Abstract Elementary Classes Abelian Groups

John T. Baldwin

University of Illinois at Chicago

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Topics

Two Goals

General

Can we extend the methods of first order stability theory to generalized logics – e.g. $L_{\omega_1,\omega}$?

Special

Can the model theory of infinitary logic solve ‘mathematical problems’ (as the model theory of first order logic has)?

Abelian Groups

1. Does the notion of AEC provide a general framework to describe some work in Abelian group theory?
2. Certain AEC of abelian groups provide interesting previously unknown examples for the general study of AEC. Can this work be extended?

A background principle

Slogan

To study a structure A, study $\mathrm{Th}(A)$.

e.g.

The theory of algebraically closed fields to investigate $(\mathcal{C}, +, \cdot)$.

The theory of real closed fields to investigate $(\mathcal{R}, +, \cdot)$.

DICTA

But there is no real necessity for the ‘theory’ to be complete.

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Strong Slogan

Classes of structures are more interesting than singleton structures.

ABSTRACT ELEMENTARY CLASSES defined

Definition

A class of L-structures, $(\mathbf{K}, \prec_{\mathbf{K}})$, is said to be an abstract elementary class: AEC if both $\mathbf{K}$ and the binary relation $\prec_{\mathbf{K}}$ are closed under isomorphism and satisfy the following conditions.

- **A1**. If $M \prec_{\mathbf{K}} N$ then $M \subseteq N$.
- **A2**. $\prec_{\mathbf{K}}$ is a partial order on $\mathbf{K}$.
- **A3**. If $\langle A_i : i < \delta \rangle$ is $\prec_{\mathbf{K}}$-increasing chain:
  1. $\bigcup_{i<\delta} A_i \in \mathbf{K}$;
  2. for each $j < \delta$, $A_j \prec_{\mathbf{K}} \bigcup_{i<\delta} A_i$
  3. if each $A_i \prec_{\mathbf{K}} M \in \mathbf{K}$ then $\bigcup_{i<\delta} A_i \prec_{\mathbf{K}} M$.

- **A4**. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$.
- **A5**. There is a Löwenheim-Skolem number $\mathrm{LS}(\mathbf{K})$ such that if $A \subseteq B \in K$ there is a $A' \in \mathbf{K}$ with $A \subseteq A' \prec_{\mathbf{K}} B$ and
  $|A'| < \mathrm{LS}(\mathbf{K}) + |\mathrm{A}|$.

Examples

1. First order complete theories with $\prec_{\mathbf{K}}$ as elementary submodel.
2. Models of $\forall\exists$-first order theories with $\prec_{\mathbf{K}}$ as substructure.
3. L^n-sentences with L^n-elementary submodel.
4. Varieties and Universal Horn Classes with $\prec_{\mathbf{K}}$ as substructure.
5. Models of sentences of $L_{\kappa,\omega}$ with $\prec_{\mathbf{K}}$ as: elementary in an appropriate fragment.
6. Models of sentences of $L_{\kappa,\omega}(Q)$ with $\prec_{\mathbf{K}}$ carefully chosen.
7. Robinson Theories with Δ-submodel
8. 'The Hrushovski Construction' with strong submodel

The group group

AIM meeting July 2006
J. Baldwin, W. Calvert, J. Goodrick, A. Villaveces, & A. Walczak-Typke, & Jouko Väänänen

Strong Submodel

Notation

Consider various subclasses $\mathbf{K}^{foo}$ of the class $\mathbf{K}^{ab}$ of all abelian groups (e.g. $foo = div, red(p), \ldots$).

1. *"$\leq$" denotes subgroup.*
2. *$G \prec_{pure} H$ means G is a pure subgroup of H:*
3. *"$G \prec_{sum} H$" means that G is a direct summand of H;*
4. *"$G \prec_{foo} H$" means that G is a pure subgroup of H and $H/G \in \mathbf{K}^{foo}$.*

Connections

Compare notion 4 with Eklof's notion of a C-filtration:

$G = \bigcup_i G_i$
and $G/G_i \in C$.

