A Short Geometry

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1 Preface

These are notes intended to illustrate the Mathematical practice of reasoning and to connect a rigorous account of synthetic Euclidean Geometry with the both the geometry content and mathematical practices of CCSS. They accompany 5 six hour sessions of a geometry workshop conducted for the Chicago Teacher Transformation Initiative.

The material is motivated by the problem: Prove that a construction taken off the internet for dividing a line segment into n equal pieces actually works. The argument uses most of the important ideas of a Geometry I class. That is, we will develop constructions, properties of parallel lines and quadrilaterals.

As a second goal we want to show Euclid VI.2: Proposition 2. If a straight line is drawn parallel to one of the sides of a triangle, then it cuts the sides of the triangle proportionally; and, if the sides of the triangle are cut proportionally, then the line joining the points of section is parallel to the remaining side of the triangle. Notoriously, this proposition seems to depend on the theory of limits and irrational numbers. However, we prove the result in synthetic geometry by defining numbers as congruence classes of segments. This approach avoids reference to limits and yields a rigorous proof in terms of concepts accessible to high school geometry students. Specifically, the crucial distinction between the development here and the normal high school geometry class is that in our treatment the arithmetic and *completeness* of the real numbers is not taken as part of the (not fully stated) axiomatic system. Rather, we rely solely upon geometrical axioms, and we prove that there is a field structure with the lengths (equivalence classes of congruent segments) of line segments as the elements. This provides a coherent "ground up" explanation for results like Euclid VI.2, without introducing limits or reductions to a concept of the real field that the students don't actually have. One of the goals is to understand how these decisions about the axiomatic foundations of the course and the manner in which they are (or are not) consistently pursued, have real pedagogical consequences. The CCSSM are agnostic about these issues. But their goal of coherence is not satisfied without making a clear choice and following it throughout the entire mathematical development.

We have indexed the common core standards [11] with the material here. Note that this implies a close connection with the CPS Content Framework [1]. In particular, the Content Framework big idea assessment: Congruence, Proof, and Assessment is addressed in the early sections. The activities referred to below are at `http://homepages.math.uic.edu/~jbaldwin/CTTIgeometry/ctti` along with slides, notes, and references. These notes build from the bottom up. The workshop frequently varied the order to motivate future definitions and results.

2 Introduction

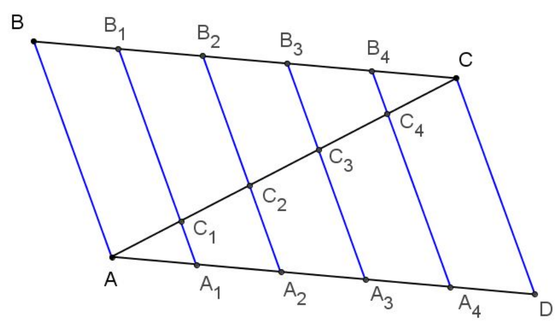
2.1 Exercise. Each group choose a number between 3 and 10. After the number is chosen, the group will be asked to fold the string to divide it into as many equal pieces as the number they chose. Other physical models will be used. Activity - Divind a line into n equal pieces.

References of the form **CCSS G-C0-12** are to the Common Core State Standards for Mathematics [11]

{linediv}

2.2 Exercise. CCSS G-C0-12 Here is a procedure to divide a line into n equal segments.

1. Given a line segment AC.
2. Draw a line through A different from AC and lay off sequentially n equal segments on that line, with end points $A_1, A_2, \ldots$. Call the last point D.
3. Construct B on the opposite side of AC from D so that $AB \cong BD$ and $CB \cong AD$.
4. Lay off sequentially n equal segments on that line, with end points $B_1, B_2, \ldots$.
5. Draw lines A_iB_i.
6. The point E_i where E_i is the intersection of A_iB_i with AB are the required points dividing AC into n equal segments.



2.3 Exercise. Show this construction used only Euclid's first 3 axioms, listed in Axiom 3.4 and Axiom 3.8 below.

These notes are a modern version of Euclid's axiomatization of geometry [6] and many of the propositions of Book 1 of Euclid. The treatment of 'betweenness' is deliberately kept semiformal but the superposition principal is made explicit by the axiom SSS. After a quick tour of the 'basics' we will prove the construction dividing a line into equal parts works. Then we will deal with the problem when the given line segments are not equal.

3 The basics of geometry

{euclid}

3.1 Common Notions

These are the common notions or *axioms* of Euclid. They apply equally well to geometry or numbers. Following modern usage we won't distinguish 'axiom' and 'postulate'.

Common notion 1. Things which equal the same thing also equal one another.
Common notion 2. If equals are added to equals, then the wholes are equal.
Common notion 3. If equals are subtracted from equals, then the remainders are equal.
Common notion 4. Things which coincide with one another equal one another.
Common notion 5. The whole is greater than the part.

3.1 Remark (Common Notion 1). Euclid used 'equal' in a number of ways: to describe congruence of segments and figures, to describe that figures had the same measure (length, area, volume). The only *numbers* for Euclid are the positive integers. He did however discuss the comparison of what we now interpret as lengths and we introduce them as 'numbers' in Section 5.

Thus, we regard the common notions as properties that describe congruence (between segments and between angles). We add that each object is equal to itself (reflexivity) and that one thing is equal to another then the second is equal to the first (symmetry).

3.2 Remark (Common Notion 4). We will develop various properties of transformations in the semiformal way common to high school geometry. But the congruence postulates are ways to make the properties of the group of rigid motions of the plane precise. In this sense two figures are congruent if and only there is a rigid motion taking one to the other. Common Notion 4 then says that the identity transformation exists and Common notion 1 says the composition of two rigid motions is a rigid motion.

3.2 Book 1

{bk1}

Activity 3.3. Standard G-C0 1. Know precise definitions[1] of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

Why is distance along a circular arc given as an undefined notion? Can we define the length (congruence) of a circular arc in terms of the length (congruence of line segments)? Remember SSS. Why is the length of the chord a less good measure than the length of the arc?

Note that we can define which arc lengths are congruent. But in general the length of an arc may not be the length of a straight line segment in the geometry. (Take the plane over the real algebraic numbers.)

In this workshop we take not distance along a line or along a circular arc as basic but: two segments are congruent or two arcs are congruent.

The Construction, Proof, and transformations activity (proofbackgr.pdf) was designed for background discussion before beginning the formal work.

We begin with a modern rendition of Euclid's axioms along with some additional axioms to fill some gaps.

{circexist}

3.4 Axiom (Euclid's first 3 axioms in modern language).

1. **Axiom 1** Given any two points there is a (unique) line segment connecting them.
2. **Axiom 2** Any line segment can be extended indefinitely (in either direction).
3. **Axiom 3** Given a point and any segment there is a circle with that point as center whose radius is the same length as the segment.

[1] Activity G-C01: definition.pdf

3.5 Exercise. CCSS G-CO.13 Prove Proposition 1 of Euclid: To construct an equilateral triangle on a given finite straight line.

3.6 Remark (Reading a diagram). There was a tradition that carried on from pre-Euclidean time until late in the 19th century that the diagram carried certain information that was part of the proof. We will continue that tradition here.

What diagrams mean.

Inexact properties can be read off from the diagram: slightly moving the elements of the diagram does not alter the property.

Intersections, betweenness and side of a line, inclusion of a segment in another are represented correctly.

What diagrams don't mean

Anything about distance, congruence, size of angle (right angle!) may be deceptive.

You can't read off a point is on a line but you can read off that two lines intersect in a point and then name that point and use the fact that it is on each line.

