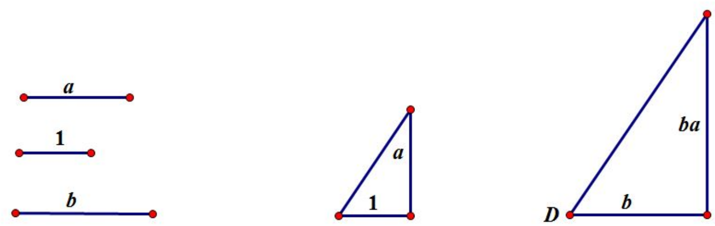
Following Hartshorne [1], here is our official definition of segment multiplication.

{segmultdef}

Definition 0.1. *[Multiplication] Fix a unit segment class* 1*. Consider two segment classes* a *and* b*. To define their product, define a right triangle*[1] *with legs of length* 1 *and* a*. Denote the angle between the hypoteneuse and the side of length* a *by* α*.*

Now construct another right triangle with base of length b *with the angle between the hypoteneuse and the side of length* b *congruent to* α*. The length of the vertical leg of the triangle is* ba*.*



{scamult}

0.2 Exercise. We now have two ways in which we can think of the product $3a$. On the one hand, we can think of laying 3 segments of length a end to end. On the other, we can perform the segment multiplication of a segment of length 3 (i.e. 3 segments of length 1 laid end to end) by the segment of length a. Prove these are the same.

Before we can prove the field laws hold for these operations we must introduce a few more geometric facts.

▌ We just did .3 through .5. They are here for reference.

{ceninsang}

Theorem 0.3. *[Euclid III.20]* **CCSS G-C.2** *If a central angle and an inscribed angle cut off the same arc, the inscribed angle is congruent to half the central angle.*

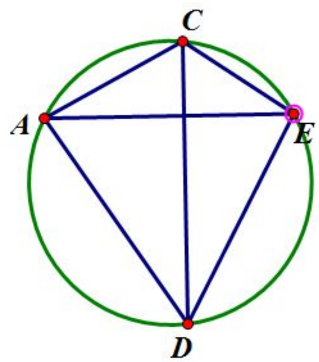
0.4 Exercise. Do the activity: Determining a curve (determinecircle.pdf).

We need proposition 5.8 of [1], which is a routine (if sufficiently scaffolded) high school problem.

{cquad}

Corollary 0.5. **CCSS G-C.3** *Let* $ACED$ *be a quadrilateral. The vertices of* A *lie on a circle (the ordering of the name of the quadrilateral implies* A *and* E *are on the same side of* CD*) if and only if* $\angle EAC \cong \angle CDE$*.*

[1] The right triangle is just for simplicity; we really just need to make the two triangles similar.



Proof. Given the conditions on the angle draw the circle determined by ABC. Observe from Lemma 0.3 that D must lie on it. Conversely, given the circle, apply Lemma 0.3 to get the equality of angles. $\square_{0.5}$

Now we want to establish the remaining properties of multiplication. Do parts 1 and 4. Work through the given proof of 2. Then do the task on the next page to understand 3.

{mult2works}

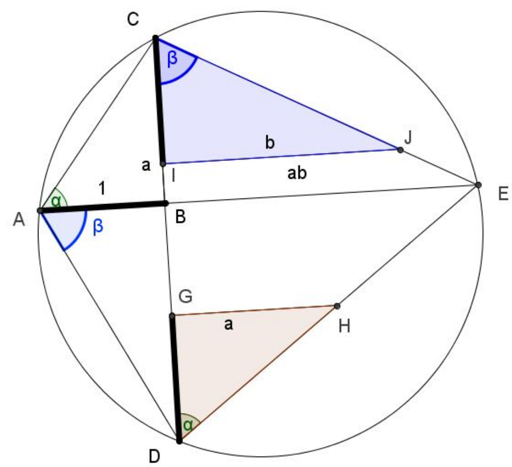
Theorem 0.6. *The multiplication defined in Definition 0.1 satisfies.*

1. *For any a, $a \cdot 1 = 1$*
2. *For any a, b*
$$ab = ba.$$
3. *For any a, b, c*
$$(ab)c = a(bc).$$
4. *For any a there is a b with $ab = 1$.*
5. *$(b + c)a = ba + ca$.*

Proof. For the moment we prove 2, since that requires some work.

