Spectrum in AEC
Depaul AMS Meeting
October 5, 2007

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Outline

ABSTRACT ELEMENTARY CLASSES

A class of L-structures, $(\mathbf{K}, \prec_{\mathbf{K}})$, is said to be an *abstract elementary class: AEC* if both $\mathbf{K}$ and the binary relation $\prec_{\mathbf{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$;

Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

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3. Downward Löwenheim-Skolem.

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AMALGAMATION PROPERTY

The class $\mathbf{K}$ satisfies the *amalgamation property* if for any situation with $A, M, N \in \mathbf{K}$:

$$\begin{tikzcd} & N \\ A \arrow[ur] \arrow[dr] & \\ & M \end{tikzcd}$$

there exists an N_1 such that

$$\begin{tikzcd} & N \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & N_1 \\ & M \arrow[ur] & \end{tikzcd}$$

Disjoint Amalgamation

For 'disjoint amalgamation' we require that the image of M intersect the image of N in the image of A.

GALOIS TYPES

If **K** has the amalgamation property then there is a 'monster' model $\mathcal{M}$.

Definition

Let $M \in \mathbf{K}$, $M \prec_{\mathbf{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The Galois type of a over M is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.

Definition

The set of Galois types over M is denoted $\mathcal{S}(M)$.

Galois Saturation

We say a Galois type p over M is realized in N with $M \prec_{\mathbf{K}} N \prec_{\mathbf{K}} \mathbb{M}$ if $p \cap N \neq \emptyset$.

Definition

The model M is μ-Galois saturated if for every $N \prec_{\mathbf{K}} M$ with $|N| < \mu$ and every Galois type p over N, p is realized in M.

Homogeneity and Saturation

Theorem

For $\lambda > \mathrm{LS}(\mathbf{K})$, The model M is λ-Galois saturated if and only if it is λ-model homogeneous.

Spectrum Problems

Studying non-elementary logics provides with many interesting spectra.
We are able to distinguish among uncountable cardinals on 'mathematical' as opposed to 'combinatorial' grounds.

spectra of P is the set of κ such that some $M \in \mathbf{K}$ with P has cardinality κ.

Tameness

Definition

1. We say **K** is (χ, μ)-*weakly tame* if for any *saturated* $N \in \mathbf{K}$ with $|N| = \mu$ if $p, q, \in \mathcal{S}(N)$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ then $q = p$.
2. We say **K** is (χ, μ)-*tame* if the previous condition holds for all N with cardinality μ.

Tameness Consequences

(χ, ∞)-tameness is an extremely strong property. Consequences for the categoricity spectrum by Grossberg-VanDieren, Shelah, Lessmann, Hyttinen-Kesala.

Tameness Spectrum

Baldwin-Shelah:

There is an AEC with amalgamation which is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame;

Tameness Spectrum

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There is an AEC with amalgamation which is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame;

Baldwin-Kolesnikov:
For each $2 < k < \omega$ there is an $L_{\omega_1,\omega}$-sentence ϕ_k such that:

1. ϕ_k has the disjoint amalgamation property;
2. For $k > 2$,
   1. ϕ_k is $(\aleph_0, \aleph_{k-3})$-tame; indeed, syntactic first-order types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
   2. ϕ_k is not $(\aleph_{k-3}, \aleph_{k-2})$-tame.

Amalgamation Spectra

In this talk we will provide examples of classes $\mathbf{K}$ with non-trivial 'disjoint-amalgamation' spectra.

Amalgamation can arrive late

(JTB) There is a sentence of $L_{\omega_1,\omega}$ that is:

1. joint embedding
2. categorical in all uncountable powers;
3. satisfies amalgamation in all uncountable powers;
4. but fails amalgamation in $\aleph_0$;
5. does not have disjoint amalgamation in any cardinal.

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Let τ contain infinitely many unary predicates P_n and one binary predicate E.
Define a first order theory T so that $P_{n+1}(x) \rightarrow P_n(x)$, E is an equivalence relation with two classes, which are each represented by exactly one point in $P_n - P_{n+1}$ for each n.
Now omit the type of two inequivalent points that satisfy all the P_i.

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Key idea: $M \prec_{\mathbf{K}} N$ requires that a certain predicate is either finite in both models or infinite in both models.

A simplified setting for two examples

The language contains infinitely many predicates $U_{n,i}$ for each n and i.

Each relation is actually on sets: holds only of distinct points and in any order.

