A GENTLE INTRODUCTION TO VON NEUMANN ALGEBRAS FOR MODEL THEORISTS

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Contents

These notes were part of my course on Continuous Model Theory at UIC in the fall of 2012. We spent several weeks on the model theory of von Neumann algebras and these notes served as an introduction to von Neumann algebras with the intended audience being model theorists who know a little bit of functional analysis. These notes borrow in great part from excellent notes of Vaughn Jones [10], Martino Lupini and Asger Tornquist [11], and Jacob Lurie [12].

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1. Topologies on $\mathcal{B}(H)$ and the double commutant theorem

In this section, H denotes a (complex) Hilbert space and $\mathcal{B}(H)$ denotes the set of *bounded* operators on H: recall that the linear operator $T : H \to H$ is bounded if $T(B_1(H))$ is bounded, where $B_1(H) := \{x \in H \ : \ \|x\| \leq 1\}$ is the closed unit ball in H. It is straightforward to check that the bounded operators are precisely the continuous linear operators.

For $T \in \mathcal{B}(H)$, we set $\|T\| := \sup\{\|Tx\| \ : \ \|x\| \leq 1\}$, and call $\|T\|$ the *(operator) norm* of T. It is easy to see that $\|T\|$ is the radius of the smallest closed ball centered at 0 in H containing $T(B_1(H))$. As the name indicates, $\|\cdot\|$ is a norm on $\mathcal{B}(H)$ and we call the resulting topology the *norm topology* on $\mathcal{B}(H)$.

Exercise 1.1. For $T \in \mathcal{B}(H)$ and $v \in H$, we have $\|Tv\| \leq \|T\| \cdot \|v\|$.

Recall that for each $T \in \mathcal{B}(H)$, there is a unique function $T^* : H \to H$ satisfyiing $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$. T^* is called the *adjoint* of T and plays a crucial role in operator algebras. It is easy to see that T^* is once again in $\mathcal{B}(H)$ and that $(T^*)^* = T$.

We now introduce two more topologies on $\mathcal{B}(H)$. First, the *strong (operator) topology* on $\mathcal{B}(H)$ is defined to be the topology on $\mathcal{B}(H)$ whose subbasic open neighborhoods are of the form

$$U_{T_0,v,\epsilon} := \{T \in \mathcal{B}(H) \ : \ \|Tv - T_0v\| < \epsilon\},$$

as T_0, v and ϵ range over $\mathcal{B}(H)$, H, and $\mathbb{R}^{>0}$ respectively.

Exercise 1.2. The strong topology is the weakest topology on $\mathcal{B}(H)$ that makes, for each $v \in H$, the map $T \mapsto \|Tv\| : \mathcal{B}(H) \to \mathbb{R}$ continuous.

The *weak (operator) topology* on $\mathcal{B}(H)$ is the topology on $\mathcal{B}(H)$ whose subbasic open neighborhoods are of the form

$$V_{T_0,v,w,\epsilon} := \{T \in \mathcal{B}(H) | : \ |\langle Tv, w\rangle - \langle T_0v, w\rangle| < \epsilon,$$

as T_0, (v, w), and $\epsilon > 0$ range over $\mathcal{B}(H)$, $H \times H$, and $\mathbb{R}^{>0}$ respectively.

Exercise 1.3. The weak topology is the weakest topology on $\mathcal{B}(H)$ that makes, for each $v, w \in H$, the map $T \mapsto \langle Tv, w\rangle : \mathcal{B}(H) \to \mathbb{R}$ continuous.

Let ONT, SOT, and WOT denote the operator norm, strong, and weak topologies respectively.

Lemma 1.4. WOT $\subseteq$ SOT $\subseteq$ ONT.

Proof. First suppose that $\mathcal{O} \in$ WOT. Fix $T_0 \in \mathcal{O}$. Fix $v_1, \ldots, v_n, w_1, \ldots, w_n \in H$ and $\epsilon > 0$ such that $\bigcap_{i=1}^n V_{T_0,v_i,w_i,\epsilon} \subseteq \mathcal{O}$. Without loss of generality, we may assume that $w_i \neq 0$ for each i. By Cauchy-Schwarz, we have $|\langle (T - T_0)v_i, w_i\rangle| \leq \|(T - T_0)v_i\| \cdot \|w_i\|$; it follows that

$$\bigcap_{i=1}^n U_{T_0,v_i,\frac{\epsilon}{\|w_i\|}} \subseteq \bigcap_{i=1}^n V_{T_0,v_i,w_i,\epsilon} \subseteq \mathcal{O}.$$

Thus, $\mathcal{O} \in \mathrm{SOT}$.

Similarly, suppose that $\mathcal{O} \in \mathrm{WOT}$ and $T_0 \in \mathcal{O}$. Fix $v_1, \ldots, v_n \in H$ and $\epsilon > 0$ such that $\bigcap_{i=1}^n U_{T_0,v_i,\epsilon} \subseteq \mathcal{O}$. Without loss of generality, each $v_i \neq 0$. Since $\|(T - T_0)v_i\| \leq \|T - T_0\|\|v\|$, it follows that

$$B_r(T_0) \subseteq \bigcap_{i=1}^n U_{T_0,v_i,\epsilon} \subseteq \mathcal{O},$$

where $r := \min_{1 \leq i \leq n} \frac{\epsilon}{\|v_i\|}$ and $B_r(T_0) := \{T \in \mathcal{B}(H) \ : \ \|T - T_0\| < r\}$. It follows that $\mathcal{O} \in \mathrm{ONT}$. $\square$

Why consider other topologies on $\mathcal{B}(H)$ other than ONT? Consider the following example:

Example 1.5. Let ℓ^2 be the separable Hilbert space, that is,

$$\ell^2 = \{x = (x_n) \in \mathbb{C}^{\mathbb{N}} \ : \ \|x\|_2 := \sum_n |x_n|^2 < \infty\}$$

with inner product $\langle x, y\rangle := \sum_n x_n \overline{y_n}$. Let $L \in \mathcal{B}(H)$ be the *left-shift operator*, that is, $L(x) = (0, x_0, x_1, \ldots)$. It is easy to see that $L^* = R$, the *right-shift operator* given by $R(x) := (x_1, x_2, \ldots)$. Notice that $L^i \to 0$ as $i \to \infty$ in the strong topology as, for $x \in \ell^2$, we have $\|L^i x\| = \sum_{n=i}^{\infty} |x_i|^2 \to 0$ as $i \to \infty$ since $v \in \ell^2$. On the other hand, $(L^i)^* = R^i \not\to 0$ as $i \to \infty$ as, for $x = (1, 0, 0, \ldots)$, we have $\|R^i x\| = 1$ for all i. Thus, we see that the map $T \mapsto T^* : \mathcal{B}(\ell^2) \to \mathcal{B}(\ell^2)$ is *not strongly continuous*, that is, not continuous with respect to SOT.

Contrast the previous example with the following lemma:

Lemma 1.6. *The map $T \mapsto T^* : \mathcal{B}(H) \to \mathcal{B}(H)$ is weakly continuous, that is, is continuous with respect to* WOT.

Proof. The inverse image of $U_{T_0,v,w,\epsilon}$ under the adjoint map is $U_{T_0^*,w,v,\epsilon}$. $\square$

Exercise 1.7. For any $S_0, T_0 \in \mathcal{B}(H)$, the maps

$$T \mapsto S_0 T, S \mapsto S T_0 : \mathcal{B}(H) \to \mathcal{B}(H)$$

are both weakly continuous; that is, composition in $\mathcal{B}(H)$ is *separately weakly continuous*. (Extra credit: Show that composition itself need not be jointly weakly continuous in general.)

For $B \subseteq \mathcal{B}(H)$, let $\overline{B}^{\mathrm{wk}}$ (resp. $\overline{B}^{\mathrm{st}}$) denote the closure of B with respect to WOT (resp. SOT).

We call $B \subseteq \mathcal{B}(H)$ a *subalgebra* of $\mathcal{B}(H)$ if B is closed under scalar multiplication, addition, and composition. If B is also closed under taking adjoint, we call B a $*$-*subalgebra* of $\mathcal{B}(H)$. If the identity operator I belongs to the subalgebra B, we say that B is a *unital subalgebra* of $\mathcal{B}(H)$.

Lemma 1.6 and Exercise 1.7 are enough to prove the following important result:

Proposition 1.8. *If $A \subseteq \mathcal{B}(H)$ is a $*$-subalgebra of $\mathcal{B}(H)$, then so is $\overline{A}^{\mathrm{wk}}$.*

For $X \subseteq \mathcal{B}(H)$, set $X' := \{T \in \mathcal{B}(H) \ : \ ST = TS \text{ for all } S \in X\}$. X' is called the *commutant* of X. We set $X'' := (X')'$, the *double commutant of* X.

Exercise 1.9. Suppose that $X \subseteq \mathcal{B}(H)$.

(1) X' is a weakly closed subalgebra of $\mathcal{B}(H)$.
(2) If X is closed under adjoints, then X' is a weakly closed $*$-subalgebra of $\mathcal{B}(H)$.
(3) $\overline{X}^{\mathrm{wk}} \subseteq X''$.
(4) $X' = X''' := (X')''$.

The following theorem can be considered the beginning of von Neumann algebra theory.

