu = r_2(\tau)$ *and* $d(a_i, a_{i+1}) < \epsilon$ *for each* $i < \nu$;
4. *For all (equiv. for some)* $\sigma, \tau \in \mathbb{R}^+_{\text{inf}}$, *there is a hyperfinite sequence* $a_0, \ldots, a_\nu$ *in* X_{inf} *such that* $a_0 = r_1(\sigma)$, $a_\nu = r_2(\tau)$ *and* $d(a_i, a_{i+1}) \in \mathbb{R}_{\text{fin}}$ *for each* $i < \nu$.

Ends as "Infinite Path Components"

- For $x \in X_{\text{inf}}$, let $[x]$ denotes the equivalence class of x with respect to the relation $\propto$.
- Let $\text{IPC}(X) := \{[x] \mid x \in X_{\text{inf}}\}$ denote the set of "infinite path components" of X.
- Fix $\sigma \in \mathbb{R}^+_{\text{inf}}$. Then what we have discussed earlier shows that we can define a map

$$\Theta : \text{Ends}(X) \to \text{IPC}(X), \quad \Theta(\text{end}(r)) := [r(\sigma)].$$

- The previous slides also show that this map is independent of the choice of σ and is injective.

Ends as "Infinite Path Components" (cont'd)

$$\Theta : \text{Ends}(X) \to \text{IPC}(X), \quad \Theta(\text{end}(r)) := [r(\sigma)].$$

Theorem

If X is a proper geodesic space, then Θ is a bijection.

Proof.

(Sketch)

- Given $x \in X_{\text{inf}}$, let $\hat{r} : [0, \sigma] \to X^*$ be an internal geodesic segment connecting x_0 and x.
- Note that $\hat{r}(t) \in X_{\text{fin}} = X_{\text{ns}}$ for $t \in \mathbb{R}^+_{\text{fin}}$, so we may define $r : [0, \infty) \to X$ by $r(t) := \text{st}(\hat{r}(t))$.
- Since $d(r(n), \hat{r}(n)) < 1$ for all $n \in \mathbb{N}$, overflow provides $\nu \in \mathbb{N}^*$ with $\nu \leq \sigma$ such that $d(r(\nu), \hat{r}(\nu)) < 1$.
- But then $r(\nu) \propto \hat{r}(\nu) \propto x$, whence $\Theta(\text{end}(r)) = [x]$.

□

Ends(X) as a topological space

- There is a natural topology on Ends(X) (X any topological space). For X a proper geodesic space, we can use Θ to translate this topology to a topology on IPC(X), which has the following description.
- For $x \in X_{\text{inf}}$ and $n > 0$, let

$$V_n([x]) := \{[x'] \mid x \propto_n x'\}.$$

- Then $(V_n([x]) \ : \ n > 0)$ forms a neighborhood base of $[x]$ in IPC(X).

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Quasi-isometries

The space of ends is a “coarse-geometric” invariant of a proper geodesic space space, i.e. it is preserved by quasi-isometries.

Definition

Suppose that (X_1, d_1) and (X_2, d_2) are metric spaces and $\lambda \geq 1$ and $\epsilon > 0$. Then a (not necessarily continuous) map $f : X_1 \to X_2$ is a (λ, ϵ)**-quasi-isometric embedding** if for all $x, y \in X_1$, we have

$$\frac{1}{\lambda} d_1(x, y) - \epsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + \epsilon.$$

If moreoever there exists $C \geq 0$ such that the C-neighborhood of $f(X_1)$ is equal to X_2, then we say that f is a (λ, ϵ)**-quasi-isometry**.

Some Standard Facts About Quasi-Isometries

Suppose $f : X_1 \to X_2$ is a quasi-isometric embedding.

- If f is a quasi-isometry, then there is a quasi-isometric embedding $g : X_2 \to X_1$ (called a *quasi-inverse for f*) and a constant $K \geq 0$ such that $d(g(f(x)), x), d(f(g(y)), y) \leq K$ for all $x \in X_1$ and $y \in X_2$.
- Suppose further that X_1 and X_2 are proper geodesic spaces. Then for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that for all $x, y \in X_1$, if $x \propto_m y$, then $f(x) \propto_n f(y)$.