This is not as formal as the rest of the argument but we will be careful when needed. The difficulty is that spelling out axiomatically the concepts 'between' and 'side of a line' is difficult and introduces complexities [9] that don't seem appropriate for K-12 instruction. The usual solution of introducing algebraic axioms (due to Birkhoff) for the real numbers into the foundations of geometry produces a system in which 40 years of experience have shown students cannot learn to prove.

3.7 Remark. Late 19th century mathematicians banished the diagram from formal mathematics. Recent research has clarified and formalized the ways in which diagrams played an essential role in mathematical proof for 2000 years. A seminal work is [12].

Euclid does not explicitly mention that a pair of circles or a circle and a line actually intersect. The following additions to Axiom 3 assert that either two circles or a circle and a line intersect. But note that this follows from the 'proper' reading of diagrams. In groups discuss some formulations of axioms to express these ideas. (Here are some our group came up with.)

{circint}

Activity 3.8 (3^+: Intersections)**.**

1. **Axiom 3'** *If a circle is drawn with radius AB and center A, it intersects any line through A other than AB in two points C and D, one on each side of AB.*

2. **Axiom 3"** *If from points A and B, circles with radius AC and BD are drawn such that each circle contains points both in the interior and in the exterior of the other, then they intersect in two points, on opposite sides of AB.*

The activity Rusty Compass Activity (pdf) lays out the geogebra construction to prove Lemma 3.9.

{EP2}

Lemma 3.9 (Euclid's Proposition 2)**.** *To place a straight line equal to a given straight line with one end at a given point.*

In modern language: Given any line segment AB and point C, one can construct a line segment of length AB and end point C.

Proof. `http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI2.html`

3.10 Remark (Extension). We can be more conservative in choosing our axioms, saying in Axiom 3: Given two points A and B, there is a circle with center A and radius AB. {wA3}

Even with this restricted form of Axiom 3, but using the congruence axiom SSS, we can prove Lemma 3.9 and then recover the original Axiom 3.

Definition 3.11. *An angle $\angle ABC$ is a pair of distinct rays from a point B. The rays BA and BC split the plane into two connected regions.*

(A region is connected if any two points are connected by a polygonal path (a sequence of segments such that successive segments share one endpoint.) The region such that any two point are connected by a segment entirely in the region is called the interior *of the angle.*

Activity 3.12. *What are at least three different units for measuring the size of an angle? (Answers include, degree, radian, turn, grad, house (astrology), Furman.)*

3.13 Remark. We differ from Euclid here in allowing straight angles. Thus we avoid the awkward locution of the 'sum of two right angles' for 'straight angle'.

3.14 Exercise. Given a point D, construct an equilateral triangle such that D is the midpoint of one side, using only the first three postulates. Note: from the first three postulates we can't prove the line segment from D to the vertex of the triangle is perpendicular to the base. See Theorem 3.24.

Definition 3.15 (Right Angle)**. CCSS G-C0-1** *When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is* right*, and the straight line standing on the other is called a* perpendicular *to that on which it stands.* {ra}

3.16 Axiom (4.Euclid's 4th postulate). **CCSS G-C0-1** All right angles are equal {ax4}

Activity 3.17. *Fold paper to make a right angle.*

3.18 Exercise (Challenge). Prove the 4th postulate from SSS.

3.19 Remark. As in Euclid, we take the notions of segment congruence ($AB \cong A'B'$) and angle congruence ($\angle ABC \cong \angle A'B'C'$)as primitive; these notions satisfy the 'equality axioms' of the common notions.

But now we define what it means for two triangles to be congruent.

Definition 3.20 (Triangle congruence)**. CCSS G-C0-7** *Two triangles are congruent if there is a way to make the sides and angles correspond so that:*
Each pair of corresponding angles are congruent.
Each pair of corresponding sides are congruent.

3.21 Axiom (4.5 The triangle congruence postulate : SSS). **CCSS G-C0-8** Let ABC and $A'B'C'$ be triangles with $AB \cong A'B'$ and $AC \cong A'C'$ and $BC \cong B'C'$ then $\triangle ABC \cong \triangle A'B'C'$ {sss}

In Euclid this result, SSS, is proved from SAS. The proof is 4 steps: Euclid Propositions 1.5 to 1.8. These 4 steps are not hard and are correct. But his proof of SAS Proposition 4 has a gap, so we have to add **one** congruence axiom; we choose to add SSS. All the other criteria for congruence (SAS, ASA, HL ...) are theorems.

{sas}

Theorem 3.22 (SAS)**. CCSS G-C0-8, G-C0-10** *Let ABC and $A'B'C'$ be triangles with $AB \cong A'B'$ and $AC \cong A'C'$ and $\angle CAB \cong \angle C'A'B'$ then $\triangle ABC \cong \triangle A'B'C'$*

Proof. Let ABC and $A'B'C'$ be triangles with $AB \cong A'B'$ and $AC \cong A'C'$ and $\angle A \cong \angle A'$. We must show $\triangle ABC \cong \triangle A'B'C'$. Draw arcs with radius AB and radius BC from A' and from B' using Axiom 3. Let them intersect at a point D on the same side of AB as C. Note that triangle $A'DB' \cong ACB$ by SSS. ($AB \cong A'B'$, $BC \cong B'D$ and $AC \cong A'D$). So $\angle CAB \cong \angle DA'B'$. But then by transitivity of equality, $\angle C'A'B' \cong \angle DA'B'$. But then D lies on $A'C'$ and in fact D must be C'. So we have proved the theorem. $\square_{3.22}$

The little box $\square_{3.22}$ signals that we have completed the proof of Theorem 3.22.

The method of proving the following important exercise is embedded in the proof of Theorem 3.22.

{moveangle}

3.23 Exercise. Let ABC be an angle. For any segment DE, choose a point F so that $\angle ABC \cong \angle DEF$.

{cp1}

Theorem 3.24 (Constructing perpendiculars I)**. CCSS G-C0-12** *Given a line AD there is a line perpendicular to the line through AD at D.*

Proof. Extend AD and let B be the intersection of that line with the circle of radius AD centered at D. Now construct an equilateral triangle with base AB by using Axiom 3.4 twice to construct the vertex C. Draw CD. SSS implies $\triangle ACD \cong \triangle BCD$; so $\angle CDA \cong \angle CDB$ and therefore $CD \perp AB$. $\square_{3.24}$

Definition 3.25 (Straight Angle)**.** *An angle $\angle ABC$ is called a* straight *angle if A, B, C lie on straight line and B is between A and C.*

{stangle}

Theorem 3.26. CCSS G-C0-9 *All straight angles are equal (congruent).*

Proof. Let $\angle ABC$ and $\angle A'B'C'$ be straight angles. Construct lines BD and $B'D'$ perpendicular to AC and $A'C'$, respectively. Now $\angle ABD + \angle DBC = \angle ABC$ and $\angle A'B'D' + \angle D'B'C' = \angle A'BC'$. By Axiom 3.16, $\angle ABD = \angle A'B'D'$ and $\angle DBC = \angle D'B'C'$. By Common Notion 2, $\angle ABC = \angle A'B'C'$. $\square_{3.26}$

Theorem 3.26 is statement about the uniformity of the plane. In terms of transformations it says any point and a line through it can be moved by a rigid motion to any other point and any line through it.