Examples

Abbreviation	Subclass of abelian groups	Section
$\mathbf{K}^{ab}$	All abelian groups	??
$\mathbf{K}^{div}$	Divisible groups	??
$\mathbf{K}^{p}$	p-groups	
$\mathbf{K}^{tor}$	torsion groups	
$\mathbf{K}^{tf}$	torsion-free groups	
$\mathbf{K}^{red(p)}$	reduced p-groups	??
$\mathbf{K}^{sep(p)}$	separable p-groups	
$\mathbf{K}^{rtf}$	reduced torsion-free groups	
$\mathbf{K}^{cyc}$	direct sums of cyclic groups	??

Some Examples

Lemma

The class $\mathbf{K}^{ab}$ of all abelian groups forms an AEC with amalgamation and joint embedding under either $\leq$ or $\prec_{pure}$, with Löwenheim-Skoelm number $\aleph_0$. Moreover, under $\leq$ it is stable in all cardinals.

But what does stable mean?

Model Homogeneity

Definition

M is μ-model homogenous if for every $N \prec_{\mathbf{K}} M$ and every $N' \in \mathbf{K}$ with $|N'| < \mu$ and $N \prec_{\mathbf{K}} N'$ there is a $\mathbf{K}$-embedding of N' into M over N.

To emphasize, this differs from the homogenous context because the N must be in $\mathbf{K}$. It is easy to show:

Monster Model

Lemma

(jep) If M_1 and M_2 are μ-model homogenous of cardinality $\mu > \mathrm{LS}(\mathbf{K})$ then
$M_1 \approx M_2$.

Theorem

If $\mathbf{K}$ has the amalgamation property and $\mu*^{<\mu*} = \mu*$ and $\mu* \geq 2^{\mathrm{LS}(\mathbf{K})}$ then there is a model $\mathcal{M}$ of cardinality $\mu*$ which is $\mu*$-model homogeneous.

GALOIS TYPES: General Form

Define:

$$(M, a, N) \cong (M, a', N')$$

if there exists N'' and strong embeddings f, f' taking N, N' into N'' which agree on M and with

$$f(a) = f'(a').$$

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$$f(a) = f'(a').$$

'The Galois type of a over M in N' is the same as 'the Galois type of a' over M in N''
if (M, a, N) and (M, a', N') are in the same class of the equivalence relation generated by $\cong$.

GALOIS TYPES: Algebraic Form

Suppose $\mathbf{K}$ has the amalgamation property.

Definition

Let $M \in \mathbf{K}$, $M \prec_{\mathbf{K}} \mathcal{M}$ and $a \in \mathcal{M}$. The Galois type of a over M is the orbit of a under the automorphisms of $\mathcal{M}$ which fix M.

We say a Galois type p over M is realized in N with $M \prec_{\mathbf{K}} N \prec_{\mathbf{K}} \mathcal{M}$ if $p \cap N \neq \emptyset$.

Galois vrs Syntactic Types

Syntactic types have certain natural locality properties.

- locality Any increasing chain of types has at most one upper bound;
- tameness two distinct types differ on a finite set;
- compactness an increasing chain of types has a realization.

The translations of these conditions to Galois types do not hold in general.

Galois and Syntactic Types

Work in $(\mathbf{K}^{ab}, \leq)$.

Lemma

Suppose that G_1 is a subgroup of both G_2 and G_3, $a \in G_2 - G_1$, and $b \in G_3 - G_1$. the following are equivalent:

1. $\text{ga-tp}(a, G_1, G_2) = \text{ga-tp}(b, G_1, G_3)$;
2. *There is a group isomorphism from $\langle G_1, a\rangle_{G_3}$ onto $\langle G_1, b\rangle_{G_3}$ that fixes G_1 pointwise;*
3. $\mathrm{tp}_{qf}(a/G_1) = \mathrm{tp}_{qf}(b/G_1)$.