Theorem 1.10 (von Neumann double commutant theorem)**.** *Suppose that $A \subseteq \mathcal{B}(H)$ is a unital $*$-subalgebra of $\mathcal{B}(H)$. Then A is strongly (and hence weakly) dense in A''.*

Before we prove Theorem 1.10, let us state the following important corollaries.

Corollary 1.11. *For A as in Theorem 1.10, we have $\overline{A}^{\mathrm{st}} = \overline{A}^{\mathrm{wk}} = A''$.*

Proof. By Exercise 1.9(3) (and the definition of the topologies), we have

$$\overline{A}^{\mathrm{st}} \subseteq \overline{A}^{\mathrm{wk}} \subseteq A''.$$

By Theorem 1.10, we have $A'' \subseteq \overline{A}^{\mathrm{st}}$. □

Corollary 1.12. *For $A \subseteq \mathcal{B}(H)$ a unital $*$-subalgebra of $\mathcal{B}(H)$, the following are equivalent:*

(1) *$A = X'$ for some $X \subseteq \mathcal{B}(H)$;*
(2) *$A = A''$;*
(3) *A is weakly closed;*
(4) *A is strongly closed.*

Proof. (1) $\Rightarrow$ (2) follows from Exercise 1.9(4). (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) follows from the previous corollary. □

Definition 1.13. A unital $*$-subalgebra of $\mathcal{B}(H)$ satisfying any of the equivalent conditions of Corollary 1.12 is called a *von Neumann algebra*.

What makes von Neumann algebras such a robust notion is the equivalence of the algebraic conditions in (1) and (2) of Corollary 1.12 with the topological conditions (3) and (4).

Proof of Theorem 1.10. Fix $T_0 \in A''$; we must show that any basic strongly open neighborhood of T_0 intersects A. We first deal with the special case of a subbasic strongly open neighborhood of T; the general case of a basic strongly open neighborhood will follow from an "amplification" trick.

We thus fix $v \in H$ and $\epsilon > 0$; we aim to prove that $U_{T_0,v,\epsilon} \cap A \neq \emptyset$. Set $A \cdot v := \{Tv \ : \ T \in A\}$ and let $P : H \to H$ denote the orthogonal projection onto the closed subspace $\overline{A \cdot v}$ of H. Notice that $v \in \overline{A \cdot v}$ (since A is unital) and that $\overline{A \cdot v}$ is A-invariant, that is, $T(\overline{A \cdot v}) \subseteq \overline{A \cdot v}$ for each $T \in A$. We claim that $P \in A'$. Towards this end, first observe that if $w \in (\overline{A \cdot v})^\perp$ and $S, T \in A$, we have $\langle Sw, Tv\rangle = \langle w, S^*TV\rangle = 0$ since $S^*Tv \in \overline{A \cdot v}$ (here we used that A is a $*$-subalgebra). It follows that $(\overline{A \cdot v})^\perp$ is A-invariant. Thus, for $T \in A$ and $w \in H$, we have

$$PTw = PTPw + PT(I - P)w = PTPw = TPw;$$

since $w \in H$ was arbitrary, this shows that $PT = TP$, and consequently, $P \in A'$, as desired. Now, since $T_0 \in A''$, we have that T_0 and P commute, whence $T_0v = T_0Pv = PT_0v$, that is, $T_0v \in \overline{A \cdot v}$. Thus, there is $T \in A$ such that $\|T_0v - Tv\| < \epsilon$, that is, $T \in U_{T_0,v,\epsilon} \cap A$.

For the general case, we now fix $v_1, \ldots, v_n \in H$ and $\epsilon > 0$; we aim to prove that $U_{T_0,v_1,\ldots,v_n,\epsilon} \cap A \neq \emptyset$. The idea is to pass from H to $H^{(n)} := \bigoplus_{i=1}^n H_i$, the direct sum of n many copies of H. Now, the tuple of vectors $(v_1, \ldots, v_n)$ from H is a *single vector* in $H^{(n)}$.

Let us make a few preliminary comments about $H^{(n)}$. First, given $T_{ij} \in \mathcal{B}(H)$, $i, j = 1, \ldots, n$, we get an operator $[T_{ij}] \in \mathcal{B}(H^{(n)})$ given by "matrix multiplication," that is, for $k \in \{1, \ldots, n\}$, the k^{th} component of $[T_{ij}](h_1, \ldots, h_n)$ is $\sum_{j=1}^n T_{kj}h_j$. We leave it to the reader to check that every element of $\mathcal{B}(H^{(n)})$ is of the form $[T_{ij}]$ for a suitable choice of elements T_{ij} from $\mathcal{B}(H)$. Now, given $T \in \mathcal{B}(H)$, we consider $T^{(n)} \in \mathcal{B}(H)$ which is the "diagonal matrix" all of whose diagonal entries are T; formall, $T^{(n)} = [T_{ij}]$, where $T_{ii} = T$ and $T_{ij} = O$ for $i \neq j$.

We now return to the proof. Set $A^{(n)} := \{T^{(n)} \ : \ T \in A\} \subseteq \mathcal{B}(H^{(n)})$, a unital $*$-subalgebra of $\mathcal{B}(H^{(n)})$. Observe now that $[T_{ij}] \in (A^{(n)})'$ if and only if each $T_{ij} \in A'$, whence $T_0^{(n)} \in (A^{(n)})'$. By the first part of the proof, there is $T \in A$ such that $T^{(n)} \in U_{T_0^{(n)},(v_1,\ldots,v_n),\epsilon} \cap A^{(n)}$. It follows that $T \in U_{T_0,v_1,\ldots,v_n,\epsilon} \cap A$. □

Remark 1.14. A unital $*$-subalgebra of $\mathcal{B}(H)$ that is closed in the norm topology is called a *C^*-algebra*. We thus see that every von Neumann algebra is a C^*-algebra; the converse does not hold. We should point out that the general theory of C^*-algebras and von Neumann algebras are wildly different.

If X is any subset of $\mathcal{B}(H)$, we call the smallest von Neumann algebra containing X the *von Neumann algebra generated by* X. By Proposition 1.8, we see that the von Neumann algebra generated by X is $(X \cup X^*)''$, where $X^* := \{T^* \ : \ T \in X\}$.

2. Examples of von Neumann algebras

We now present a list of examples of von Neumann algebras. Of course, for any Hilbert space H, $\mathcal{B}(H)$ is a von Neumann algebra; in particular, for any $n \in \mathbb{N}$, $M_n(\mathbb{C})$ is a von Neumann algebra (this is isomorphic to $\mathcal{B}(H)$ when $\dim(H) = n$). We now turn to less trivial examples.

2.1. **Abelian von Neumann algebras.** A von Neumann algebra A is said to be *abelian* if $TS = ST$ for all $S, T \in A$. For example, suppose that (X, μ) is a σ-finite measure space. We set

$$L^\infty(X,\mu) := \{f : X \to \mathbb{C} \ : \ f \text{ is measurable and } \operatorname{essup} f < \infty\}$$

and

$$L^2(X,\mu) := \{f : X \to \mathbb{C} \ : \ f \text{ is measurable and } \int_X |f|^2 d\mu < \infty\}.$$

We then have an algebra embedding $L^\infty(X,\mu) \to \mathcal{B}(L^2(X,\mu)$ given by $f \mapsto m_f$, where $m_f(g) := fg$. We often identify $L^\infty(X,\mu)$ with its image under this embedding.

Exercise 2.1. $L^\infty(X,\mu)' = L^\infty(X,\mu)$. (Hint: In showing that $L^\infty(X,\mu)' \subseteq L^\infty(X,\mu)$, first assume that $\mu(X) < \infty$. In that case, $1 \in L^2(X,\mu)$; for $T \in L^\infty(X,\mu)'$, set $f := T(1)$. Show that $f \in L^\infty(X,\mu)$ and that $T(g) = fg$ for all $g \in L^\infty(X,\mu)$, which is sufficient since $L^\infty(X,\mu)$ is dense in $L^2(X,\mu)$. The general σ-finite case reduces to the case of finite measure by considering restrictions to suitable finite measure sets.)

Consequently, we see that $L^\infty(X,\mu)$ is a von Neumann algebra. Moreover, $m_f m_g = m_{fg}$, we see that it is an abelian von Neumann algebra. It is a fact that all abelian von Neumann algebras are of the form $L^\infty(X,\mu)$ for some (X,μ). It is for this reason that von Neumann algebra theory is sometimes regarded as *noncommutative measure theory*.

Example 2.2. Suppose that $X = \mathbb{S}^1$ equipped with its haar measure and suppose that $f \in L^\infty(\mathbb{S}^1)$. Let $s_n := \sum_{k=-n}^{n} c_n e^{ik\theta}$ be the n^{th} partial sum of the Fourier series for f. Since s_n converges to f in $L^2(\mathbb{S}^1)$, we have that m_{s_n} converges to m_f strongly in $\mathcal{B}(L^2(\mathbb{S}^1))$. It follows that $L^\infty(\mathbb{S}^1)$ is generated (as a von Neumann algebra) by the functions $e^{ik\theta}$ for $k \in \mathbb{Z}$.

2.2. **Group von Neumann algebras and representation theory.** Recall that $U(H)$ consists of the elements U of $\mathcal{B}(H)$ for which $U^* = U^{-1}$. It is clear that $U(H)$ is a group (under composition) and is referred to as the unitary group of H.