Definition 3.27. *If two lines cross the angles which do not share a common side are called* vertical angles*.*

Deduce from Theorem 3.26:

3.28 Exercise. CCSS G-C0-9 Vertical Angles are equal.

Definition 3.29 (Isosceles)**.** *A triangle is* isosceles *if at least two sides have the same length. The angles opposite the equal sides are called the* base angles*.*

Activity 3.30. G-CO 11,12 *Make a) an isosceles and b) equilateral triangle in Geogebra using translations.*

Activity 3.31. G-CO 10 *Activity: Isosceles triangle and exterior angle theorem (Euclid Translation Activity.pdf) Compare 'paragraph' and 'two column' proof.*

{isosbase}

Theorem 3.32. CCSS G-C0-10 *The base angles of an isosceles triangle are equal (congruent).*

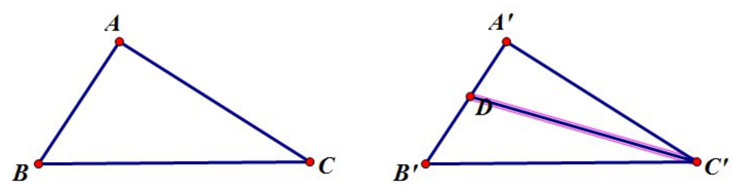
Proof. Let ABC be an isosceles triangle with $AC \cong BC$. We will prove $\angle CAB \cong \angle CBA$. The trick is to prove $ABC \cong BAC$. (BAC is obtained from ABC by flipping the triangle over its altitude.) We have two ways to prove the congruence. We know $BC \cong AC$ and $BC \cong AC$. We can also note $AB \cong AB$ and use SSS or $\angle ACB \cong \angle BCA$ and use SAS. In any case, since the triangles are congruent $\angle CAB \cong \angle CBA$. $\square_{3.32}$

Activity 3.33. *Prove the angles of an equilateral triangle are equal. (Note that there are two proofs, using either SSS or SAS, and they are distinguished by which correspondences are made in defining the congruence. Explain this by considering the theorem in terms of rotational or reflective symmetry.)*

{asa}

3.34 Exercise. CCSS G-C0-8, G-C0-10 Generalize the argument for Theorem 3.32 to show ASA: if two triangles have two angles and the included side congruent, then the triangles are congruent.

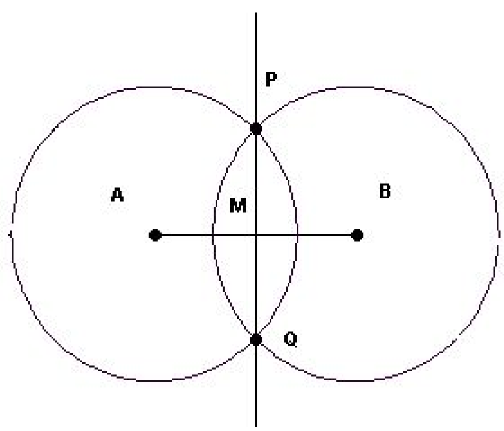
Solution Suppose ABC and $A'B'C'$ satisfy $\angle ABC = \angle A'B'C'$, $\angle ACB = \angle ACB$ and $BC = B'C'$. We will show the triangles are congruent.



Choose D on $A'B'$ so that $AB \cong B'D$ (We'll assume D is between A' and B' for contradiction. If A' is between B' and D, there is a similar proof.) Now, $AB \cong B'D$, $BC = B'C'$ and $\angle ABC = \angle A'B'C'$ so by SAS, $\triangle ABC \cong \triangle A'B'C'$. Since the angles correspond, $\angle DC'B' \cong \angle ACB$ and so by Common Notion 1, $\angle DC'B' \cong \angle A'C'B'$. But this is absurd since $\angle DC'B'$ is a proper subangle of $\angle A'C'B'$.

{cp2}

Theorem 3.35 (Constructing Perpendicular Bisectors)**. CCSS G-C0-12** *For any line segment AB there is a line PQ such that PQ is perpendicular to AB at the point of intersection M and M is the midpoint of AB.*



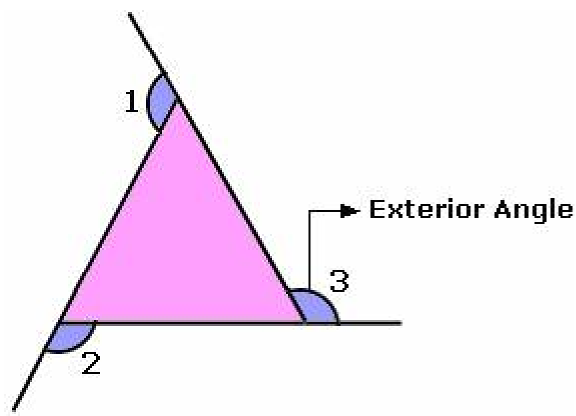
Proof. Set a compass at any length at least that of AB and draw two circles centered at A and B respectively. Let the two circles intersect at P above AB and Q below AB and let M be the intersection of AB and PQ.

To show PQ perpendicular to AB, note first that $\triangle APQ \cong \triangle BPQ$ by SSS. So $\angle APM \cong \angle BPM$. Then by SAS, $\triangle APM \cong \triangle BPM$. Thus $\angle AMP \cong \angle BMP$. And therefore these are each right angles by Definition 3.15. But also $\triangle AEC \cong \triangle BEC$ implies $AM \cong BM$ so M bisects AB. $\square_{3.35}$

3.36 Remark. Note we could be more prescriptive and just as correct by requiring in the proof of Theorem 3.35 that the circle have radius AB. But this is an unnecessary additional requirement.

Definition 3.37. *If D is in the interior of angle $\angle ACB$, line CD bisects the angle $\angle ACB$ if $\angle ACD \cong \angle BCD$.*

Theorem 3.38 (Exterior Angle Theorem, Euclid I.16)**.** *An exterior angle of a triangle is greater than either of the interior and opposite angles*



Angle1.jpg

Proof.

Here is Euclid's proof. `http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI16.html` But there is a subtle dependence on betweenness. See the treatment in [8] on page 36.

3.3 The Parallel Postulate

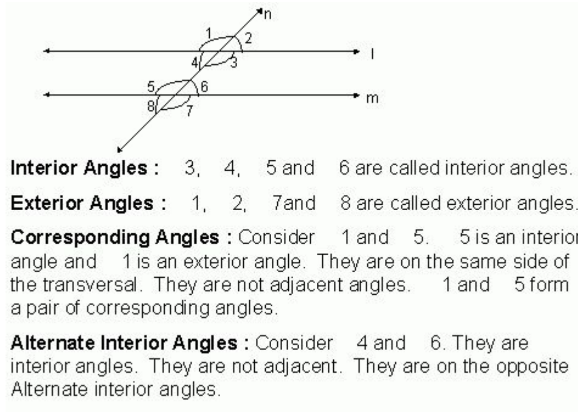
Definition 3.39. *Two lines are parallel if they do not intersect.*

{altint}

3.40 Notation (Alternate Interior Angles). When a line crosses two others, it is called a transversal.

If a transversal crosses to lines the angles between the lines and the transversal that are on opposite side of the transversal are called *alternate interior angle*; see the following diagram.

The definitions of corresponding, interior, and exterior angles are spelled out in the diagram below.



The difference between several statements which are close to the parallel postulate provides interesting historical and pedagogical background [3].

Theorem 3.41 (Euclid I.27)**.** *If two lines are crossed by a third and alternate interior angles are equal, the lines are parallel.*

Proof. The hypothesis says the exterior angle to triangle EFG is equal to the interior angle FEB. That contradicts the exterior angle theorem.

`http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI27.html`

Now consider the converse. {E5}

3.42 Axiom (5. Euclid's 5th postulate). If two parallel lines are cut by a transversal then the alternate interior angles are equal.