But this equivalence is far from true of all AEC's of Abelian groups.

Stability

Corollary

The AEC of Abelian groups under subgroup is stable in all cardinals.

Compare with the first order notion where there are Abelian groups e.g. Z^ω that are stable in λ only when $\lambda^\omega = \lambda$.

Sums of torsion cyclic groups

Definition

$\mathbf{K}^{cyc}$ is the class of all groups that are isomorphic to

$$\bigoplus_{p\in\Pi}\bigoplus_{k\in\Sigma_p}\left(\mathbb{Z}_{p^k}\right)^{\lambda_{p,k}},$$

for some subset Π of the prime numbers, subsets Σ_p of $\mathcal{N}$, and cardinals $\lambda_{p,k}$ (which may be finite or infinite).

Sums of torsion cyclic groups - Non-AEC

Fact

$(\mathbf{K}^{cyc}, \leq)$ and $(\mathbf{K}^{cyc}, \prec_{pure})$ are not AEC's.

An example shows the class $(\mathbf{K}^{cyc}, \prec_{pure})$ is not closed under unions of chains, which serves as a counterexample for both classes.

Sums of torsion cyclic groups-AEC??

Fact

(Follows from Kaplansky, Theorems 1 and 13) If $G \in \mathbf{K}^{cyc}$ and $H \leq G$, then $H \in \mathbf{K}^{cyc}$.

Lemma

Suppose that $G_1, G_2, G_3 \in \mathbf{K}^{cyc}$ and $G_1 \prec_{sum} G_2$, $G_1 \prec_{sum} G_3$, and $a \in G_2 - G_1$, $b \in G_3 - G_1$. Then, working within $\mathbf{K}^{cyc}$, the following are equivalent:

1. $\text{ga-tp}(a, G_1, G_2) = \text{ga-tp}(b, G_1, G_3)$;

2. There are $n, k \in \omega$ and $g \in G_1$ such that $\mathrm{ht}_{G_2}(a) = k = \mathrm{ht}_{G_3}(b)$, $na = g = nb$, and for any $m < n$, neither ma nor mb are in G_1.

Properties of $(\mathbf{K}^{cyc}, \prec_{sum})$

Fact

Abbreviating $(\mathbf{K}^{cyc}, \prec_{sum})$ as $\mathbf{K}^{cyc}$, we ought to be able to prove the following:

- $\mathbf{K}^{cyc}$ *is not an elementary class.*
- $\mathbf{K}^{cyc}$ *is a tame AEC with amalgamation and Löwenheim-Skolem number $\aleph_0$.*
- $\mathbf{K}^{cyc}$ *is not categorical.*
- $\mathbf{K}^{cyc}$ *has a universal model at every infinite cardinal λ.*
- $\mathbf{K}^{cyc}$ *is (galois-)stable at every cardinal.*
- $I(\mathbf{K}^{cyc}, \aleph_d) = |d + \omega|^{\omega}$.

Still not an AEC

Let $G = \prod_i \mathcal{Z}/2^i$.

Let $A = \sum_i \mathcal{Z}/2^i$; $A_j = \sum_{i<j} \mathcal{Z}/2^i$

Let b_i be the sequence in G consisting of i 0's followed by $\langle 1, 2, 4, 8 \ldots \rangle$.

Let B the subgroup of G generated by the b_i.

Now $\bigcup_j A_j = A$ is a not a direct summand of B although each A_j is.

Reflection

Why do we want A.3.3?