For the rest of this subsection, we suppose that Γ is an arbitrary countable (discrete) group. A *unitary group representation of* Γ is a group homomorphism $\alpha : \Gamma \to U(H)$. The von Neumann algebra generated by $\alpha(\Gamma)$, which is just $\alpha(\Gamma)''$ as $\alpha(\Gamma)^* = \alpha(\Gamma)$, is referred to as the *group von Neumann algebra of the representation* α.

There is a compelling reason for studying $\alpha(\Gamma)''$. A major concern in representation theory is to understand the invariant subspaces of a representation, that is, the closed subspaces H_0 of H such that $\alpha(\gamma)(H_0) \subseteq H_0$ for each $\gamma \in \Gamma$. First notice that if H_0 is an invariant subspace of α and

$P : H \to H$ is the orthogonal projection onto H_0, then $P \in \alpha(\Gamma)'$. Indeed, for $\gamma \in \Gamma$, and $x = x_1 + x_2 \in H$, with $x_1 \in H_0$ and $x_2 \in H_0^\perp$, we have

$$P\alpha(\gamma)(x) = P\alpha(\gamma)(x_1) + P\alpha(\gamma)(x_2) = \alpha(\gamma)(x_1) = \alpha(\gamma)(Px_1).$$

In the above calculation, we used that both H_0 and $H_0^\perp$ are invariant (the latter is invariant since each $\alpha(\gamma)$ is unitary). Conversely, it is easy to see that if $P : H \to H$ is an abstract projection, that is, $P^2 = P^* = P$, which lies in $\alpha(\Gamma)'$, then $P(H)$ is an invariant subspace of H.

It follows that an understanding of the invariant subspaces of α amounts to an understanding of the projections of the von Neumann algebra $\alpha(\Gamma)'$. Since the projections of a von Neumann algebra generate the von Neumann algebra (we will discuss in Subsection 3.1), it follows that understanding the invariant subspaces of α amounts to understanding $\alpha(\Gamma)'$, or somewhat equivalently, understanding $\alpha(\Gamma)''$.

There is a particular unitary representation of Γ that is of extreme importance, the so-called *left regular representation*. To define this, we first set $\ell^2(\Gamma) := \{f : \Gamma \to \mathbb{C} \ : \ \sum_{\gamma\in\Gamma} |f(\gamma)|^2 < \infty\}$. We make $\ell^2(\Gamma)$ into a Hilbert space by equipping it with the inner product $\langle f, g\rangle := \sum_{\gamma\in\Gamma} f(\gamma)\overline{g(\gamma)}$. (This is precisely the same thing as ℓ^2, the only difference being that we index our sequences by Γ rather than $\mathbb{N}$.) For $\gamma \in \Gamma$, we define $u_\gamma : \ell^2(\Gamma) \to \ell^2(\Gamma)$ by $u_\gamma(f)(\eta) := f(\gamma^{-1}\eta)$. It is easy to see that u_γ is a linear operator and that $u_\gamma^* = u_\gamma^{-1} = u_{\gamma^{-1}}$, so that each $u_\gamma \in U(\ell^2(\Gamma))$. Moreover, $u_\gamma u_\rho = u_{\gamma\rho}$, whence we see that $u : \Gamma \to U(\ell^2(\Gamma))$ given by $u(\gamma) := u_\gamma$ is a unitary representation of Γ, referred to as the *left regular representation*. (As one might imagine, if we had used the right action rather than the left action, we would then be defining the *right regular representation*.) It is easy to check that $\{u_\gamma \ : \ \gamma \in \Gamma\}$ is a linearly independent set in $\mathcal{B}(\ell^2(\Gamma))$, whence the algebra generated by the $u(\Gamma)$ is isomorphic to the group ring $\mathbb{C}\Gamma$. The von Neumann algebra associated to the left regular representation is simply called the *group von Neumann algebra corresponding to* Γ and is denoted by $L(\Gamma)$.

For $\gamma \in \Gamma$, let $\epsilon_\gamma \in \ell^2(\Gamma)$ be the standard basis vector corresponding to γ, that is, $\epsilon_\gamma(\eta) = \delta_{\gamma\eta}$. Then $\{\epsilon_\gamma \ : \ \gamma \in \Gamma\}$ form an orthonormal basis for $\ell^2(\Gamma)$ and, for $f \in \ell^2(\Gamma)$, we write $f = \sum_{\gamma\in\Gamma} f(\gamma)\epsilon_\gamma$. What does the "matrix representation" for u_γ look like with respect to the aforementioned basis for $\ell^2(\Gamma)$? Well, notice that $u_\gamma(\epsilon_\rho) = \epsilon_{\gamma\rho}$, so the (η, ρ) entry of the matrix for u_γ is 1 if $\gamma\rho = \eta$ and 0 otherwise, that is, the (η, ρ) entry of u_γ is 1 if $\eta\rho^{-1} = \gamma$ and 0 otherwise. Let us call an infinite matrix *quasidiagonal* if the (η, ρ) entry only depends on $\eta\rho^{-1}$. It follows that the matrix for u_γ is quasidiagonal. Notice also that any linear combination of quasidiagonal matrices is also quasidiagonal. Less obvious is the fact that if $T \in L(\Gamma)$, then the matrix for T is quasidiagonal. Indeed, the (η, ρ) entry of T is $\langle T\rho, \eta\rangle$, which is a limit of a net of the form $\langle T_\alpha\rho, \eta\rangle$, where each T_α is in the algebra generated by $u(\Gamma)$.

Based on the last observation of the previous paragraph, we write elements of $L(\Gamma)$ as formal sums $\sum_{\gamma\in\Gamma} c_\gamma u_\gamma$, where c_γ represents the constant value of the "diagonal" $\eta\rho^{-1} = \gamma$. (Of course, if $c_\gamma \neq 0$ for only finitely many γ, that then signifies that we are looking at an element of the group algebra.) In general, it is quite difficult to establish what sequences $(c_\gamma)_{\gamma\in\Gamma}$ are the coefficients of an element of $L(\Gamma)$, but it is useful to at least observe the following:

Lemma 2.3. *If $\sum_{\gamma\in\Gamma} c_\gamma u_\gamma$ represents an element of $L(\Gamma)$, than the function $\gamma \mapsto c_\gamma$ belongs to $\ell^2(\Gamma)$.*

Proof. Notice that $(\sum_{\gamma\in\Gamma} c_\gamma u_\gamma)(\epsilon_{\mathrm{id}})(\eta) = c_\eta$; here id denotes the identity of the group. Since $(\sum_{\gamma\in\Gamma} c_\gamma u_\gamma)(\epsilon_{\mathrm{id}}) \in \ell^2(\Gamma)$, it follows that $\eta \mapsto c_\eta$ belongs to $\ell^2(\Gamma)$. □

Here is a case we can fully analyze:

Example 2.4. Suppose that $\Gamma = \mathbb{Z}$. The map $V : \ell^2(\Gamma) \to L^2(\mathbb{S}^1)$ defined by $V(\sum_k c_k\epsilon_k) := \sum_k c_k e^{ik\theta}$ is an (isometric) isomorphism. Observe also that $(Vu_lV^{-1})(e^{ik\theta}) = e^{il\theta}e^{ik\theta}$, that is, $Vu_lV^{-1} = m_{e^{il\theta}}$ (since the $e^{ik\theta}$ form an orthonormal basis for $L^2(\mathbb{S}^1)$). By Example 2.2, we know that the operators $m_{e^{ik\theta}}$ generate the von Neumann algebra $L^\infty(\mathbb{S}^1)$, and, for $f \in L^\infty(S^1)$, we have that $V^{-1}m_fV = \sum_k c_k u_k$, where $c_k =$ the Fourier coefficients for f. It follows that $L(\Gamma)$ consists of all formal sums $\sum_k c_k u_k$, where (c_k) is the Fourier coefficients for an element of $L^\infty(\mathbb{S}^1)$.

For an arbitrary von Neumann algebra A, set

$$\mathcal{Z}(A) := \{T \in A \ : \ TS = ST \text{ for all } S \in A\},$$

the *center* of A.

Example 2.5. Suppose that $\Gamma = \mathbb{F}_n$, the free group on n generators, for $n \geq 2$. Consider the question: what is the center of $L(\Gamma)$? Note that $\sum_\gamma c_\gamma u_\gamma \in \mathcal{Z}(L(\Gamma))$ if and only if $(\sum_\gamma c_\gamma u_\gamma)u_\rho = u_\rho(\sum_\gamma c_\gamma u_\gamma)$; it is routine to check that this latter condition is equivalent to $c_{\gamma\rho\gamma^{-1}} = c_\rho$ for all $\gamma, \rho \in \Gamma$. In other words, $\sum_\gamma c_\gamma u_\gamma \in \mathcal{Z}(L(\Gamma))$ if and only if the function $\gamma \mapsto c_\gamma$ is constant on conjugacy classes. In our case, all nontrivial conjugacy classes, that is, all conjugacy classes other than $\{\mathrm{id}\}$, are infinite. Since $\gamma \mapsto c_\gamma$ belongs to $\ell^2(\Gamma)$ (by Lemma 2.3), it follows that $c_\gamma = 0$ for $\gamma \neq \mathrm{id}$. Consequently, we have that $\mathcal{Z}(L(\Gamma)) = \{cu_{\mathrm{id}} \ : \ c \in \mathbb{C}\} = \mathbb{C} \cdot I$.