Some Nonstandard Facts About Quasi-Isometries

$$\frac{1}{\lambda}d(x,y)-\epsilon \leq d(f(x),f(y)) \leq \lambda d(x,y)+\epsilon$$

Lemma

Suppose that X and Y are proper geodesic spaces and $f : X \to Y$ is a quasi-isometric embedding. Then:

1. *For all $x, x' \in X^*$, $d(x,x') \in \mathbb{R}_{\text{fin}}$ if and only if $d(f(x),f(x')) \in \mathbb{R}_{\text{fin}}$;*
2. *If $x, x' \in X_{\text{inf}}$ are such that $x \propto x'$, then $f(x) \propto f(x')$. Moreover, if f is a quasi-isometry, then for all $x, x' \in X_{\text{inf}}$, we have $x \propto x'$ if and only if $f(x) \propto f(x')$.*

Quasi-Isometries Preserve Infinite Path Components

Theorem

Suppose that X and Y are proper geodesic spaces. Then every quasi-isometry $f : X \to Y$ induces a homeomorphism $f_e : \mathrm{IPC}(X) \to \mathrm{IPC}(Y)$.

Proof.

- For $[x] \in \mathrm{IPC}(X)$, let $f_e([x]) := [f(x)]$. The previous two slides show that we can define such a map and that it is continuous.
- If $g : Y \to X$ is also a quasi-isometry, then $g_e(f_e([x])) = g_e([f(x)]) = [g(f(x))]$.
- If g were a *quasi-inverse* to f, then $d(g(f(x)), x) \in \mathbb{R}_{\text{fin}}$, so $g(f(x)) \propto x$, and $g_e \circ f_e = \mathrm{Id}_{\mathrm{IPC}(X)}$.

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Cayley Graphs

- For the rest of the talk, G is an infinite finitely generated group.
- Suppose that S is a finite generating set for G. Form a graph, called the **Cayley graph of G with respect to S**, denoted by $\mathrm{Cay}(G, S)$, whose vertices are the elements of the group and with edges connecting elements of the form g and gs for $s \in S^{\pm 1}$.
- Obtain a metric space X_S from $\mathrm{Cay}(G, S)$ by viewing each edge as an isometric copy of $[0, 1]$ and then defining the distance between two points to be the infimum of the lengths of paths connecting them.
- X_S then becomes a proper geodesic space and if S' is another finite generating set for G, one has that X_S and $X_{S'}$ are quasi-isometric.
- We may thus define $\mathrm{Ends}(G) := \mathrm{Ends}(X_S)$, $\mathrm{IPC}(G) := \mathrm{IPC}(X_S)$, where S is any finite generating set S for G.

IPC(G)

- From now on, fix a finite generating set S for G and let $X := X_S$. We take $x_0 = e$, the identity of G, as our basepoint.
- For $g \in G$, let $|g| := d(g, e)$.
- Notice that $X_{\text{fin}} \cap G^* = G$. Define $G_{\text{inf}} := G^* \setminus G$, so for all $g \in G^*$, $g \in G_{\text{inf}}$ if and only if $|g| > \mathbb{N}$.
- By transfer, for all $x \in X^*$, there is $g \in G^*$ such that $d(x, g) \leq 1$, whence every infinite path component is represented by an element of G_{inf}, i.e.

$$\text{IPC}(X) := \{[g] \mid g \in G_{\text{inf}}\}.$$

- For $g, g' \in G_{\text{inf}}$, can we find a group-theoretic condition for when $g \propto g'$?

Algebraic Interpretation of $\propto$

Lemma

For $g, g' \in G_{\text{inf}}$, we have $g \propto g'$ if and only if there is a hyperfinite sequence $s_0, \cdots, s_\nu$ in $S^{\pm 1}$ such that $g' = gs_0 \cdots s_\nu$ and $gs_0 \cdots s_i \in G_{\text{inf}}$ for each $i \leq \nu$.