Given a, b, first make a right triangle $\triangle ABC$ with legs 1 for AB and a for BC. Let α denote $\angle BAC$. Extend BC to D so that BD has length b. Construct DE so that $\angle BDE \cong \angle BAC$ and E lies on AB extended on the other side of B from A. The segment BE has length ab by the definition of multiplication.

Since $\angle CAB \cong \angle EDB$ by Corollary 0.5, $ACED$ lie on a circle. Now apply the other direction of Corollary 0.5 to conclude $\angle DAE \cong \angle DCA$ (as they both cut off arc AD. Now consider the multiplication beginning with triangle $\triangle DAE$ with one leg of length 1 and the other of length b. Then since $\angle DAE \cong \angle DCA$ and one leg opposite $\angle DCA$ has length a, the length of BE is ba. Thus, $ab = ba$.



$\square_{0.6}$

Now to prove associativity use the following diagrams.

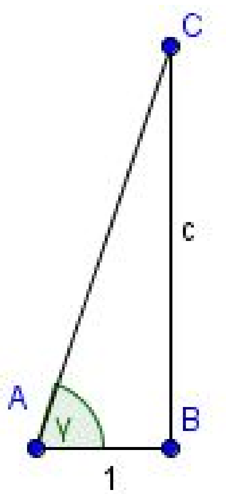


Figure 1: multiply by c

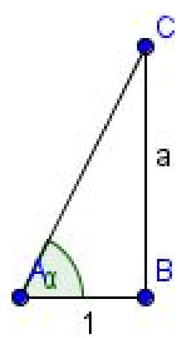
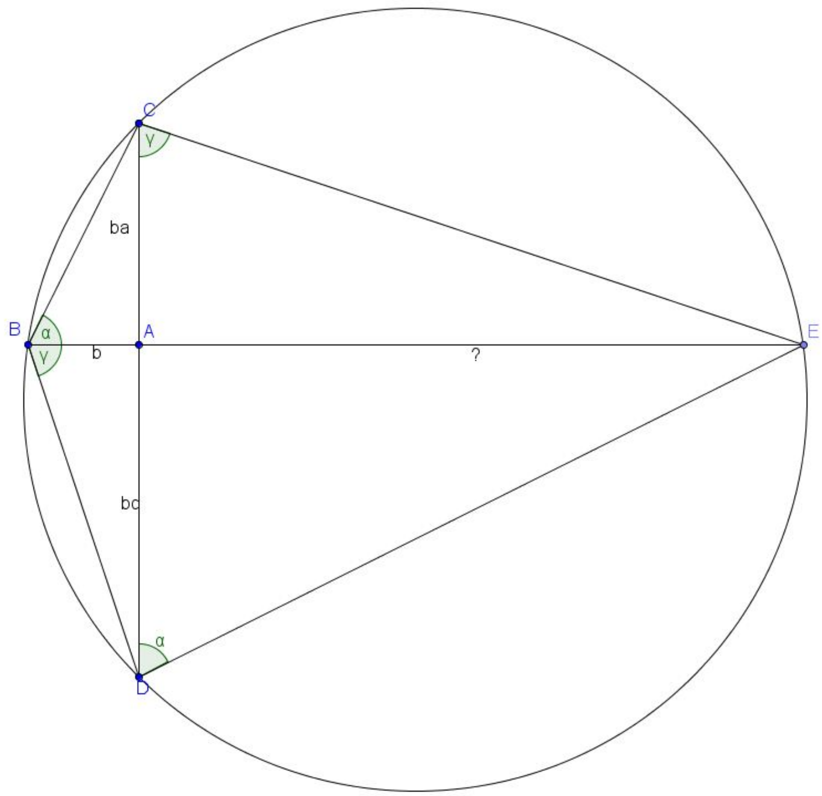


Figure 2: multiply by a

α and γ are associated with right multiplication by a and c by the previous page.

In the diagram below, what is the value of the ?, That is, what is the length of AE? Calculate it in two ways. Note that the heavy lines have length 1.



Prove that multiplication distributes over addition.

References

[1] Robin Hartshorne. *Geometry: Euclid and Beyond*. Springer-Verlag, 2000.