The phenomenon exhibited in the previous example is important enough to merit a definition.

Definition 2.6. A von Neumann algebra A is called a *factor* if $\mathcal{Z}(A) = \mathbb{C} \cdot I$.

Observe that $\mathbb{C} \cdot I$ is always contained in $\mathcal{Z}(A)$, so being a factor means that the scalar operators are the only elements of the center. Observe also that the only property of $\mathbb{F}_n$ used to conclude that $L(\mathbb{F}_n)$ was a factor that all nontrivial conjugacy classes are infinite; we call a group with this property *ICC* (for infinite conjugacy classes). We have thus established:

Proposition 2.7. *If Γ is an ICC group, then $L(\Gamma)$ is a factor.*

Another example of a factor is $\mathcal{B}(H)$ for any Hilbert space H.

Fact 2.8. Every von Neumann algebra is a *direct integral* of factors. It is in this sense that the factors are the "prime" or "indecomposable" objects in von Neumann algebra theory and one often proves facts about arbitrary von Neumann algebras by first proving the result for factors.

We end this section with one of the most difficult open problems in von Neumann algebra theory:

Question 2.9. If $m, n \geq 2$ are distinct, is $L(\mathbb{F}_m) \cong L(\mathbb{F}_n)$?

Even though $\mathbb{F}_m$ and $\mathbb{F}_n$ are clearly not isomorphic (nor are their group rings for that matter), the group von Neumann algebra includes so many new elements (as belonging to the weak closure is, well, a weak condition) that it becomes much harder to distinguish things at the von Neumann algebra level.

2.3. **The Hyperfinite II$_1$ factor $\mathcal{R}$.** In this subsection, we introduce arguably the most important von Neumann algebra. Towards this end, it will be important to recall the *Kronecker product* of matrices. Suppose that A is an $m \times n$ matrix and B is a $p \times q$ matrix. Then the Kronecker product of A and B is the $mp \times nq$ block matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \vdots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}.$$

We now consider the chain of inclusions

$$M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C}) \hookrightarrow M_8(\mathbb{C}) \hookrightarrow \cdots,$$

where the inclusion $M_{2^n}(\mathbb{C}) \hookrightarrow M_{2^{n+1}}(\mathbb{C})$ is given by $B \mapsto B \otimes I$. We set $M := \bigcup_n M_{2^n}(\mathbb{C})$. The *hyperfinite II$_1$ factor* $\mathcal{R}$ will be the von Neumann algebra generated by M once we can view M as a set of operators on some Hilbert space.

For each n, let $\mathrm{tr}_n : M_n(\mathbb{C}) \to \mathbb{C}$ denote the *normalized trace*, namely $\mathrm{tr}_n := \frac{1}{n}\,\mathrm{tr}$, where tr is the usual trace on matrices (so $\mathrm{tr}(I) = 1$). Notice that $\mathrm{tr}_{2^n}(B) = \mathrm{tr}_{2^{n+1}}(B \otimes I)$ (this is the reason we work with normalized traces), so that we get a unique function $\mathrm{tr} : M \to \mathbb{C}$ extending the individual tr_{2^n}'s. We can now define an inner product on M by $\langle A, B\rangle := \mathrm{tr}(B^*A)$. (This is the extension of the usual *Frobenius inner product* on the space of matrices.)

Let H be the completion of M (with respect to the norm inherited from the aforementioned inner product). Then H is a Hilbert space and we can view elements of M as operators on H. Indeed, given $A \in M$, the action of left multiplication by A, $B \mapsto A \cdot B : M \to M$, extends to a bounded operator on H. (Exercise) In this way, we get an algebra embedding $M \hookrightarrow \mathcal{B}(H)$ and we let $\mathcal{R}$ be the von Neumann algebra generated by the (image) of M.

Exercise 2.10. $\mathcal{R}$ is a factor.

The adjective "II_1" in the name of $\mathcal{R}$ will be defined later in these notes. The word "hyperfinite" refers to the fact that $\mathcal{R}$ is the von Neumann algebra generated by the increasing union of a countable chain of finite-dimensional $*$-subalgebras. It is a theorem of Murray and von Neumann that $\mathcal{R}$ is the unique II_1 factor with these properties. It follows that if Γ is an infinite ICC group which is the increasing union of a countable chain of finite subgroups, then $L(\Gamma)$ is isomorphic to $\mathcal{R}$. An example of a group with this property is $S_\infty^{\mathrm{fin}}(\mathbb{N})$, which is the group of permutations of $\mathbb{N}$ with finite support.

We will come to see that $\mathcal{R}$ plays a special role in the theory of von Neumann algebras (both from the operator algebra perspective as well as the model-theoretic perspective).

3. Projections, Type Classification, and Traces

3.1. **Projections and the spectral theorem.** Recall that a bounded operator $P : H \to H$ is called a *projection* if $P^2 = P^* = P$. For example, if $P : H \to H$ is the orthogonal projection onto a closed subspace of H, then P is a projection. Conversely, if P is a projection, then P is the orthogonal projection onto the closed subspace $P(H)$ of H.

The goal of this subsection is to prove the following theorem:

Theorem 3.1. *If A is a von Neumann algebra, then A is generated by the projections in A.*

It is for this reason that the study of von Neumann algebras relies on a deep understanding of the projections in the algebra; we will discuss this in the next subsection. For now, let us discuss the proof of Theorem 3.1. A good reference for what is to follow is Conway's book [4].

Recall that an operator $N : H \to H$ is *normal* if $NN^* = N^*N$. The climax of a good undergraduate course in linear algebra is the following:

Theorem 3.2 (Spectral Theorem for Finite-Dimensional Operators)**.** *Suppose that* $\dim(H) < \infty$ *and $N : H \to H$ is a normal operator. If $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of N and $P_i : H \to H$ denotes the orthogonal projection onto the eigenspace corresponding to λ_i, then $N = \sum_{i=1}^k \lambda_i P_i$.*

Theorem 3.1 will follow from a suitable infinite-dimensional version of the Spectral Theorem. In infinite-dimensional Hilbert spaces, rather than discussing the eigenvalues of an operator, we will need to refer to its *spectrum*, and instead of decomposing a normal operator as a sum of multiples of projections, we will need to decompose it into an integral of such operators.

Definition 3.3. If X is a set, Ω is a σ-algebra of subsets of X, and H is a Hilbert space, then a *spectral measure* for (X, Ω, H) is a function $E : \Omega \to \mathcal{B}(H)$ such that:

(1) For each $C \in \Omega$, $E(C)$ is a projection;
(2) $E(\emptyset) = 0$ and $E(X) = I$;
(3) $E(C_1 \cap C_2) = E(C_1)E(C_2)$;
(4) If $\{C_n : n \in \mathbb{N}\}$ are pairwise disjoint elements of Ω, then $E(\bigcup_n C_n) = \sum_n E(C_n)$ in the strong topology.

Suppose that E is a spectral measure for (X, Ω, H). Then for $v, w \in H$, the function $E_{v,w}(C) := \langle E(C)v, w\rangle$ defines a countably additive measure on Ω. Moreover, if $f : X \to \mathbb{C}$ is a bounded, Ω-measurable function, then it is possible to define $\int f dE$, which is an element of $\mathcal{B}(H)$ satisfying the important identity $\langle (\int f dE)v, w\rangle = \int f dE_{v,w}$ for all $v, w \in H$.

For $T \in \mathcal{B}(H)$, recall that the *spectrum* of T is

$$\sigma(T) := \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is not invertible}\}.$$

Recall that $\sigma(T)$ is a nonempty, compact subset of $\mathbb{C}$.

We can now state the Spectral Theorem.

Theorem 3.4 (Spectral Theorem)**.** *Suppose that $N : H \to H$ is a normal operator. Then there is a unique spectral measure on the Borel subsets of $\sigma(N)$ such that $N = \int z dE$.*

An important by-product of the proof of the Spectral Theorem is the following:

Theorem 3.5. *Suppose that $N : H \to H$ is a normal operator and E is the spectral measure for N. For $T \in \mathcal{B}(H)$, we have $[TN = NT$ and $TN^* = N^*T]$ if and only if $TE(C) = E(C)T$ for every Borel subset C of $\sigma(N)$.*

Proof of Theorem 3.1. First, we notice that every element is a sum of two normal operators. Indeed, given $T \in \mathcal{B}(H)$, set $\Re(T) := \frac{T+T^*}{2}$ and $\Im(T) := \frac{T-T^*}{2i}$. Then $T = \Re(T) + i\Im(T)$ and $\Re(T)$ and $\Im(T)$ are normal (in fact, self-adjoint). Moreover, if $T \in A$, then so are $\Re(T)$ and $\Im(T)$, whence A is generated by its normal elements. Thus, it suffices to show that every normal element of A is in the weak closure of the algebra generated by the projections in A.