The rest of this subsection is devoted to the technical remark that assuming the first 4 axioms the two versions of the 5th postulate, Axiom 3.42 and Axiom 3.45 are equivalent.

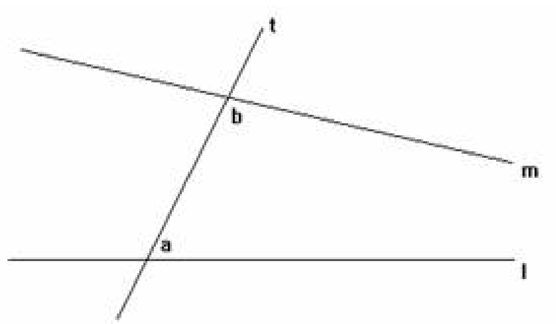
Definition 3.43 (Contraposition)**.** *Let A and B be mathematical statements.*

The contrapositive *of A implies B is $\neg B$ implies $\neg A$*

Theorem 3.44 (Logical fact)**.** *Any implication is* equivalent *to its contrapositive.* {HE5}

3.45 Axiom. Heath's statement of Euclid's 5th postulate:

If a straight line crosses two straight lines in such a way that the interior angles of the same side are less than two right angles, then, if the two straight lines are extended, they will meet on the side on which the interior angles are less than two right angles.



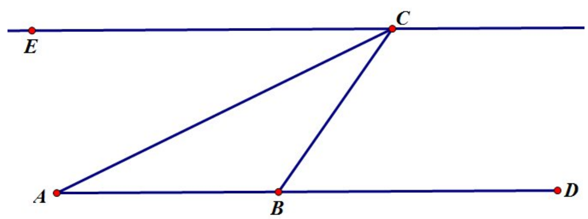
Using the earlier axioms this statement is seen to be equivalent to my phrasing of Euclid's 5th postulate Axiom 3.42. The contrapositive (and so equivalent) to Heath's version of Euclid's 5th postulate reads: If two straight lines are parallel the two consecutive interior angles are not less than two right angles.

But since all straight angles are equal, it is easy to check: 'two consecutive interior angles are not less than two right angles' is equivalent to 'the alternate interior angles are equal'.

3.4 Degrees in a triangle and classifying quadrilaterals

Theorem 3.46. CCSS G-C0-10 *The sum of the angles of a triangle is* 180^o. {sumdeg}

Proof. That is, we must show the sum of the angles of a triangle is a straight angle.



1. Draw EC so that $\angle BCE \cong \angle DBC$. (by move angles)
2. Then $EC \parallel AD$. (by getting parallel lines)
3. So $\angle BAC \cong \angle ACE$ (by the parallel postulate)
4. So $\angle BAC + \angle ACB = \angle DBC$
5. But $\angle ABC + \angle DBC$ is a straight angle.
6. So $\angle ABC + \angle BAC + \angle ACB$ is a straight angle.

$\square_{3.46}$

Definition 3.47. *A parallelogram is a quadrilateral such that the opposite sides are parallel.*

Theorem 3.48. CCSS G-CO.11 *If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.* {eqpar}

Proof. Suppose $ABCD$ is the parallelogram; draw diagonal AC. Then ABC and ACD are congruent by SSS. Therefore $\angle BAC \cong \angle ACD$. Now since alternate interior angles are equal, $AB \parallel DC$. Similarly (which angles?) $BC \parallel AD$. $\square_{3.48}$

Theorem 3.49 (Euclid I.34)**. CCSS G-CO.11** *In any parallelogram the opposite sides and angles are equal. Moreover the diagonal splits the parallelogram into two congruent triangles.* {oppsideeq}

Immediate from our results on parallelogram and the congruence theorems.

3.50 Exercise. CCSS G-CO.11 If a pair of opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram .

4 Proof the division of a line into n equal parts succeeds

4.1 Exercise. Let ABCD be an arbitrary quadrilateral? Let DEFG be the midpoints of the sides. What can you say about the quadrilateral DEFG? ([Varignon's Theorem[2]])

Here are some possibilities that came up in class. See Sidesplitter Exloration (Sidesplitter motivationjb2.docx):

1. The area of the inner quad is 1/2 the area of the outer.
2. The inner quadrilateral is a parallelogram.
3. Under further conditions on outer quad (rectangle?, square?), the inner quadrilateral is a rectangle.

We began this excursion into axiomatic geometry by trying to prove that we could divide a line into n equal segments. We did the construction from Exercise 2.2 in class. The diagram is on the first page of the notes.

This construction used only Euclid's first 3 axioms. *We need to show the segments cut off by the C_i are actually equal.* In the sidesplitter motivation activity we gave several arguments for this. Here is one which is easy but uses a powerful tool.

{cutn}

Lemma 4.2. *The segments C_iC_{i+1} constructed above all have the same length.*

Since a quadrilateral whose opposite sides are equal is a parallelogram, $ACBD$ is a parallelogram (Theorem 3.48). We DO NOT know that A_4B_4BD is a parallelogram. It follows from the following lemma.

{getpar}

Lemma 4.3. *If $ABCD$ is a parallelogram and two points X, Y are chosen on the opposite sides AB and CD so that $XB \cong YD$ then $XBDY$ is a parallelogram.*

Proof. Draw the diagonals XD and YB. Since $AB \parallel CD$, by alternate interior angles $XBY \cong BYD$ and $BXD \cong XDY$. Label the intersection of the diagonals as E.

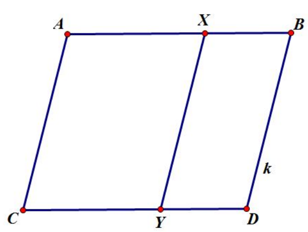


Figure 1: Lemma 4.3

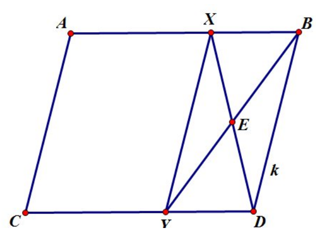


Figure 2: Draw the diagonals

Since $XB \cong YD$ by ASA, $\triangle YED \cong \triangle XEB$. So by corresponding sides, $BE \cong EY$ and $XE \cong ED$. $\angle XEY \cong \angle BED$ since they are vertical angles. By SAS $\triangle XEY \cong \triangle BED$. By corresponding sides $XY \cong BD$. So $XYBD$ is a parallelogram and $XY \parallel BD$. $\square_{4.3}$

Now we return to the proof of Lemma 4.2: the construction divided the line segment AB into n equal pieces. By repeating the argument for Lemma 4.3, we show all the lines $A_iC_iB_i$ are parallel. In particular

[2]See `http://en.wikipedia.org/wiki/Varignon's_theorem`. Note generalizations to 3 dimensions.

the line C_4B_4 cuts the triangle B_3C_3B and is parallel to the base B_3C_3. By Theorem 7.2, the side-splitter theorem,

$$\frac{BB_4}{BB_3} = \frac{BC_4}{BC_3}.$$

But we constructed $B_4B \cong B_3B_4$, so $B_4B \cong C_3C_4$, which is what we are trying to prove. Now move along AB, successively applying this argument to each triangle. $\square_{4.2}$

5 Introducing Arithmetic

{num}

The difficulty with the side-splitter theorem is that we don't really know what ratio means when the sides are *incommensurable*. The following activity introduces this notion.

Activity 5.1. *See goldenratio.pdf and [14].*

Suppose that we wanted to divide a line into three segments in proportions that were not commensurable. How could we do that?