All axioms will be universal. So if two models can be amalgamated they can be amalgamated with the existing points.

Defining the class will also require omitting some types.

k-systems for disjoint amalgamation

An independent (λ, k)-system is set of models of cardinality λ indexed by proper subsets u of k such that

1. each $N_u \in \mathbf{K}_k$ and $|N_u| = \lambda$;
2. if $u \subseteq v$ then $N_u \subseteq N_v$:
3. $N_u \cap N_v = N_{u \cap v}$:

$(<\lambda, k)$-amalgamation

K has $(<\lambda, k)$-amalgamation (or the $(<\lambda, k)$-existence property) if for any independent $(<\lambda, k)$-system $\overline{N}$ there is a model $M \in \mathbf{K}$ such that for every $u \subset k_i$, $N^i_u \in \mathbf{K}_{\overline{M}\restriction i}$:

1. M_i extends each N^i_u;
2. $|M_i| = \bigcup_{u \subset k_i} |N^i_u|$.

Special amalgamation

A *special* (λ, k)-system $\overline{N}$ for **K** is a (λ, k)-system with a *special* sequence of elements $\{a_\ell : \ell < k\}$ such that:

1. for each $u \subset k$, $|N_u| = N_\emptyset \cup \{a_\ell : \ell \in u\}$ and
2. $||N_\emptyset|| = \lambda$.

We say that **K** has *special* (λ, k)*-amalgamation* (or the *special* (λ, k)*-existence property*) if any special (λ, k)-system can be amalgamated.

Disjoint amalgamation up to $\aleph_k$

Theorem: (B-Kolesnikov-Shelah

For each natural number k,

1. $\mathbf{K}_k$ has no models of cardinality $> \beth_{k+1}$.
2. $\mathbf{K}_k$ has the disjoint amalgamation property on models of cardinality $\leq \aleph_{k-3}$.
3. $\mathbf{K}_k$ has models of cardinality $\aleph_{k-1}$.
4. there is a $\kappa \leq \beth_{k+1}$ where disjoint amalgamation fails.

Special amalgamation implies disjoint amalgamation

Lemma

If $\mathbf{K} \subset \mathbf{K}_k$ is closed under increasing union and has special (λ, k)-amalgamation then it has (λ, k)-amalgamation.

The $\aleph_k$-example

1. Let τ contain n-ary predicates $U_{n;i}$ for each $n, i < \omega$.
2. For any k, Let $\mathbf{K}_k$ be the class of τ-structures (possibly empty) such that:
    1. for each n, the $U_{n;i}$ partition the n-element subsets of M.
    2. there is no sequence of $k+1$ elements of M that are indiscernible for quantifier-free τ-formulas.

Construction

Disjoint Amalgamation on finite models
Show $\mathbf{K}_k$ has $(< \aleph_0, \leq k)$-amalgamation by force.
That is, just do a Fraïssé construction proving simple amalgamation.
At each stage we have finitely many new tuples that we have to label without creating long indiscernible sets.
We just use new labels.

Disjoint Amalgamation up to $\aleph_k$

Key Step

$(< \aleph_m, k)$-amalgamation implies $(< \aleph_{m+1}, k-1)$-amalgamation.

This is the underlying picture from Shelah's work on excellent classes. (Sketch 2-case on board).

Now induct to get the result.

Bound on Size

The Erdos-Rado theorem yields a bound on the size of models. If we partition the $k+1$ element subsets of a set of cardinality $> \beth_{k+1}$ into countably many sets determined by the $P_{n;i}$, there is an uncountable set homogeneous for the partition.

Disjoint amalgamation up to $\aleph_\alpha$

Theorem: (B-Kolesnikov-Shelah

For each countable ordinal α, there is a class $\mathbf{K}_\alpha$ such that:

1. $\mathbf{K}_\alpha$ has no models of cardinality $> \beth_{\omega_1}$.
2. $\mathbf{K}_\alpha$ has the disjoint amalgamation property on models of cardinality $\leq \aleph_\alpha$.
3. $\mathbf{K}_\alpha$ has models of cardinality $\aleph_\alpha$.
4. there is a $\kappa \leq \beth_{\omega_1}$ where disjoint amalgamation fails.

The upper bound of $\beth_{\omega_1}$ can be made more precise.