Suppose that N is a normal element of A. By Theorem 3.5, we see that the spectral projections $E(C)$ of N lie in the von Neumann algebra generated by N and thus in A. It thus suffices to show that N is in the weak closure of the algebra generated by its spectral projections. Fix $v, w \in H$. Then $\langle Nv, w\rangle = \langle (\int z dE)v, w\rangle = \int z dE_{v,w}$. Now $\int z dE_{v,w}$ is approximated by the integral of a simple function $\sum c_i \chi_{C_i}$, which is $\sum c_i E_{v,w}(C_i) = \langle \sum c_i E(C_i)v, w\rangle$, finishing the proof. $\square$

3.2. **Type classification of factors.** For the rest of these notes, we keep things simple and assume that all Hilbert spaces are assumed to have dimension $\leq \aleph_0$.

Notation: We are going to switch from denoting elements of von Neumann algebras by uppercase letters T and P and rather start using lowercase letters x and p as we are now thinking of them as elements rather than operators.

If $u \in \mathcal{B}(H)$, we say that u is a *partial isometry* if u^*u and uu^* are projections; in this case we call u^*u and uu^* the *support projections* and *range projections* respectively and we call the spaces $(u^*u)(H)$ and $(uu^*)(H)$ the *initial space* and *final space* respectively.

The next exercise helps explain the terminology.

Exercise 3.6. If $u \in \mathcal{B}(H)$ is a partial isometry with initial space H_0 and final space H_1, then $u = u'P$, where $P : H \to H$ is the orthogonal projection onto H_0 and $u' : H_0 \to H_1$ is an (isometric) isomorphism of H_0 onto H_1.

For the remainder of this subsection, M denotes a von Neumann algebra. Set $P(M)$ to be the set of projections in M. For $p, q \in P(M)$, we write $p \leq q$ to mean $p(H) \subseteq q(H)$.

Definition 3.7. For $p, q \in P(M)$ we say that p and q are *Murray-von Neumann equivalent*, denoted $p \sim q$, if there is a partial isometry $u \in M$ such that $u^*u = p$ and $uu^* = q$.

Thus p and q being Murray-von Neumann equivalent means that the spaces $p(H)$ and $q(H)$ are isomorphic *and* M knows that they are isomorphic.

We write $p \preceq q$ to mean $p \sim p'$ for some $p' \leq q$.

Exercise 3.8 (Murray-von Neumann Schröder-Bernstein)**.** If $p \preceq q$ and $q \preceq p$, then $p \sim q$.

Exercise 3.9. $\preceq$ induces a partial order on $P(M)/\sim$.

Example 3.10. If $M = \mathcal{B}(H)$, then $p \sim q$ if and only if $\dim(p) = \dim(q)$ as $\mathcal{B}(H)$ knows about all isomorphisms. Thus, $\preceq$ defines a *linear* ordering on $P(\mathcal{B}(H))/\sim$, which is order isomorphic to:

- $\{0, 1, \ldots, n\}$ if $\dim(H) = n$, or
- $\omega + 1$ if $\dim(H) = \aleph_0$.

The fact that $\preceq$ was a linear ordering on $P(\mathcal{B}(H))/\sim$ was not an accident:

Fact 3.11. M is a factor if and only if $\preceq$ is a linear order on $P(M)/\sim$.

Definition 3.12. Suppose that $p \in P(M)$. Then p is called:

- *finite* if p is not equivalent to a proper subprojection;
- *infinite* if it is not finite;
- *purely infinite* if it has no nonzero finite subprojection;

- *semifinite* if it is infinite but is the supremum of an increasing family of finite subprojections;
- *minimal* if it is nonzero but has no proper nonzero subprojection.

For any of the adjectives $\diamond$ above, we say that M is $\diamond$ if I is $\diamond$ (viewed as an element of $P(M)$).

Example 3.13. $\mathcal{B}(H)$ is finite if $\dim(H) < \infty$ and is otherwise semifinite. (Remember that we are assuming that H has dimension at most $\aleph_0$.)

Which of the above adjectives applies to $L(\Gamma)$? We will soon see the answer to that.

Definition 3.14 (Factor classification)**.** Suppose that M is a factor. We say that M is of type

- I_n if M is finite and $P(M)/\sim$ is order isomorphic to $\{0, 1, \ldots, n\}$ for $n \in \mathbb{N}$;
- I_∞ if M is infinite and $P(M)/\sim$ is order isomorphic to $\omega + 1$;
- II_1 if M is finite and $P(M)/\sim$ is order isomorphic to $[0, 1]$;
- II_∞ if M is semifinite and $P(M)/\sim$ is order isomorphic to $[0, \infty]$;
- III if M is purely infinite and $P(M) \sim$ is order isomorphic to $\{0, \infty\}$.

Of course $[0, 1]$ and $[0, \infty$ are order isomorphic; the point of writing them this way is to indicate that I is finite in the former case and infinite in the latter case. The same remark applies to $\{0, 1\}$ and $\{0, \infty\}$.

Facts 3.15.

(1) Every factor is of one of the above types.
(2) There are factors of each type.
(3) Every type I_n factor is isomorphic to $M_n(\mathbb{C})$.
(4) Every type I_∞ factor is isomorphic to $\mathcal{B}(H)$, for $\dim(H) = \aleph_0$.
(5) Classification of type II and type III factors is very difficult. For example, the isomorphism problem for either type II or type III factors is not classifiable by countable structures (in the sense of descriptive set theory).

We will be primarily concerned with type II_1 factors. In fact, there will be an excellent model-theoretic reason for doing so, as we will see later on in these notes.

3.3. **Traces.** Throughout, M denotes a von Neumann algebra.

Definition 3.16. A *(faithful, normal) trace* on M is a function $\tau : M \to \mathbb{C}$ satisfying:

- τ is a linear map;
- τ is *positive*, that is, $\tau(x^*x) \geq 0$ for each $x \in M$;
- (normality) τ is weakly continuous;
- (faithful) $\tau(x^*x) = 0$ if and only if $x = 0$;
- (trace property) $\tau(xy) = \tau(yx)$.

It often becomes convenient to normalize the trace and assume that $\tau(1) = 1$.

Example 3.17. If $\dim(H) = n$, it is not hard to see that every trace on $\mathcal{B}(H)$ is a scalar multiple of the usual trace. If $\dim(H) = \aleph_0$, then there is no trace on $\mathcal{B}(H)$. Indeed, towards a contradiction, suppose that τ was a trace on $\mathcal{B}(H)$. Let p_n denote the orthogonal projection onto the subspace spanned by the first n elements of some fixed orthogonal basis for H and let q_n denote the orthogonal projection onto the subspace spanned by the n^{th} element of the basis (so $p_n = q_1 + \cdots q_n$). By faithfulness, we see that $\tau(q_i) > 0$ for each i. Since $q_i \sim q_j$ for all i, j, we will shortly see (Lemma 3.21) that $\tau(q_i) = \tau(q_j)$ for all i, j, whence $\tau(p_n) = n\tau(q_1)$. Since p_n converges strongly (and hence weakly) to I, normality would imply that $\tau(p_n) \to \tau(I)$; but $\tau(p_n) \to \infty$, a contradiction.

Fact 3.18. A factor M is of type II_1 if and only if it is infinite-dimensional and posseses a trace, which is then necessarily unique if one assumes the trace to be normalized.

Example 3.19. Recall that the hyperfinite II_1 factor $\mathcal{R}$ was the completion of $\bigcup_n M_{2^n}(\mathbb{C})$ with respect to an appropriate inner product. Recall that the inner product stemmed from a function $\mathrm{tr} : \bigcup_n M_{2^n}(\mathbb{C}) \to \mathbb{C}$ which was the extension of the normalized trace on each $M_{2^n}(\mathbb{C})$. It is relatively straightforward to show that tr extends to a trace on $\mathcal{R}$, showing that $\mathcal{R}$ is a II_1 factor (whence the name is indeed appropriate).

Example 3.20. Let's consider $L(\Gamma)$, where Γ is an ICC group. We claim that $L(\Gamma)$ is a II_1 factor. Since $L(\Gamma)$ is certainly infinite-dimensional, it remains to find a trace on $L(\Gamma)$. Recall that we were thinking of elements of $L(\Gamma)$ as infinite pseudodiagonal matrices with respect to the standard basis on $\ell^2(\Gamma)$. If Γ were finite, then the normalized trace of such a pseudodiagonal matrix would be the complex number that appears along the actual diagonal of the matrix; thinking of the element of $L(\Gamma)$ as a formal sum, it would be the coefficient of u_{id}. This suggests we define a function $\tau : L(\Gamma) \to \mathbb{C}$ by $\tau(\sum_{\gamma\in\Gamma} c_\gamma u_\gamma) := c_{\mathrm{id}}$; alternatively, thinking of an element of $L(\Gamma)$ as an operator T on $\ell^2(\Gamma)$, we define $\tau(T) := \langle T\epsilon_{\mathrm{id}}, \epsilon_{\mathrm{id}}\rangle$. It is readily verified that τ is a trace on $L(\Gamma)$, whence we see that $L(\Gamma)$ is a II_1 factor. In particular, this answers the question from before, namely that $L(\Gamma)$ is a finite factor (in the sense of Definition 3.12).