Activity 5.2. *Divide a line AB into three segments whose lengths are proportional to the sides of a* $30-60-90$ *triangle. (Irrational sidesplitter Motivation.pdf)*

The construction is actually the same as before. But how do we know it works? See [13] and [2] for a discussion of how this problem affected the 20th century high school mathematics curriculum in the U.S. For this we introduce *segment arithmetic*. This topic appears in Euclid, gets a different interpretation in Descartes and still another in the 19th arithmetic of real numbers.

We want to define the multiplication of 'lengths'. Identify the collection of all congruent line segments as having a common 'length' and choose a representative segment OA for this class. There are then three distinct historical steps. (See in particular [7] and Heath's notes to Euclid VI.12 (`http://aleph0.clarku.edu/~djoyce/java/elements/bookVI/propVI12.html`.) In Greek mathematics numbers (i.e. 1, 2, 3 ...) and magnitudes (what we would call length of line segments) were distinct kinds of entities and areas were still another kind.

5.3 Remark. *From geometry to numbers*

1. Euclid shows that the area of a parallelogram is jointly proportional to it base and height. [3]
2. Descartes defines the multiplication of line segments to give another segment[4]. Hilbert shows the multiplication on segments satisfies the field[5] axioms.
3. Identify the points of the line with (a subfield) of the real numbers. Now addition and multiplication can be defined on points[6].

The standard treatment in contemporary geometry books is to begin with stage 3, taking the operations on the real numbers as basic. We will pass rather from geometry to number, concentrating on stage 2. Thus, not all real numbers may be represented by points on the line in some planes.

---

[3] In modern terms this means the area is proportional to the base times the height. But Euclid never discusses the multiplication of magnitudes.

[4] He refers to the fourth proportional ('ce qui est meme que la multiplication'[4])

[5] In [9], the axioms for a semiring (no requirement of an additive inverse are verified.

[6] And thus all axioms for a field are obtained. Hilbert had done this in lecture notes in 1894[10]

We first introduce an addition and multiplication on line segments. Then we will prove the geometric theorems to show that these operations satisfy the field axioms except for the existence of an additive inverse. We note after Definition 5.16 how to remedy this difficulty by the passing to points as in stage 3.

Activity 5.4. CCSS 8.F.1, F-IF.1 *Prepare for the definition of segment addition with the worksheet 1fnact-geo.pdf concerning the meaning of equivalence relation and function and the connections between them.*

{segeq}

5.5 Notation. Note that congruence forms an equivalence relation on line segments. We fix a ray ℓ with one end point 0 on ℓ. For each equivalence class of segments, we consider the unique segment $0A$ on ℓ in that class as the representative of that class. We will often denote the class (i.e. the segment $0A$ by a. We say a segment (on any line) CD has length a if $CD \cong 0A$.

{segadddef}

Definition 5.6 (Segment Addition)**.** *Consider two segment classes a and b. Fix representatives of a and b as OA and OB in this manner: Extend OB to a straight line, and choose C on OB extended (on the other side of B from A) so that so that $BC \cong OA$. OC is the sum of OA and OB.*

Diagram for adding segments

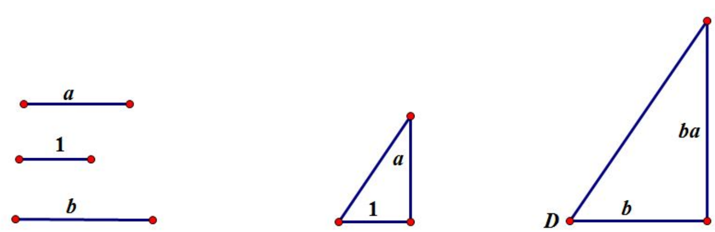
Activity 5.7. *Prove that this addition is associative and commutative*

Of course there is no additive inverse if our 'numbers' are the lengths of segments which must be positive. We discuss finding an additive inverse after Definition 5.16. Following Hartshorne [8], here is our official definition of segment multiplication.

{segmultdef}

Definition 5.8. *[Multiplication] Fix a unit segment class* 1. *Consider two segment classes a and b. To define their product, define a right triangle*[7] *with legs of length 1 and a. Denote the angle between the hypoteneuse and the side of length a by α.*

Now construct another right triangle with base of length b with the angle between the hypoteneuse and the side of length b congruent to α. The length of the vertical leg of the triangle is ab.



[7] The right triangle is just for simplicity; we really just need to make the two triangles similar.

{scamult}

5.9 Exercise. We now have two ways in which we can think of the product $3a$. On the one hand, we can think of laying 3 segments of length a end to end. On the other, we can perform the segment multiplication of a segment of length 3 (i.e. 3 segments of length 1 laid end to end) by the segment of length a. Prove these are the same.

Before we can prove the field laws hold for these operations we introduce a few more geometric facts.

{ceninsang}

Theorem 5.10. *[Euclid III.20]* **CCSS G-C.2** *If a central angle and an inscribed angle cut off the same arc, the inscribed angle is congruent to half the central angle.*

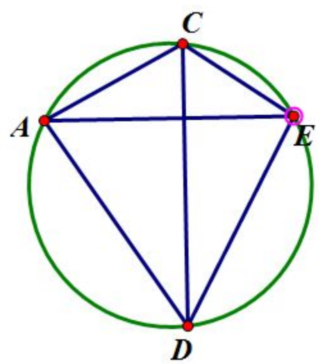
5.11 Exercise. Do the activity: Determining a curve (determinecircle.pdf).

We need proposition 5.8 of [8], which is a routine (if sufficiently scaffolded) high school problem.

Activity 5.12. *Prove a central angle is twice and inscribed angle that inscribes the same arc. How many diagrams must you consider? Activity: central angle is twice inscribed angle (centralinscribed - both pdf and geogebra.)*

{cquad}

Corollary 5.13. CCSS G-C.3 *Let $ACED$ be a quadrilateral. The vertices of A lie on a circle (the ordering of the name of the quadrilateral implies A and E are on the same side of CD) if and only if $\angle EAC \cong \angle CDE$.*



Proof. Given the conditions on the angle draw the circle determined by ABC. Observe from Lemma 5.10 that D must lie on it. Conversely, given the circle, apply Lemma 5.10 to get the equality of angles. $\square_{5.13}$

Activity 5.14. *Do activity Segment arithmetic (multpropact.pdf).*

Here is a write-up of the solution.

{mult2works}

Theorem 5.15. *The multiplication defined in Definition 5.8 satisfies.*

1. *For any a, $a \cdot 1 = 1$*
2. *For any a, b*

$$ab = ba.$$

3. *For any* a, b, c

$$(ab)c = a(bc).$$

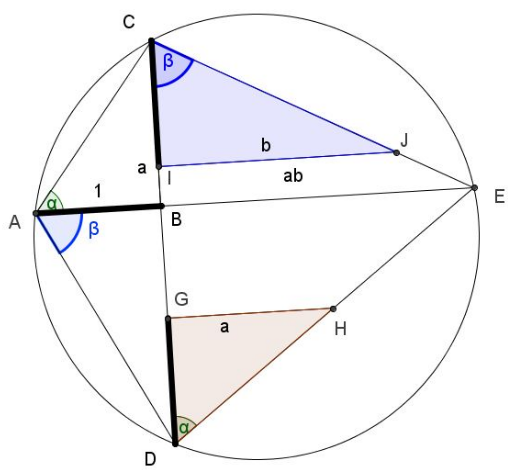
4. *For any* a *there is a* b *with* $ab = 1$.

