Geometry, the Common Core, and Proof

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Outline

Agenda

1. G-C0-1 – Context.
2. Activity: Divide a line into n pieces -with string; via construction
3. Reflection activity (geometry/ proof/definition/ common core)
4. mini-lecture Axioms and Definitions in Euclid
5. Activity Construct an equilateral triangle. Prop 1.
6. Diagrams and proofs
7. lunch/Discussion: How do these differ from axioms in high school texts
8. Activity - rusty compass theorem (30 min)
9. congruence as a basic notion; SSS
10. Discussion of G-C0 1

Common Core

G-C01

Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

Why is word undefined in this standard? What and how do we know about the undefined notions?

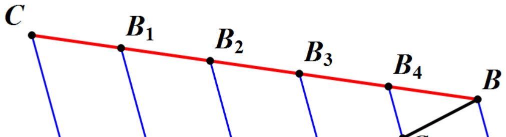
Activity: Dividing a line into n-parts: Informal

See Handout

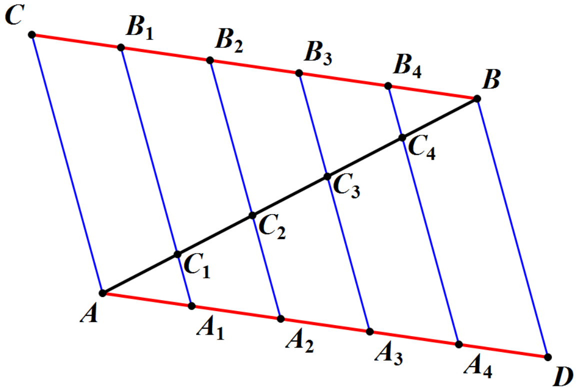
Activity: Dividing a line into n-parts: Construction

Here is a procedure to divide a line into n equal segments.

1. Given a line segment AB.
2. Draw a line through A different from AB and lay off sequentially n equal segments on that line, with end points $A_1, A_2, \ldots$. Call the last point D.
3. Construct C on the opposite side of AB from D so that $AC \cong BD$ and $CB \cong AD$.
4. Lay off sequentially n equal segments on that line, with end points $C, B_1, B_2, \ldots B$.
5. Draw lines A_iB_i.
6. The point C_i where C_i is the intersection of A_iB_i with AB are the required points dividing AB into n equal segments.



Activity: Dividing a line into n-parts: Diagram



Why did it work?

Do exercise 2.2 in text.

Reflection: Constructions, proofs, definitions

See activity worksheet.

Common Notions

These common notions or *axioms* of Euclid apply equally well to geometry or numbers. We won't make a big deal over the difference between 'axiom' and 'postulate'.

Common notion 1. Things which equal the same thing also equal one another.
Common notion 2. If equals are added to equals, then the wholes are equal.
Common notion 3. If equals are subtracted from equals, then the remainders are equal.
Common notion 4. Things which coincide with one another equal one another.
Common notion 5. The whole is greater than the part.

return

Terminology

Modern (new math) text books make a big deal about the difference between congruence and equality. Numbers are central - so equalities are only between numbers while line segments or figures are congruent.

Geometry before Number

Euclid did not have numbers as distinct objects - He'd say line segments are congruent where we'd say have the same length.

So equality can be replaced by congruence in understanding the common notions.

Euclid's first 3 axioms in modern language

1. **Axiom 1** Given any two points there is a line segment connecting them.
2. **Axiom 2** Any line segment can be extended indefinitely (in either direction).
3. **Axiom 3** Given a point and any segment there is a circle with that point as center whose radius is the same length as the segment.

What are these axioms telling us?

Discussion:

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Discussion:
implicit definition

Activity: Construct an equilateral triangle

Do it with straightedge and compass.

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Suppose the base line is between $(-1, 0)$ and $(1, 0)$. What are coordinates of the vertex?

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Does anyone have any doubts?

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Suppose I just take a plane with rational coordinates. What's up?

Axioms on the intersection of circles

1. **Axiom 3'** If a circle is drawn with radius AB and center A, then it intersects any line through A other than AB in two points C and D, one on each side of the line AB.
2. **Axiom 3''** If from points A and B, two circles C_1 and C_2 are drawn so that each circle contains points both in the interior and in the exterior of the other, then they intersect in two points, each on opposites sides of AB." then they intersect in two points, one on each side of AB.

We used 'inexact' properties to phrase the axiom. It could be done by more detailed conditions on the radii.

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Hilbert et al banished the diagram from formal mathematics. This requires complicated axioms about betweenness which have never successfully been taught to high school students.

The Cure is worse than the disease

The 'new math' tried to replace the diagram with algebra.
`http://www.glencoe.com/sec/math/studytools/cgi-bin/msgQuiz.php4?isbn=0-07-829637-4&chapter=2&lesson=7&quizType=1&headerFile=6&state=il`

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Why does it take six steps to show: If two line segments have the same length and equal line segments are taken away from each, the resulting segments have the same length.

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Common Notions

A different cure: properties of diagrams

Inexact properties

Properties that are not changed by minor variations in the diagram.

subsegment, inclusion of one figure in another, two lines intersect, betweenness

Exact properties

Properties that can be changed by minor variations in the diagram.

Whether a curve is a straight line, congruence, a point is on a line

We can rely on inexact properties from the diagram. We must write exact properties in the text.

The fly in the ointment

Depending on the particular diagram that you draw, after a construction, the diagram may have different inexact properties.

The solution is case analysis.
We will spend time on this later in the workshop.

Choice of Axioms matters

Students learned to prove from Euclid.

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Students learned to prove from Euclid.
Student are buried by trivialities and don't learn to prove from Moise-Birkhoff.

The challenge

We will try in this workshop to develop ways of using diagrams in a semi-formal way to support the line-by-line argument.
This worked for 2000 years.
But it hasn't been rigorously formulated nor implemented with modern students.
(Miller book)

Activity: Euclid Proposition 2

Understanding the reason for and proof of Proposition 2.

What is congruence

Undefined concept: Congruence.

of angles, of line segments
We have axioms about congruence.

The common notions apply. In the Common Notions 'congruence' can be substituted for equal.

Congruence

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Definition: Figure congruence

CCSS G-C0-7 Two figures are congruent if there is a way to make the sides and angles correspond so that:
Each pair of corresponding angles are congruent.
Each pair of corresponding sides are congruent.

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Note this is really about polygons. And we haven't defined angle yet.

Congruence postulate

The triangle congruence postulate : SSS

CCSS G-C0-8 Let ABC and $A'B'C'$ be triangles with $AB \cong A'B'$ and $AC \cong A'C'$ and $BC \cong B'C'$ then $\triangle ABC \cong \triangle A'B'C'$

Homework Challenge

PROVE:

Theorem SAS

CCSS G-C0-8, G-C0-10 Let ABC and $A'B'C'$ be triangles with $AB \cong A'B'$ and $AC \cong A'C'$ and $\angle CAB \cong \angle C'A'B'$ then $\triangle ABC \cong \triangle A'B'C'$

We will discuss the proof next week.

Common Core

G-C01

Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

We have seen how axioms are necessary to specify undefined notions.

Now let's address the standard.

Defining the important concepts

Using the undefined notions

of point, line, congruent line segment, congruent angles,

Define

line segment

1. group 1 angle, right angle, perpendicular line,
2. group 2 circle, arc
3. parallel line

Explain each notion with physical models or transformations. A group can use the notions defined by groups with smaller numbers.

Defining the measure of an angle

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For this we need the undefined notion of ‘length of an arc’ (i.e. congruence of circles)