THE PRESENTATION THEOREM

Every AEC is a PCΓ

More precisely,

Theorem

If K is an AEC with Lowenheim number $\mathrm{LS}(\mathbf{K})$ (in a vocabulary τ with $|\tau| \leq \mathrm{LS}(\mathbf{K})$), there is a vocabulary τ', a first order τ'-theory T' and a set of $2^{\mathrm{LS}(\mathbf{K})}$ τ'-types Γ such that:

$$\mathbf{K} = \{M' \restriction L : M' \models T' \text{ and } M' \text{ omits } \Gamma\}.$$

Moreover, if M' is an L'-substructure of N' where M', N' satisfy T' and omit Γ then
$M' \restriction L \prec_{\mathbf{K}} N' \restriction L$.

Still a PCΓ

$(\mathbf{K}^{cyc}, \prec_{sum})$ is a PCΓ class by adding a predicate for a basis and using omitting types to translate $L_{\omega_1,\omega}$-axioms.

Andrew Coppola introduces the notion of a Q-AEC which generalizes the notion and still allows the presentation theorem to hold. This notion might be relevant here although the motivation was very different - equicardinality quantifiers.

Questions

This is a toy example.

Are there natural classes of Abelian groups that form AEC under an appropriate notion of substructure?

Why should it matter?

PCΓ-classes have models generated by sequences of indiscernibles - EM-models. This is a powerful tool for studying categoricity.

Tameness

Grossberg and VanDieren focused on the idea of studying 'tame' abstract elementary classes:

Definition

We say $\mathbf{K}$ is (χ, μ)-tame if for any $N \in \mathbf{K}$ with $|N| = \mu$ if $p, q, \in \mathcal{S}(N)$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ then $q = p$.

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

If for every $N \prec_{\mathbf{K}} M$ with $|N| \leq \chi$ there exists $\alpha \in \mathrm{aut}_{\mathrm{N}}(\mathcal{M})$ with $\alpha(a) = b$,

then there exists $\alpha \in \mathrm{aut}_{\mathrm{M}}(\mathcal{M})$ with $\alpha(a) = b$.

Consequences of Tameness

Suppose **K** has arbitrarily large models and amalgamation.

Theorem (Grossberg-Vandieren)

If **K** *is* λ^+*-categorical and* $(<\lambda, \infty)$*-tame then* **K** *is categorical in all* $\theta \geq \lambda^+$*.*

Theorem (Lessmann)

If **K** *is* $\aleph_1$*-categorical and* $(\aleph_0, \infty)$*-tame then* **K** *is categorical in all uncountable cardinals*

Fact

In studying categoricity of short exact sequences, Zilber has proved equivalences between categoricity in uncountable cardinals and 'arithmetic properties' of algebraic groups. These are not proved in ZFC but an independent proof of tameness would put them in ZFC.

Two Examples that are not tame

1. 'Hiding the zero'
   For each $k < \omega$ a class which is $(\aleph_k, \infty)$-tame but not $(\aleph_{k+1}, \aleph_{k+2})$-tame. Baldwin-Kolesnikov (building on Hart-Shelah)
2. Coding EXT
   A class that is not $(\aleph_0, \aleph_1)$-tame.
   A class that is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame. (Baldwin-Shelah)

Categoricity does not imply tameness

Theorem For each $k < \omega$ there is an $L_{\omega_1,\omega}$ sentence ϕ_k such that:

1. ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
2. ϕ_k is not $\aleph_{k-2}$-Galois stable;
3. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$;
4. ϕ_k has the disjoint amalgamation property;
5. ϕ_k is $(\aleph_0, \aleph_{k-3})$-tame; indeed, syntactic types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
6. ϕ_k is not $(\aleph_{k-3}, \aleph_{k-2})$-tame.

Locality and Tameness

Definition

K has (κ, λ)-local galois types if for every continuous increasing chain $M = \bigcup_{i<\kappa} M_i$ of members of **K** with $|M| = \lambda$ and for any $p, q \in \mathcal{S}(M)$: if $p \restriction M_i = q \restriction M_i$ for every i then $p = q$.