Sketch.

- The backwards direction is clear.
- For the forward direction, choose a hyperfinite sequence $a_0, \cdots, a_\eta$ from X_{inf} such that $a_0 = g$, $a_\eta = g'$ and $d(a_i, a_{i+1}) < \frac{1}{2}$.
- Let (b_i) be the sequence obtained by inserting elements of G^* which are strictly in between consecutive a_i's (if there are any).
- Let (c_i) be the sequence obtained from (b_i) by deleting elements which are in $X^* \setminus G^*$ and then deleting consecutive repetitions..
- Since $d(c_i, c_{i+1}) = 1$ for all i, define $s_i := c_i^{-1} c_{i+1}$.

End Stabilizers

- Notice that for all $g \in G$, left multiplication by g preserves edges, so extends to an isometry of X.
- We thus obtain a group homomorphism

$$g \mapsto ([x] \mapsto [gx]) : G \to \text{Homeo}(\text{IPC}(X)),$$

  where $x \in G_{\text{inf}}$.
- We let H be the kernel of this homomorphism and call it the **end stabilizer** of G. (Note that it is independent of the choice of S.)
- So for all $h \in H$ and all $x \in G_{\text{inf}}$, we have $hx \in G_{\text{inf}}$ and $hx \propto x$.
- Since G_{inf} and $\propto$ are *external* notions, we cannot conclude that for all $h \in H^*$ and all $x \in G_{\text{inf}}$, we have $hx \in G_{\text{inf}}$ and $hx \propto x$. We can partially overcome this hurdle.

The Nonstandard Extension of the End Stabilizer

Lemma

Let $W \subseteq G_{\text{inf}}$ be internal. Then there is $\nu \in \mathbb{N}^* \setminus \mathbb{N}$ such that $hx \in G_{\text{inf}}$ and $hx \propto x$ for all $x \in W$ and all $h \in H^*$ with $|h| \leq \nu$.

Proof.

- For each n, let $A_n := \{\eta \in \mathbb{N}^* \mid \eta > n\}$;
- For each $h \in H$, let $B_h := \{\eta \in \mathbb{N}^* \mid hx \propto_\eta x \text{ for all } x \in W\}$;
- By saturation, there is $\gamma \in \bigcap_n A_n \cap \bigcap_h B_h$;
- Let $C := \{\eta \in \mathbb{N}^* \mid (\forall h \in H^*)(\forall x \in W)(|h| \leq \eta \rightarrow hx \propto_\gamma x\}$;
- By assumption, $\mathbb{N} \subseteq C$, so by overflow, there is $\nu \in C$ with $\nu > \mathbb{N}$.

Hopf's Theorem on the Number of Ends

Let us use our description of things to give a slick proof of Hopf's theorem on the number of ends of a group.

Theorem (Hopf)

G has either $1, 2$ or infinitely many ends.

Proof

- Suppose, towards a contradiction, that G has at least 3 ends, but only a finite number of them.
- Let $x_1, x_2 \in G_{\text{inf}}$ be such that $x_1 \not\propto x_2$.
- Fix $\nu \in \mathbb{N}^* \setminus \mathbb{N}$ such that $hx_i \propto x_i$ for $i = 1, 2$ and all $h \in H^*$ with $|h| \leq \nu$. (This is possible by the previous lemma.)
- Fix $h \in H_{\text{inf}}$ such that $|h| \leq \nu$ and $h \not\propto x_1$, $h \not\propto x_2$. This is possible because H has finite index in G, whence every element of G^* is within a fixed finite distance from an element of H^*.