Traces are useful in factors because they detect Murray-von Neumann equivalence:

Lemma 3.21. *If M is a factor and $\tau : M \to \mathbb{C}$ is a positive, faithful linear functional that satisfies the trace property, then for $p, q \in P(M)$, we have $p \sim q$ if and only if $\tau(p) = \tau(q)$.*

Proof. First suppose that $p \sim q$. Then there exists $u \in M$ such that $p = u^*u$ and $q = uu^*$. But then

$$\tau(p) = \tau(u^*u) = \tau(uu^*) = \tau(q).$$

Conversely, suppose that $\mathrm{tr}(p) = \mathrm{tr}(q)$. Since M is a factor, we may assume (by switching the roles of p and q if necessary) that $p \preceq q$. Then there is $p' \in P(M)$ such that $p \sim p'$ and $p' \leq q$. Note that $\mathrm{tr}(p') = \mathrm{tr}(p) = \mathrm{tr}(q)$ by the first part of the proof. Meanwhile, we have $q = p' + q(1 - p')$, whence we have $\mathrm{tr}(q) = \mathrm{tr}(p') + \mathrm{tr}(q(1 - p'))$ and consequently $\mathrm{tr}(q(1 - p')) = 0$. Since $q(1 - p')$ is a projection, we have $q(1 - p') = (q(1 - p'))^*(q(1 - p'))$. By faithfulness, we see that $q(1 - p') = 0$, that is $p' = q$, whence $p \sim q$. □

We can think of traces as a dimension-type function in factors. However, in II_1 factors, $\{\mathrm{tr}(p) \ : \ p \in P(M)\} = [0, \mathrm{tr}(1)]$, whence the dimension takes on a continuum number of values.

The following fact will be utilized repeatedly later on in these notes:

Fact 3.22. $\mathcal{R}$ embeds into *any* II_1 factor.

Proof. (Sketch) Let M be a II_1 factor; we will show how to find a copy of $M_n(\mathbb{C})$ inside of M. (An elaboration on this idea allows one to embed all of $\mathcal{R}$ into M.) First choose $p_1, \ldots, p_n \in P(M)$ of trace $\frac{1}{n}$ that sum up to I; this is possible because M is a II_1 factor. Since $p_1 \sim p_j$ for each $j \in \{1, \ldots, n\}$, there are partial isometries v_{1j} in M such that $v_{1j}^* v_{1j} = p_1$ and $v_{1j} v_{1j}^* = p_j$. For $i \in \{2, \ldots, n\}$, set $v_{ij} := v_{1i}^* v_{1j}$. It is now an exercise to see that the von Neumann alebra algebra generated by $\{v_{ij} \ : \ 1 \leq i, j \leq n\}$ is isomorphic to $M_n(\mathbb{C})$. □

4. Tracial ultrapowers and the Connes Embedding Problem

4.1. **The 2-norm.** From now on, by a *tracial von Neumann algebra* we will mean a von Neumann algebra A equipped with a trace τ. We will also assume that the trace is normalized so that $\tau(1) = 1$. We will often abuse notation and just write A for a tracial von Neumann algebra and suppress mention of the trace. (If A is a II_1 factor, then this trace is unique and so there is no loss of information in this notation.)

In the rest of these notes, we will be exclusively concerned with tracial von Neumann algebras. Suppose that (A, τ) is a tracial von Neumann algebra. We then get an inner product on A given by $\langle x, y \rangle_\tau := \tau(y^*x)$ (this is reminiscent of the inner product we considered when discussing $\mathcal{R}$) which then induces a norm on A, called the *2-norm* on A and denoted $\|\cdot\|_2$, that is, $\|x\|_2 := \sqrt{\tau(x^*x)}$.

Unlike the operator norm on A, the 2-norm is not submultiplicative, that is, $\|xy\| \leq \|x\| \cdot \|y\|$ but the corresponding fact need not hold for the 2-norm. Nevertheless, the following inequality is often useful:

Lemma 4.1. *For any $x, y \in A$, we have $\|xy\|_2 \leq \|x\| \cdot \|y\|_2$.*

Proof. We will need two standard bits of functional analysis. The first is that in any C^* algebra (and particular in a von Neumann algebra), we have $\|yy^*\| = \|y\|^2$. Secondly, for any $w \in A$, we have $w \leq \|w\| \cdot I$ (this follows from the *functional calculus*); consequently, if $x \in A$, then $wx^*x \leq \|w\|x^*x$ since x^*x is a positive element. We are now ready:

$$\|xy\|_2^2 = \tau(y^*x^*xy) = \tau(yy^*x^*x) \leq \tau(\|yy^*\|x^*x) = \|y\|^2\tau(x^*x) = \|x\|_2^2 \cdot \|y\|^2.$$

Note that the second equality follows from the trace property while the inequality follows from positivity and linearity. □

In particular, by setting $y = I$, we have that $\|x\|_2 \leq \|x\|$ for all $x \in A$.

Fact 4.2. A is *not* complete with respect to the 2-norm. However, any operator norm closed and bounded subset of A is complete with respect to the 2-norm.

We call a tracial von Neumann algebra *separable* if A is separable with respect to the metric induced by the 2-norm. For example, $\mathcal{R}$ and $L(\Gamma)$ are separable.

4.2. **Tracial ultraproducts.** Operator algebraists like taking ultraproducts/ultrapowers (almost) as much as model theorists do. Here is the definition of the tracial ultraproduct.

Definition 4.3. Suppose that $(A_n \ : \ n \in \mathbb{N})$ is a sequence of tracial von Neumann algebras and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. Set

$$\mathcal{F}((A_n)) := \{(a_n) \in \prod A_n \ : \ \sup_n \|a_n\| < \infty\},$$

and

$$\mathcal{I}((A_n)) := \{(a_n) \in \prod A_n \ : \ \lim_{\mathcal{U}} \|a_n\|_2 \to 0\}.$$

$\mathcal{F}$ and $\mathcal{I}$ are to remind us of the words *finite* and *infinitesimal.* It is straightforward to check that $\mathcal{F}((A_n))$ is an algebra and $\mathcal{I}((A_n))$ is an operator norm closed two-sided ideal. The quotient $\mathcal{F}((A_n))/\mathcal{I}((A_n))$ is referred to as the *tracial ultraproduct* of (A_n) with respect to $\mathcal{U}$ and is denoted $\prod_{\mathcal{U}} A_n$. If $A_n = A$ for each n, then we refer to the tracial ultraproduct as the *tracial ultrapower of* A and denote it by $A^{\mathcal{U}}$.

Remarks 4.4.

(1) Recall that if (r_n) is a bounded sequence of real numbers, then $\lim_{\mathcal{U}} r_n = r$ means, for every $\epsilon > 0$, we have $\{n \in \mathbb{N} \ : \ |r_n - r| < \epsilon\} \in \mathcal{U}$. It is a standard fact that the ultralimit of a bounded sequence of real numbers always exist.

(2) It is relatively straightforward to show that $\prod_{\mathcal{U}} A_n$ is once again a tracial von Neumann algebra whose trace is given by $\mathrm{tr} = \lim_{\mathcal{U}} \mathrm{tr}_n$. Less trivially, if each A_n is a factor (resp. II_1 factor), then so is $\prod_{\mathcal{U}} A_n$. This will also follow from the ability to axiomatize these concepts in (continuous) first-order logic; see Subsection 5.1.

(3) The uses of $\|\cdot\|$ in $\mathcal{F}((A_n))$ and $\|\cdot\|_2$ in $\mathcal{I}((A_n))$ is not a typo. Indeed, if one replaced $\|\cdot\|_2$ with $\|\cdot\|$ in the definition of $\mathcal{I}((A_n))$, then one would be performing the C^* ultraproduct; unless one is in a trivial situation, the C^* ultraproduct of a sequence of von Neumann algebras is *not* a von Neumann algebra again. (See [13] for a wonderful discussion of this issue.)

Remark 4.5. There are two bits of notational nuances that model theorists should be aware of. First of all, operator algebraists like to use the notation $\beta\mathbb{N}\setminus\mathbb{N}$ to denote the set of all nonprincipal ultrafilters on $\mathbb{N}$. Secondly, operator algebraists like to use the notation ω for an element of $\beta\mathbb{N}\setminus\mathbb{N}$; this clearly causes confusion for logicians and we will refrain from this practice.

We will see later that the tracial ultraproduct is precisely the model-theoretic ultraproduct in an appropriate continuous logic.

4.3. **The Connes Embedding Problem.** In 1976, Connes proved the following result:

Theorem 4.6. *Fix $\mathcal{U}\in\beta\mathbb{N}\setminus\mathbb{N}$. Then $L(\mathbb{F}_n)$ embeds into $\mathcal{R}^{\mathcal{U}}$.*

He then remarked that the previous fact "ought to be true for any separable II_1 factor." This remark has yet to be proven and the resulting problem is known as the *Connes Embedding Problem* (note the use of the word "problem" rather than "conjecture"). It is arguably the most important unsolved problem in operator algebras. It has an unbelievable number of surprising equivalent formulations (see [3]). It also has an interesting interaction with the model theory of tracial von Neumann algebras, as we will see a bit later.

Let us introduce some terminology that will be useful. Once the model-theoretic appartus is set up in the next section, we will see that, for $\mathcal{U},\mathcal{V}\in\beta\mathbb{N}\setminus\mathbb{N}$ and A a separable tracial von Neumann algebra, A embeds into $\mathcal{R}^{\mathcal{U}}$ if and only if A embeds into $\mathcal{R}^{\mathcal{V}}$. For that reason, we will say that A is *$\mathcal{R}^\omega$-embeddable* if A embeds into some (equivalently any) nonprincipal ultrapower of $\mathcal{R}$. (Yes, I know, I am acquiescing to the operator algebraist notation.) Thus, the Connes Embedding Problem (which we will abbreviate CEP) asks whether every separable II_1 factor is $\mathcal{R}^\omega$-embeddable.