5. $a(b + c) = ab + ac$.

Proof. For the moment we prove 2, since that requires some work.

Given a, b, first make a right triangle $\triangle ABC$ with legs 1 for AB and a for BC. Let α denote $\angle BAC$. Extend BC to D so that BD has length b. Construct DE so that $\angle BDE \cong \angle BAC$ and E lies on AB extended on the other side of B from A. The segment BE has length ab by the definition of multiplication.

Since $\angle CAB \cong \angle EDB$ by Corollary 5.13, $ACED$ lie on a circle. Now apply the other direction of Corollary 5.13 to conclude $\angle DAE \cong \angle DCA$ (as they both cut off arc AD. Now consider the multiplication beginning with triangle $\triangle DAE$ with one leg of length 1 and the other of length b. Then since $\angle DAE \cong \angle DCA$ and one leg opposite $\angle DCA$ has length a, the length of BE is ba. Thus, $ab = ba$.



$\square_{5.15}$

Now to prove associativity use the following diagram.

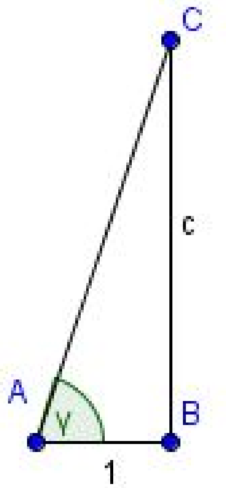


Figure 3: multiply by a

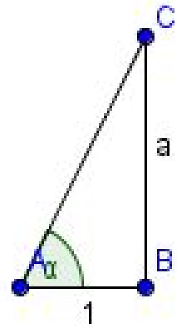
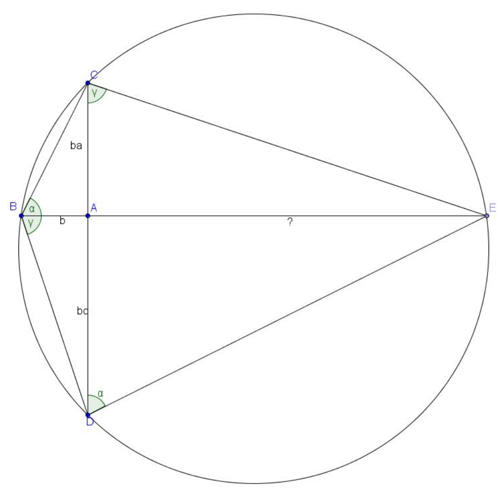


Figure 4: multiply by c



Note that AE can be represented as either $(ba)c$ or $(bc)a$ use the commutative law twice to complete the proof. $\square_{5.15}$

The remainder of this section is a modification to identify points on the line with numbers and so have additive inverses.

{pointadd}

Definition 5.16 (Adding points)**.** *Recall that a line is a set of points. Fix a line ℓ and a point 0 on ℓ. We define an operations $+$ on ℓ. Recall that we identify a with the (directed length of) the segment 0a.*

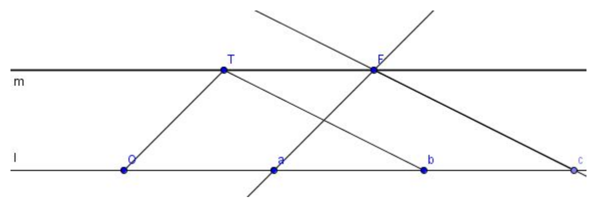
For any points a, b on ℓ, we define the operation $+$ on ℓ:

$$a + b = c$$

if c is constructed as follows.

1. *Choose T not on ℓ and m parallel to ℓ through T.*
2. *Draw $0T$ and BT.*
3. *Draw a line parallel to $0T$ through a and let it intersect m in F.*
4. *Draw a line parallel to bT through a and let it intersect ℓ in c.*

Diagram for point addition



$$0b \cong ac$$

Problem 5.17. *Add a and b (i.e. construct c) when a is to the left of 0 on ℓ. What is the inverse of a? 0. and the additive inverse of a is a' provided that $a'0 \cong 0a$ where a' is on ℓ but on the opposite side of 0 from a.*

6 Area of Parallelograms and triangles

An activity (Area of triangle week#3.pdf), based on one of Mark Driscoll, uses area to emphasize that an altitude of triangle does not have to lie inside the triangle.

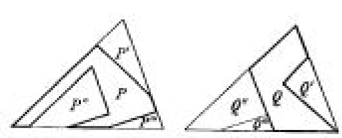
We begin with thinking about where area makes sense and when two 'figures' have the same area.

Definition 6.1. *A (rectilineal) figure is a finite union of disjoint triangles.*

Definition 6.2 (Scissor Congruence)**.** *Two polygons are* scissor-congruent *if you can cut one up (on straight lines) into a finite number of triangle which can be rearranged to make the second.*

See exercises on scissors-congruence from pages 174-175 of CME geometry [5]. Cut and Paste Activity Week#3.pdf (These illustrate **CCSS 6.G.1**)

Definition 6.3 (Equal content)**.** *Two figures P, Q have* equal content[8] *if there are figures $P'_1 \ldots P'_n$, $Q'_1 \ldots Q'_n$ such that none of the figures overlap, each P'_i and Q'_i are scissors congruent and $P \cup P'_1 \ldots \cup P'_n$ is scissors congruent with $Q \cup Q'_1 \ldots \cup Q'_n$.*



[8]The diagram is taken from [9].

Theorem 6.4 (Euclid I.35, I.38)**.** *Parallelograms on the same base and in the same parallels have the same area.*

Triangles on the same base and in the same parallels have the same area.

6.5 Exercise. Prove these results in your groups. Did you use scissors-congruence or equal content?

`http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI35.html`
`http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI38.html`

We want to define a formula to compute the area of a polygonal figure; this will take two steps. First show that equal content is an equivalence relation that satisfies the intuitive properties (Axiom 6.6) for figures to have the same area. Then we define the area of a unit square to be 1 (sq unit) and then show that this justifies the standard formulas (with multiplication as segment multiplication) for the area of rectangles, parallelograms, and triangle.

{areaax}

6.6 Axiom (Area Axioms). The following properties of area are used in Euclid I.35 and I.38. We take them from pages 198-199 of [5].

1. Congruent figures have the same area.
2. The area of two 'disjoint' polygons (i.e. meet only in a point or along an edge) is the sum of the two areas of the polygons.
3. Two figures that have equal content [9] have the same area.
4. [10] If one figure is properly contained in another then the area of the difference (which is also a figure) is positive.

Note that the first of these is really common notion 4.

6.7 Exercise. Draw a scalene triangle such that only one of the three altitudes lies within the triangle. Compute the area for each choice of the base as b (and the corresponding altitude as h).

Our argument below shows that the function assigning bh as the area of a rectangle does not depend on which choice of base and altitude is made. The argument would have worked just as well if we had taken $17bh$ as the function. The reason we choose bh is so that we generalize the simple counting argument that a 3×5 rectangle contains 15 squares, each with area 1 square unit.

The activity, A crucial lemma, cruclemma.pdf gives teachers an opportunity to work on Lemma **??** before seeing the solution. This lemma uses the 4th of the area axioms. The goal is prove Claim 6.9.