Lemma

If $\lambda \geq \kappa$ and $\mathrm{cf}(\kappa) > \chi$, then (χ, λ)-tame implies (κ, λ)-local. If particular, $(\aleph_0, \aleph_1)$-tame implies $(\aleph_1, \aleph_1)$-local.

Whitehead Groups

Definition

We say A is a Whitehead group if $Ext(A, \mathcal{Z}) = 0$. That is, every short exact sequence

$$0 \to \mathcal{Z} \to H \to A \to 0,$$

splits or in still another formulation, H is the direct sum of A and $\mathcal{Z}$.

Key Example

Shelah constructed (page 228 of Eklof-Mekler, first edition) of a group with the following properties.

Fact

There is $\aleph_1$-free group G of cardinality $\aleph_1$ which is not Whitehead. Moreover, there is a countable subgroup R of G such that G/R is p-divisible for each prime p.

THE AEC EXAMPLE

Let $\mathbf{K}$ be the class of structures $M = \langle G, Z, I, H \rangle$, where each of the listed sets is the solution set of one of the unary predicates $(\mathbf{G}, \mathbf{Z}, \mathbf{I}, \mathbf{H})$.
G is a torsion-free Abelian Group. Z is a copy of $(Z, +)$. I is an index set and H is a family of infinite groups.

Each model in $\mathbf{K}$ consists of

1. a torsion free group G,
2. a copy of $\mathcal{Z}$
3. and a family of extensions of Z by G.

Each of those extensions is coded by single element of the model so the Galois type of a point of this kind represents a specific extension. The projection and embedding maps from the short exact sequence are also there.

$M_0 \prec_{\mathbf{K}} M_1$ if

M_0 is a substructure of M,

but $\mathbf{Z}^{M_0} = \mathbf{Z}^{M}$

and $\mathbf{G}^{M_0}$ is a pure subgroup of $\mathbf{G}^{M_1}$.

FACTS

Definition

We say the AEC $(\mathbf{K}, \prec_{\mathbf{K}})$ admits closures if for every $X \subseteq M \in \mathbf{K}$, there is a minimal closure of X in M. That is, $M \restriction \bigcap\{N : X \subseteq N \prec_{\mathbf{K}} M\} = \mathrm{cl}_M(X) \prec_{\mathbf{K}} M$.

The class $(\mathbf{K}, \prec_{\mathbf{K}})$ is an abstract elementary class that admits closures and has the amalgamation property.

NOT LOCAL

Lemma

$(\mathbf{K}, \prec_{\mathbf{K}})$ is not $(\aleph_1, \aleph_1)$-local. That is, there is an $M^0 \in \mathbf{K}$ of cardinality $\aleph_1$ and a continuous increasing chain of models M^0_i for $i < \aleph_1$ and two distinct types $p, q \in \mathcal{S}(M^0)$ with $p \restriction M^0_i = q \restriction M_i$ for each i.

Let G be an Abelian group of cardinality $\aleph_1$ which is $\aleph_1$-free but not a Whitehead group. There is an H such that,

$$0 \rightarrow \mathcal{Z} \rightarrow H \rightarrow G \rightarrow 0$$

is exact but does not split.

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z \rangle$

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z \rangle$

$M_1 = \langle G, \mathcal{Z}, \{a, t_1\}, \{G \oplus Z, H\} \rangle$

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z \rangle$

$M_1 = \langle G, \mathcal{Z}, \{a, t_1\}, \{G \oplus Z, H\} \rangle$

$M_2 = \langle G, \mathcal{Z}, \{a, t_2\}, \{G \oplus Z, G \oplus Z\} \rangle$

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z \rangle$

$M_1 = \langle G, \mathcal{Z}, \{a, t_1\}, \{G \oplus Z, H\} \rangle$

$M_2 = \langle G, \mathcal{Z}, \{a, t_2\}, \{G \oplus Z, G \oplus Z\} \rangle$

Let $p = \mathrm{tp}(t_1/M^0, M^1)$ and $q = \mathrm{tp}(t_2/M^0, M^2)$.
Since the exact sequence for $\mathbf{H}^{M^2}$ splits and that for $\mathbf{H}^{M^1}$ does not, $p \neq q$.