Proof of Hopf's Theorem (Continued)

- Write $x_1 = s_0 \cdots s_\eta$, each $s_i \in S^{\pm 1}$ and $|s_0 \cdots s_i| = i + 1$ for all $i \leq \eta$.
- Since $hx_1 \propto x_1 \not\propto h$, we must have $hs_0 \cdots s_i \in G$ for some $i < \eta$.
- Similarly, write $x_2 = t_0 \cdots t_\zeta$, each $t_j \in S^{\pm 1}$, $|t_0 \cdots t_j| = j + 1$ for all $j \leq \zeta$ and conclude that $ht_0 \cdots t_j \in G$ for some $j < \zeta$.
- Since $h \in G_{\text{inf}}$, we must have $s_0 \cdots s_i, t_0 \cdots t_j \in G_{\text{inf}}$, whence $i, j > \mathbb{N}$.
- Since $s_i^{-1} \cdots s_0^{-1} t_0 \cdots t_j \in G$, we have $d(s_0 \cdots s_i, t_0 \cdots t_j) \in \mathbb{R}_{\text{fin}}$, and thus $s_0 \cdots s_i \propto t_0 \cdots t_j$.
- But $x_1 \propto s_0 \cdots s_i \propto t_0 \cdots t_j \propto x_2$, a contradiction.

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Realizing the Possiblities in Hopf's Theorem

All three possibilities for the number of ends of a finitely generated group are actually realized.

- If G_1 and G_2 are both infinite, finitely generated groups, then $G_1 \times G_2$ has one end.
- It is another theorem of Hopf that G has two ends if and only if G is *virtually* $\mathbb{Z}$, i.e. if and only if G has an infinite cyclic subgroup of finite index.
- An example of a group with infinitely many ends is the free product $G := \mathbb{Z} * \mathbb{Z}$. Indeed, if a and b are generators for the two copies of $\mathbb{Z}$, then for distinct $m, n > 0$ and $\nu \in \mathbb{N}^* \setminus \mathbb{N}$, we have $a^n b^\nu \not\asymp a^m b^\nu$.
- One can actually completely characterize when a group has infinitely many ends.

Stalling's Theorem

Theorem (Stallings, Bergman)

G has infinitely many ends if and only if G "splits" over a finite subgroup.

Here, "splits over a finite subgroup" means that G is either an *amalgamated free product* $A *_C B$ or an *HNN extension* $A*_C$, where C is finite.

This theorem still does not have a simple proof, even in the torsion-free case. Can one give a nonstandard proof of this result?

Groups with Multiplicative Ends

A natural question arose: Which groups have *multiplicative ends*, i.e. for which G is it true that if $g, g' \in G_{\mathrm{inf}}$ are such that $g \propto g'$, then $gg' \in G_{\mathrm{inf}}$ and $gg' \propto g$?

Theorem

Let G be an infinite, finitely generated group. Then the following are equivalent:

1. *G has multiplicative ends;*
2. *for all $g \in G_{\mathrm{inf}}$, $g \not\propto g^{-1}$;*
3. *G has two ends and equals its own end stabilizer;*
4. *G is a semidirect product of a finite group and an infinite cyclic group;*
5. *G has an infinite cyclic central subgroup.*

Relative Cayley Graphs

Definition

Suppose that G is generated by the finite set S and H is a subgroup of G such that $[G : H] = \infty$. Then the **relative Cayley graph of G with respect to S and H** has as its vertices the *right* cosets of H in G with edges connecting cosets Hg and Hgs, where $s \in S^{\pm 1}$. We make it into a metric graph just as in the case of the ordinary Cayley graph.

Little is known about the ends of relative Cayley graphs in comparison with ordinary Cayley graphs. Can one use nonstandard methods to make progress on this subject?

Relative Cayley Graphs (cont'd)

Analyzing what the notion $\propto$ means in $(G/H)_{\text{inf}}$, I can prove the following analogue of Hopf's Theorem for Relative Ends:

Theorem (D. Farley ?)

If G is a finitely generated group and H is a subgroup of G of infinite index such that H has infinite index in its normalizer $N_G(H)$, then $\mathrm{Cay}(G, H)$ has 1, 2, or infinitely many ends.

Reference

You can find a preliminary version of this material on my webpage at:

http://www.math.uiuc.edu/˜ igoldbr2/Ends.pdf