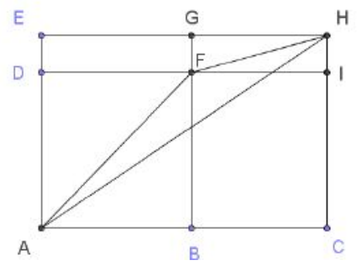
{areadiag}

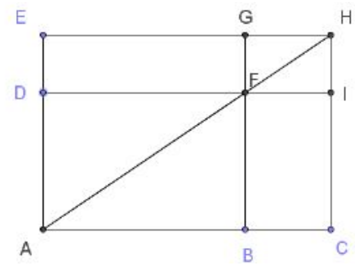
Lemma 6.8. *If two rectangles $ABGE$ and $WXYZ$ have equal content there is a rectangle $ACID$, congruent to $WXYZ$ and satisfying the following diagram. Further the diagonals AF and FH are collinear.*

Proof. Suppose AB is less than WX and YZ is less than AE. Then make a copy of $WXYZ$ as $ACID$ below. The two triangles are congruent. Let F be the intersection of BG and DI. Construct H as the intersection of EG extended and IC extended. Now we prove F lies on AH.

[9] CME reads 'scissor-congruent' but relies on the assumption about the real numbers just before the statement of Postulates 3.3 and 3.4. That is, on Hilbert's argument [9] that for geometries over Archimedean fields, scissors-congruent and equal content are the same.

[10] Hartshorne (Sections 19-23 of [8]) proceeds in a more expeditious manner and avoids the need to axiomatize the properties of area.



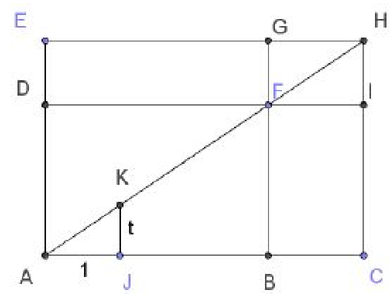


Suppose F does not lie on AH. Subtract $ABFD$ from both rectangles, then $DFGE$ and BCIF have the same area. AF and FH bisect $ABFD$ and $FIHG$ respectively. So $AFD \cup DFGE \cup FHG$ has the same content as $ABF \cup BCIF \cup FIH$, both being half of rectangle $ACHE$ (Note that the union of the six figures is all of $ACHE$. Here, $AEHF$ is properly contained in AHE and $ACHF$ properly contains ACH. This contradicts Axiom 6.6.4; hence F lies on AH. $\square_{6.8}$

{diagmult}

Claim 6.9. *If $ABGE$ and $ACID$ are as in the diagram (in particular, have the same area, then in segment multiplication $(AB)(BG) = (AC)(CI)$.*

Proof. Let the lengths of AB, BF, AC, CH, JK be represented by a, b, c, d, t respectively and let AJ be 1. Now $ta = b$ and $tc = d$, which leads to $b/a = d/c$ or $ac = bd$, i.e.$(AB)(BG) = (AC)(CI)$.



By congruence, we have $(AE)(AB) = (WZ)(XY)$ as required. $\square_{6.9}$

We have shown that for any rectangles with equal areas the products (in segment arithmetic) of the base and the height are the same. This condition would be satisfied if the area were kbh for any k representing a segment class. In order to agree with the intuitive notion that this area should be same as the number of unit squares in a rectangle we define.

Definition 6.10. *The area of a square 1 unit on a side is (segment arithmetic) product of its base times its height, that is one square unit.*

{areaformrect}

Theorem 6.11. *The area of a rectangle is the (segment arithmetic) product of its base times its height.*

Proof. Note that for rectangles that have integer lengths this follow from Exercise 5.9. For an arbitrary rectangle with side length c and d, apply the identity law for multiplication and associativity. $\square_{6.11}$

Activity 6.12. *The activity fndef.pdf explores some examples of 'well-defined' notions.*

We can show that any triangle is scissor congruent to half of a rectangle. So the area of a triangle should be $\frac{1}{2}$ base x height. But why is this well-defined? Could the value of $\frac{1}{2}$ base x height depend on the choice of the base?

{triformula}

Theorem 6.13. *Any of the three choices of base for a triangle give the same value for the product of the base and the height.*

Proof. Consider the triangle ABC is figure 1. The rectangles in figures 1,3, and 5 are easily seen to be scissors congruent. By Claim 6.9, each product of height and base for the triangle is the same. That is, $(AB)(CD) = (AC)(BJ) = (BC)(AM)$. But these are the three choices of base/altitude pair for the triangle ABC.

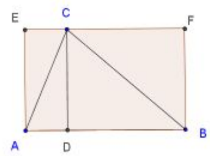


Figure 1

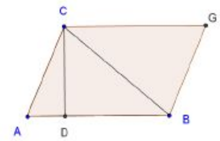


Figure 2

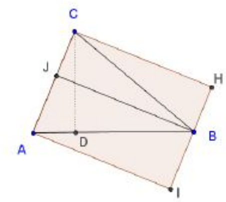


Figure 3

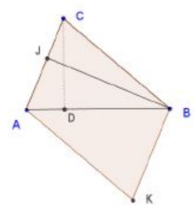


Figure 4

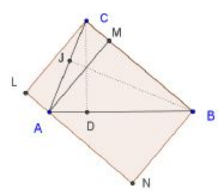


Figure 5

$\square_{6.13}$

We have the following immediate corollary which is the key to what CME calls the side-splitter theorems. Note that proof of this lemma is *purely algebraic* (once we have established the area formulas) and requires using the *associative law* several times as well the existence of multiplicative inverses.

{htareaprop}

Corollary 6.14. *If two triangles have the same height, the ratio of their areas equals the ratio of the length of their corresponding bases.*

In Euclid this result holds for irrationals only by the method of Eudoxus, which is a precursor of the modern theory of limits, but did not envision the existence of arbitrary real numbers. In contrast the development here shows that for any triangles which occur in a geometry satisfying the axioms here [11] the areas and their ratios are represented by line segments in the field.

Activity 6.15. *Consider various proofs of the Pythagorean Theorem Activity: Pythagorean Theorem (pythag.pdf). Reconstruct Garfield's diagram (Garfield.pdf has a copy of the original article.) and work out his proof of the Pythagorean theorem. ('On the hypoteneuse cb of the right angled triangle abc, draw the half cube cbe', means 'draw the triangle cbe such that be is the diagonal of a square one side cb.)*

[11] Crucially, neither Archimedean, nor complete, is assumed.

7 Similarity: Euclid book 6, CME Side-splitter

Theorem 7.1. *Euclid VI.2* **CCSS G-SRT.4** *If a line is drawn parallel to the base of triangle the corresponding sides of the two resulting triangles are proportional and conversely.* {sidespl}

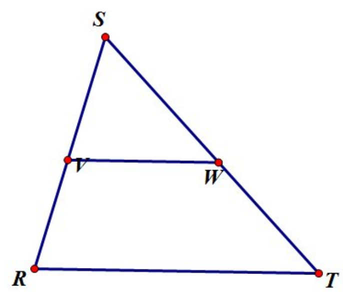
The proof is the 'side-splitter theorem' from pages 313 and 315 of CME geometry [5] (Side-splitter activity–Week#3.pdf.

Here is the CME reformulation of the first part of Euclid VI.2; we will discuss the converse later.