NOT $\aleph_1$-LOCAL

But for any countable $M_0' \prec_{\mathbf{K}} M_0$, $p \upharpoonright M_0' = q \upharpoonright M_0'$, as

$$0 \to Z \to H_i' \to G' \to 0. \tag{1}$$

splits.
$G' = \mathbf{G}(M_0')$, $H' = \pi^{-1}(t_i, G')$.

NOT $\aleph_0$-TAME

It is easy to see that if $(\mathbf{K}, \prec_{\mathbf{K}})$ is $(\aleph_0, \aleph_0)$-tame then it is $(\aleph_1, \aleph_1)$-local, so
$(\mathbf{K}, \prec_{\mathbf{K}})$ is not $(\aleph_0, \aleph_0)$-tame.
So in fact, $(\mathbf{K}, \prec_{\mathbf{K}})$ is not $(\chi, \aleph_0)$-tame for any χ.

NOT κ-TAME

With some use of diamonds, for each successor cardinal κ, there is a κ-free but not free group of cardinality κ which is not Whitehead. This shows that, consistently,
For arbitrarily large κ,
$(\mathbf{K}, \prec_{\mathbf{K}})$ is not (κ, κ^+)-tame for any κ.

Question

Could this example be formulated more naturally as
$\{Ext(G, Z) : G \text{is torsion-free } \}$
(with projection and injection maps?

Incompactness

Theorem

Assume $2^{\aleph_0} = \aleph_1$, *and* $\diamondsuit_{\aleph_1}, \diamondsuit_{S_1^2}$ *where*

$$S_1^2 = \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}.$$

Then, the last example fails either $(\aleph_1, \aleph_1)$ *or* $(\aleph_2, \aleph_2)$*-compactness.*

BECOMING TAME ??

Grossberg and Van Dieren asked for $(\mathbf{K}, \prec_{\mathbf{K}})$, and $\mu_1 < \mu_2$ so that $(\mathbf{K}, \prec_{\mathbf{K}})$ is not (μ_1, ∞)-tame but is (μ_2, ∞)-tame.

Tameness gained

Theorem

There is an AEC with the closure property in a countable language with Lowenheim-Skolem number $\aleph_0$ which is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame.

Proof Sketch: Repeat the previous example but instead of letting the quotient be any torsion free group

1. insist that the quotient is an $\aleph_1$-free group;
2. add a predicate R for the group R G/R is divisible by every prime p where G is Shelah's example of a non-Whitehead group.

This forces $|G| \leq 2^{\aleph_0}$ and then we get $(2^{\aleph_0}, \infty)$-tame. But $\aleph_1$-free groups fail amalgamation ??

Lemma

For any AEC $(\mathbf{K}, \prec_{\mathbf{K}})$ which admits closures there is an associated AEC $(\mathbf{K}', \prec_{\mathbf{K}})$ with the same (non) locality properties that has the amalgamation property.

Theorem

There is an AEC with the amalgamation property in a countable language with Lowenheim-Skolem number $\aleph_0$ which is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame.

Summary

The true significance of first order stability theory became clear when one found a wide variety of mathematically interesting theories at various places in the stability hierarchy.
We are trying to find analogous examples of AEC.

References

Much is on the web at www.math.uic.edu/jbaldwin including:

1. Categoricity: a 200 page monograph introducing AEC,
2. Some examples of Non-locality (with Shelah)
3. Categoricity, amalgamation and Tameness (with Kolesnikov)
4. And see Grossberg, VanDieren, Shelah

jbaldwin@uic.edu in Barcelona until December.