Fact 4.7. Any tracial von Neumann algebra embeds into a II_1 factor. Indeed, if A is a tracial von Neumann algebra, then $A * L(\mathcal{Z})$ (free product) is a II_1 factor and $A\subseteq A * L(\mathcal{Z})$.

Thus, we may replace the word "II_1 factor" by "tracial von Neumann algebra" in the statement of the CEP.

5. Some Model Theory of tracial von Neumann algebras

In this section, we will highlight some of the most striking aspects of the model theory of tracial von Neumann algebras. We should remark that we are still at the early stages of our understanding of the situation. We will be brief with our discussion, referring the reader to the appropriate parts of the literature where these things are discussed in greater detail.

5.1. **Axiomatizability.** The first task is to make the tracial von Neumann algebras the models of a theory in an appropriate logic. The approach developed in [7] is to work in a version of continuous logic (a la [1]) that contains "domains of quantification." The metric on a structure is required to be complete with respect to each domain of quantification and the language has to specify a modulus of uniform continuity for the symbols on each domain.

For tracial von Neumann algebras, the metric is the one induced by the 2-norm. Since the 2-norm is complete with respect to operator norm closed and bounded sets, the domains of quantification D_n correspond to the ball of radius n around 0 with respect to the operator norm. We include a symbol for the 2-norm, but *not the operator norm*. Indeed, the operator norm is not uniformly continuous with respect to the metric induced by the 2-norm and could thus not be asked to be a distinguished predicate.

However, writing down a set of axioms that captures all of the structures that are the result of viewing tracial von Neumann algebras in this way is tricky business. Indeed, while asking that the structure is a tracial unital $*$-algebra is not so difficult, requiring that the domains D_n correspond to the operator norm balls is difficult considering that one is not allowed to refer to the operator norm in the axioms! Nevertheless, it is shown in [7] that one can axiomatize the class of structures that are the result of viewing tracial von Neumann algebras in the way specified above. Since the class of tracial von Neumann algebras are closed under subalgebras, the resulting axiomatization should be universal. (This was not the case in the original version of [7].) We will henceforth refer to the theory of tracial von Neumann algebras in this logic as T_{vNa}.

We should mention that verifying the correctness of the axioms for tracial von Neumann algebras is nontrivial and uses some facts from von Neumann algebra theory, including *functional calculus*, the *Kaplansky density theorem*, the *Russo-Dye theorem*, and the *GNS construction*.

One feature of the logic used to study tracial von Neumann algebras is that the corresponding ultraproduct construction, when applied to tracial von Neumann algebras, is precisely the tracial ultraproduct construction. This yields a proof of the fact that the tracial ultraproduct of a family of tracial von Neumann algebras is once again a tracial von Neumann algebra.

Since the weak closure of a union of a chain of factors is once again a factor, the class of factors should be $\forall\exists$-axiomatizable. An explicit set of $\forall\exists$-axioms is given in [7]; proving the correctness of these axioms requires the *Dixmier Averaging Theorem*. Once again, this yields as a corollary that the tracial ultraproduct of a family of factors is once again a factor.

Similarly, the class of II_1 factors is also $\forall\exists$-axiomatizable. The axiom one needs to add to the axioms for factors in order to get the axioms for II_1

factors is quite easy to write down. In fact, here it is:

$$\inf_{x\in D_1}(\|xx^* - (xx^*)^2\|_2 + |\operatorname{tr}(xx^*) - \frac{1}{\pi}|) = 0.$$

If M is a II_1 factor, then by the remark following Lemma 3.21, we know that there is a projection in M of trace $\frac{1}{\pi}$, whence this projection witnesses that M satisfies this axiom. On the other hand, suppose that M is a tracial factor that satisfies the above axiom but is not a II_1 factor. From our discussion in Section 3.3, we see that M is thus a type I_n factor for some n, whence isomorphic to $M_n(\mathbb{C})$ for some n. Since $D_1(M_n(\mathbb{C}))$ is compact, the inf axiom is actually realized rather than approximately realized, say by $a \in D_1(M_n(\mathbb{C}))$. Let $p = aa^*$; then p is a projection of trace $\frac{1}{\pi}$. However, the traces of projections in $M_n(\mathbb{C})$ are of the form $\frac{k}{n}$ for $k \in \{0, 1, \ldots, n\}$, a contradiction. (We thus see that the actual value of $\frac{1}{\pi}$ was not relevant, only that it was irrational.) Once again, we see that the tracial ultraproduct of a family of II_1 factors is once again a II_1 factor.

The fact that the class of II_1 factors is $\forall\exists$-axiomatizable has a very interesting model theoretic consequence. Recall that a model M of a theory T is an *existentially closed model of* T if for any extension $N \supseteq M$ that is also a model of T and any formula $\varphi(x)$ with parameters from M, we have $(\inf_x \varphi(x))^M = (\inf_x \varphi(x))^N$. Since any model of T_{vNa} extends to a II_1 factor (Lemma 4.7) and the theory of II_1 factors is $\forall\exists$-axiomatizable, we see that

Proposition 5.1. *Any existentially closed tracial von Neumann algebra is a II_1 factor.*

This proposition was the model theoretic reason for appreciating II_1 factors alluded to at the end of Subsection 3.2.

We should mention that we only know of one concrete example of a separable existentially closed II_1 factor, namely $\mathcal{R}$, although we know that there are continuum many separable existentially closed II_1 factors; see [5].

While we are on the topic of axiomatizability, let us discuss one interesting side note. In order to show that $\mathcal{R}$ and $L(\mathbb{F}_2)$ are not isomorphic, Murray and von Neumann isolated a property of II_1 factors that $\mathcal{R}$ has and that $L(\mathbb{F}_2)$ does not. The property, called property (Γ), says that, for every finite tuple $\bar{x}$ and every $\epsilon > 0$, there is a trace 0 unitary u such that $[x_i, u] := x_i u - u x_i$ has 2-norm less than ϵ. It turns out that property (Γ) is axiomatizable in continuous first-order logic. Indeed, for each $n \geq 1$, let σ_n be the sentence

$$\sup_{\bar{x}} \inf_y (\|y^* y - I\|_2 + |\tau(y)| + \sum_{i=1}^n \|[x_i, y]\|_2),$$

where $\bar{x}$ is an n-tuple. Notice that $\sigma_n^M = 0$ does not at first glance guarantee that there exists a trace 0 unitary that almost commutes with each x_i, but that there is an "almost" unitary of small trace that almost commutes with each x_i (as inf's are not necessarily realized in an arbitrary structure); however, a standard functional calculus trick allows one to move that near witness to an actual trace 0 unitary with the desired property. It follows that a II_1 factor satisfies property (Γ) if and only if it makes each σ_n equal to 0. As a consequence, we see that $\mathcal{R} \not\equiv L(\mathbb{F}_2)$.

The following weaker version of Question 2.9 is still open:

Question 5.2. If $m, n \geq 2$ are distinct, is $L(\mathbb{F}_m) \equiv L(\mathbb{F}_n)$?

5.2. **The model theoretic version of CEP.** Suppose that A is an $\mathcal{R}^\omega$-embeddable tracial von Neumann algebra. Then, by standard model theory, we have that $A \models \mathrm{Th}_\forall(\mathcal{R})$. Conversely, suppose that $A \models \mathrm{Th}_\forall(\mathcal{R})$ is separable. Let $M \models \mathrm{Th}(\mathcal{R})$ be separable such that A embeds into M; we may choose M separable by Downward Löwenheim-Skolem. Then since $\mathcal{R}^\mathcal{U}$ is an $\aleph_1$-saturated model of $\mathrm{Th}(\mathcal{R})$, we have that M embeds into $\mathcal{R}^\mathcal{U}$, whence A is $\mathcal{R}^\omega$-embeddable. In other words:

Lemma 5.3. *If $A \models T_{\mathrm{vNa}}$ is separable, then A is $\mathcal{R}^\omega$-embeddable if and only if $A \models \mathrm{Th}_\forall(\mathcal{R})$.*

Since T_{vNa} is universally axiomatizable, we see that:

Corollary 5.4. *CEP is equivalent to the statement $T_{\mathrm{vNa}} = \mathrm{Th}_\forall(\mathcal{R})$.*

We should mention that, using "abstract model-theoretic nonsense," Hart, Farah, and Sherman proved in [8] that there exists a separable II_1 factor $\mathcal{S}$ such that $T_{\mathrm{vNa}} = \mathrm{Th}_\forall(\mathcal{S})$; they call such an $\mathcal{S}$ *locally universal*. In fact, once there exists one locally universal II_1 factor, then there exists many locally universal II_1 factors, e.g. if $\mathcal{S}$ is locally universal, so is $\mathcal{S} * A$ for any $A \models T_{\mathrm{vNa}}$. CEP asks whether or not $\mathcal{R}$ is locally universal.