The Setting

Fix an ordinal α with $\alpha < \aleph_1$.
Let τ contain unary function symbols $P_{n,\beta}$ for $n < \omega$ and $\beta < \alpha$.
$\mathbf{K}'_\alpha$ is the class of τ-structures M such that $[M]^n$ is partitioned by the $P_{n,\beta}$.

Ranking Indiscernibles

Definition

ID is the set of all types of finite indiscernible sequences. Thus each member of *ID* is a collection p of formulas $P_{1,\alpha_1}(x_1), P_{2,\alpha_2}(x_1, x_2) \ldots P_{m,\alpha_m}(x_1, \ldots x_m)$.

A rank is a *partial* map f from *ID* to α such that if $p \subset q$, $f(p) > f(q)$.

Punchline first

If every finite indiscernible sequence in a model M is ranked, then $|M| < \beth_{\omega_1}$ since if it is larger it contains an infinite indiscernible sequence by Morley. But then f gives us a decreasing sequence of ordinals.

Notations

An approximation pair $(\mathcal{M}, f)$ consists of a set $\mathcal{M}$ of μ finite models and a rank f such that every p that is realized in $\mathcal{M}$ is in the domain of f.

A special $(\aleph_\beta, k, f)$-system is a special $(\aleph_\beta, k)$-system that is ranked by f.

Formal Indiscernibility

Let $\overline{N}$ be a special $(\aleph_\beta, \mathcal{P}^-(k), f)$-system.

1. We say $\langle a_1, \ldots a_k\rangle$ is *formally indiscernible* if for every $u, v \subset k$ with $|u| = |v|$, letting $a_u = \langle a_i : i \in u\rangle$, $\mathrm{tp}_{qf}(a_u, N_u) = \mathrm{tp}_{qf}(a_v, N_v)$.
2. For any class of models $\mathcal{M}$, $\overline{N}$ is an $\mathcal{M}$-special system if each $N_u \in \mathcal{M}$.

$(\aleph_\beta, k, f)$-existence

We say that $(\mathcal{M}, f)$ has the $(\aleph_\beta, k, f)$-existence property if for every special $(\aleph_\beta, k, f)$-system $\overline{N}$ and every $\gamma \geq \beta$:

if **a** is formally indiscernible and for one (any) $u \subset k$ with $|u| = k - 1$, $f(a_u) > \gamma$
then there is an amalgam N' of $\overline{N}$ with $f(\mathbf{a}) \geq \gamma$.

Taking Stock

The rest of the proof has two steps. We first show that there is a nice class of finite models. Then we show it is 'nice' up $\aleph_\alpha$.

Main Construction

Lemma There is an approximation pair $(\mathcal{M}, f)$ such that $(\mathcal{M}, f)$ satisfies the $(< \aleph_0, k, f)$-existence property for all $k < \omega$.

Proof Sketch

Begin induction: Let $\mathcal{M}_0$ be any single model M in $\mathbf{K}_k$ and f any rank defined on the set of $p \in ID$ that are realized in M. The key point is:

Inductive Step

Claim

Suppose $(\mathcal{M}, f)$ is a approximation pair. Then there is a approximation pair $(\mathcal{M}', f')$ extending $(\mathcal{M}, f)$ such that any standard special $(< \aleph_0, k, f)$-system, $\overline{N}$, with each $N^u \in \mathcal{M}$, with respect to $\{a_\ell : \ell < k\}$, $\overline{N}$ can be amalgamated as a $\mathbf{K}_{\mathcal{M}'}$-special system.

The proof is by induction on β.

Move f-amalgamation up

Lemma

If $(\mathcal{M}, f)$ satisfies the $(< \aleph_0, k, f)$-existence property for all $k < \omega$, then for any $\beta < \alpha$, $(\mathcal{M}, f)$ satisfies the $(\aleph_\beta, k, f)$-existence property for all $k < \omega$.

induction on β.

At the successor stage we invoke:

Lemma

If $(\mathcal{M}, f)$ satisfies the $(\aleph_\beta, k, f)$-existence property then it satisfies the $(\aleph_{\beta+1}, k-1, f)$-existence property.

From f-amalgamation to amalgamation

Theorem

Suppose $(\mathcal{M}, f)$ is defined as above. If $\beta < \alpha$ any two models of cardinality $\aleph_\beta$ can be disjointly amalgamated.

Disjoint amalgamation up to $\beth_\alpha$

Using a generalization of Martin's axiom one can make the disjoint amalgamation go up to $\beth_\alpha$.
There are still only a set of models.