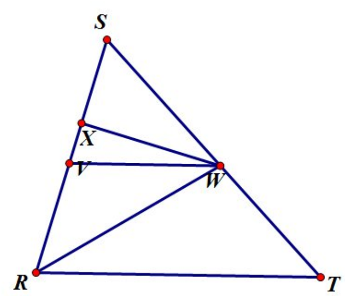
Theorem 7.2. *Side-splitter theorem* **CCSS G-SRT.4** *If a segment with end points on two sides of a triangle is parallel to the third side of the triangle, then it splits the sides it intersects proportionally.* {sidespl}

Proof. We show that in the diagram below if $VW \parallel RT$ then

$$\frac{SV}{VR} = \frac{SW}{WT}.$$



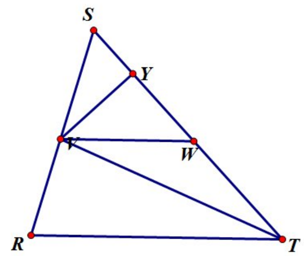
In the figure below triangles SVW and RVW have the same height WX.



$\triangle SVW$ and $\triangle RVW$ share a vertex W and their bases SV and RV are on the straight line SVR. So the height of each triangle is WX. Thus by Theorem 6.14,

$$\frac{\text{area}(\triangle SVW)}{\text{area}(\triangle RVW)} = \frac{SV}{VR}.$$

Now consider the diagram:



$\triangle SVW$ and $\triangle TVW$ share a vertex V and their bases SW and WT are on the straight line SWT. So the height of each triangle is VY. So we have by Theorem 6.14,

$$\frac{\text{area}(\triangle SVW)}{\text{area}(\triangle TVW)} = \frac{SW}{WT}.$$

Now $\frac{\text{area}(\triangle SVW)}{\text{area}(\triangle TVW)}$ and $\frac{\text{area}(\triangle SVW)}{\text{area}(\triangle RVW)}$ are two fractions with the obviously the same numerator. But since $\triangle TVW$ and $\triangle RVW$ share the same base and are between parallel lines, they also have the same area. So since the two ratios of areas are the same, so are the two ratios of sides.

$$\frac{SW}{WT} = \frac{SV}{VR}.$$

$\square_{7.2}$

We now have a hard and an easy exercise.

7.3 Exercise. Prove the converse to the side-splitter theorem. **CCSS G-SRT.4** If a segment with end points on two sides of a triangle splits the sides it intersects proportionally, then it is parallel to the third side of the triangle.

7.4 Exercise. We are given from the first part of the problem that $\frac{SV}{VR} = \frac{SW}{WT}$. Show {extendproport}

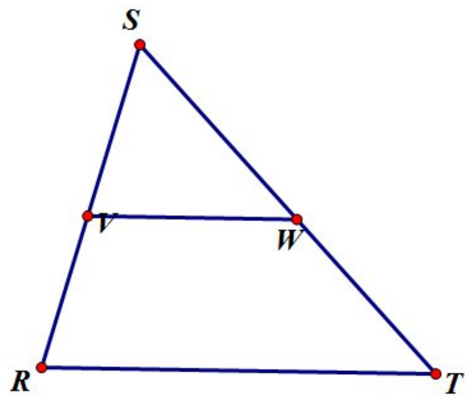
$$\frac{SR}{SV} = \frac{ST}{SW}.$$

Note that Exercise 7.4 is most easily done entirely as algebra.

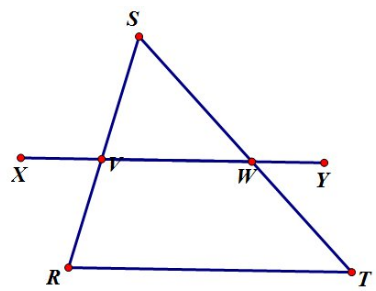
Definition 7.5. *Two triangles $\triangle ABC$ and $\triangle A'B'C$ are similar if under some correspondence of angles, corresponding angles are congruent; e.g. $\angle A' \cong \angle A$, $\angle B' \cong \angle B$, $\angle C' \cong \angle C$.*

Activity 7.6. *Various texts define 'similar' as we did, or as corresponding sides are proportional or require both. Discuss the advantages of the different definitions. Why are all permissible?*

Lemma 7.7. *Suppose a line VW is drawn connecting points on two sides a triangle SRT. VW is parallel to RT if and only if SVW is similar to SRT.* {simpar}



Proof. Extend VW to a line and pick points X and Y on VW on opposite sides of the triangle.



Now $\angle XVR$ and $\angle VRT$ are alternate interior angles for the transversal RS crossing the two lines XY and RT. So $\angle XVR \cong \angle VRT$ if and only $VW \parallel RT$. But $\angle XVR \cong \angle SVW$ since they are vertical angles. So $\angle SVW \cong \angle VRT$ if and only if $VW \parallel RT$.

By a similar argument, $\angle SWV \cong \angle STR$ if and only if $VW \parallel RT$.

Since the sum of the angles of a triangle is 180^0, two corresponding angles congruent implies the third pair is as well. Thus, SVW is similar to SRT if and only if $VW \parallel RT$. $\square_{7.7}$

7.8 Remark. Note that what we have really proved is that for a transversal cutting two lines corresponding angles are equal if and only if alternate interior angles are equal. If we accepted that fact, the proof of Lemma 7.7 would just be to noticed that the equalities betweem the pairs of corresponding angles $\angle SWV \cong \angle STR$ and $\angle SWV \cong \angle STR$ hold if and only if $VW \parallel RT$ if and only if SVW is similar to SRT.

7.9 Remark (BIG PICTURE). It is standard in middle school mathematics to compute areas and to solve proportionality problems for similar triangles. We are showing that these topics are intimately related and indeed that a natural way to prove the proportionality property for similar triangle is to use area.

7.10 Exercise. Show that if points on the sides of triangle split the sides proportionally, then the segment joining them is parallel to the base.

{simprop}

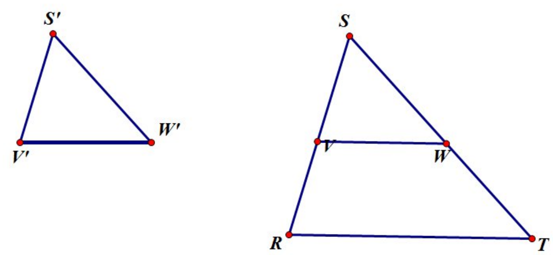
Theorem 7.11. *Similar triangles have proportional sides.*

Proof. We have done most of the hard work. The trick is to put it together. Here are the four big steps.

1. Triangles in the same parallels have the same area. (We can abbreviate this as $A = \frac{1}{2}bh$. (Euclid I.35, I.38)

2. Theorem 7.2, if a segment with endpoints on two sides of a triangle is parallel to the third side of the triangle, then it splits the sides proportionally.

3. Suppose two triangles, $\triangle S'V'W'$ and $\triangle SRT$ are similar. Construct a triangle congruent to the smaller on the larger. (See picture below.)



4. By the side-splitter theorem we have that the sides of $\triangle SVW$ and $\triangle SRT$ are proportional. Since $\triangle S'V'W' \cong \triangle SVW$ the sides of $\triangle S'V'W'$ and $\triangle SRT$ are proportional.

This completes the proof that similar triangles have proportional sides. Note that there is no use of limits in the proof. We have shown the theorem holds for whatever segment lengths happen to be in the geometry under consideration. Of course, the proof that the real numbers actually include 'all' the irrationals requires the completeness axiom and constructing a model explicitly requires a theory of limits.

7.12 Exercise. Use the same ideas as in the proof of Theorem 7.11 to show its converse: If corresponding sides of two triangles are proportional, the triangles are similar.

7.13 Exercise (CCSS G-SRT4). Prove the Pythagorean formula using similarity.

Activity 7.14. *The activity incenter.pdf contains some 'real-world' applications of incenter and Hartshorne's direct proof of the side-splitter theorem for segment arithmetic (Proposition 20.1 of [8]) without using area.*

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