5.3. **Model companions and connection to CEP.** Based on work of Nate Brown [2], the following appears in [9]:

Theorem 5.5. $\mathrm{Th}(\mathcal{R})$ *does not have quantifier elimination.*

We also noticed that the proof of the preceding theorem applied to a wider class of von Neumann algebras. First, we need a definition. Call a II_1 factor A *McDuff* if $A \otimes \mathcal{R} \cong \mathcal{R}$. (We have not defined the tensor product of von Neumann algebras, but it works as one might guess; see [10] for more details.) For example, $\mathcal{R}$ is McDuff. Indeed, a fancier way of describing our construction of $\mathcal{R}$ was that $\mathcal{R} = \bigotimes_{n=1}^\infty M_2(\mathbb{C})$; it seems quite plausible (and is in fact the truth) that the tensor product of two infinite tensor powers of $M_2(\mathbb{C})$ would be isomorphic to one such infinite tensor power. More generally, if A is a II_1 factor, then A embeds into a McDuff II_1 factor, namely $A \otimes \mathcal{R}$.

Theorem 5.6. *If $\mathcal{S}$ is locally universal and McDuff, then* $\mathrm{Th}(\mathcal{S})$ *does not have quantifier elimination.*

In [8], it is shown that, for separable II_1 factors, being McDuff is an $\forall\exists$-axiomatizable property. Since every II_1 factor embeds into a McDuff II_1 factor (by tensoring with $\mathcal{R}$), it follows that every existentially closed II_1

factor is McDuff. (On a side note, if $\mathcal{S}$ is locally universal, so are $S \otimes \mathcal{R}$ and $S * \mathcal{R}$; since the first is McDuff and the second is not, we see that not all locally universal II_1 factors are elementarily equivalent.)

The previous observation, combined with the fact that T_{vNa} has the amalgamation property, yields the following negative result:

Theorem 5.7 (G., Hart, Sinclair [9])**.** *T_{vNa} does not have a model companion.*

Proof. Suppose that $\mathrm{Th}(\mathcal{S})$ is the model companion of T_{vNa}, where $\mathcal{S}$ is separable. Since $\mathrm{Th}(\mathcal{S})$ is model complete, it follows that $\mathcal{S}$ is existentially closed, whence a McDuff II_1 factor. Moreover, since $T_{\mathrm{vNa}} = \mathrm{Th}_\forall(\mathcal{S})$, we see that $\mathcal{S}$ is locally universal. On the other hand, since T_{vNa} has the amalgamation property, it follows that the model companion has quantifier elimination, a contradiction. $\square$

Instead of asking for a model companion of T_{vNa}, one might ask for a model complete theory of II_1 factors. This question has an interesting connection to CEP. First, two preparatory results.

Fact 5.8. Every embedding $\mathcal{R} \to \mathcal{R}^{\mathcal{U}}$ is unitarily conjugate to the diagonal embedding. In particular, every embedding $\mathcal{R} \to \mathcal{R}^{\mathcal{U}}$ is elementary.

Exercise 5.9. Use Fact 5.8 to prove that $\mathcal{R}$ is the prime model of its theory.

Lemma 5.10 ([9])**.** *If M is an $\mathcal{R}^\omega$-embeddable II_1 factor such that* $\mathrm{Th}(M)$ *is $\forall\exists$-axiomatizable (in particular, if* $\mathrm{Th}(M)$ *is model-complete), then $M \equiv \mathcal{R}$.*

Proof. (Sketch) This is a "sandwiching chain" argument. To begin the argument, one embeds $\mathcal{R}$ into M (Fact 3.22) and M into $\mathcal{R}^{\mathcal{U}}$ and uses the fact that the composition of these embeddings is elementary by Fact 5.8. One continues the chain by taking the ultrapowers of the aforementioned maps. $\square$

Thus, the only possible $\forall\exists$-axiomatizable complete theory of II_1 factors is $\mathrm{Th}(\mathcal{R})$. It is still currently unknown whether or not $\mathrm{Th}(\mathcal{R})$ is $\forall\exists$-axiomatizable.

Theorem 5.11. *If CEP has a positive solution, then there is no model complete theory of II_1 factors.*

Proof. Suppose that $\mathrm{Th}(M)$ is model complete, where M is a II_1 factor. By CEP, M is $\mathcal{R}^\omega$-embeddable, whence, by Lemma 5.10, $\mathrm{Th}(M) = \mathrm{Th}(\mathcal{R})$. By CEP again, we see that $\mathrm{Th}(\mathcal{R})$ is the model companion of T_{vNa}, contradicting Theorem 5.7. $\square$

A more careful analysis led to the following sharpening of the Theorem 5.11:

Theorem 5.12 (Farah, G., Hart [5])**.** *If* $\mathrm{Th}_\forall(\mathcal{R})$ *has the amalgamation property, then there is no model complete theory of II_1 factors.*

The previous result is indeed sharper than Theorem 5.11 as the fact that T_{vNa} has the amalgamation property shows that CEP implies that $\mathrm{Th}_{\forall}(\mathcal{R})$ has the amalgamation property. In fact, in [5], we exhibit a concrete $\mathcal{R}^{\omega}$-embeddable tracial von Neumann algebra such that the ability to amalgamate over it while staying $\mathcal{R}^{\omega}$-embeddable yields the fact that there is no model complete theory of II_1 factors.

5.4. **Stability.** A major impetus for understanding the model theory of tracial von Neumann algebras stemmed from many questions of the form: "How canonical are tracial ultraproducts and ultrapowers" For example, if M is a separable tracial von Neumann algebra and $\mathcal{U}, \mathcal{V} \in \beta\mathbb{N} \setminus \mathbb{N}$, is $M^{\mathcal{U}} \cong M^{\mathcal{V}}$? (There were many variants of this questions asked by many different people, but let us focus on this particular question.) Now, if the continuum hypothesis holds, then $M^{\mathcal{U}}$ and $M^{\mathcal{V}}$ are saturated elementarily equivalent structures of the same uncountable density character. Thus, a familiar back and forth argument shows that they are isomorphic. Thus, the question is only interesting if we assume the negation of the continuum hypothesis.

The following theorem (in its classical form) is model-theoretic folklore, but a proof can be found in [7]:

Theorem 5.13. *Suppose that the continuum hypothesis fails and M is a separable structure in a countable language. Then the following are equivalent:*

(1) *For every $\mathcal{U}, \mathcal{V} \in \beta\mathbb{N} \setminus \mathbb{N}$, we have $M^{\mathcal{U}} \cong M^{\mathcal{V}}$.*
(2) $\mathrm{Th}(M)$ *is stable.*

A word about the proof: the direction (1) $\Rightarrow$ (2) proceeds by showing that a witness to the order property allows one to encode the partial order on increasing sequences of natural numbers (modulo almost everywhere agreement) by almost everywhere domination into the structure and then use the fact that there are nonprincipal ultrafilters whose associated partial orders have distinct coinitalities (a result due independently to Dow and Shelah). The other direction proceeds by showing that stability still allows one to conclude that $M^{\mathcal{U}}$ is saturated: one realizes a type over a large set by finding Morley sequences for its restriction to a countable set of size continuum in $M^{\mathcal{U}}$ and then using some forking calculus and statonarity of types to find an actual realization of the original type.

Suppose that M is a II_1 factor. By Fact 3.22, one can find arbitrarily large matrix algebras in M. A straightforward computation with matrices allows one to find an order in M, whence II_1 factors are unstable and hence have nonisomorphic ultrapowers (assuming the negation of the continuum hypothesis). Similar reasoning allows one to show that there are nonisomorphic ultraproducts of matrix algebras. (Details for these two facts can be found in [6].)

Which tracial von Neumann algebras are stable? That question is also answered in [6]:

Theorem 5.14 (Farah, Hart, Sherman)**.** *If $M \models T_{\mathrm{vNa}}$, then M is stable if and only if it is of type I.*

A word is in order about the statement of Theorem 5.14. We only defined the type classification for factors, but there is also a type classification for arbitrary von Neumann algebras. We will not explain this in detail, but only say as much as is needed to understand the previous result. A tracial von Neumann algebra M can be written $M = M_{II_1} \oplus M_{I_2} \oplus M_{I_3} \oplus \cdots$, where the subscript tells us the type of the algebra. A type I_n A algebra is of the form $M_n(\mathbb{C}) \otimes \mathcal{Z}(A)$ and M as above is said to be of type I if its type II_1 component is 0.

A word about the proof of Theorem 5.14. If M is not of type I_1, then by our earlier discussion, M is unstable. It remains to show that type I algebras are stable; equivalently, we show that all ultrapowers of type I algebras are isomorphic. It can be shown that ultrapowers and direct sums "commute", so it suffices to show that ultrapowers of type I_n algebras are isomorphic. It can be shown that $(M_n(\mathbb{C}) \otimes \mathcal{Z}(A))^{\mathcal{U}} \cong M_n(\mathbb{C})^{\mathcal{U}} \otimes \mathcal{Z}(A)^{\mathcal{U}}$. Since $(M_n(\mathbb{C}))^{\mathcal{U}} \cong M_n(\mathbb{C})$ (compact things are isomorphic to their ultrapowers), it suffices to show that the ultrapowers of abelian von Neumann algebras are all isomorphic. There are many ways to explain this, but perhaps the easiest is to recall that there is an equivalence of categories between the category of abelian von Neumann algebras and the category of atomless probability algebras, respecting the ultraproduct construction; the latter theory is known to be stable (see [1]), finishing the proof of the thoerem.

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