Using Set theory in model theory

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Today's Topics

Smallness

Definition

1. A τ-structure M is L^*-small for L^* a countable fragment of $L_{\omega_1,\omega}(\tau)$ if M realizes only countably many $L^*(\tau)$-types (i.e. only countably many $L^*(\tau)$-n-types for each $n < \omega$).
2. A τ-structure M is called small or $L_{\omega_1,\omega}$-small if M realizes only countably many $L_{\omega_1,\omega}(\tau)$-types.

Why Smallness matters

Fact

1. Each small model satisfies a Scott-sentence, a complete sentence of $L_{\omega_1,\omega}$.
2. There is a 1-1 correspondence between the models of Scott sentence in a vocabulary τ and the class of atomic models of a first order theory T in an expanded vocabulary τ^*.

The Theorem

Main Theorem

If $\boldsymbol{K}_T$ fails 'density of pseduominimal types' (algebraic symmetry) then $\boldsymbol{K}_T$ has $2^{\aleph_1}$ models of cardinality $\aleph_1$.

Proof Outline

1. Start with a model $\mathcal{N}_0$ of enough set theory and an infinitary τ-sentence ψ that fails algebraic symmetry.
2. Force a generic extension $\mathcal{N}_1$ of $\mathcal{N}_0$ that satisfies Martin' axiom, MA.
3. Expand the vocabulary τ to a τ^* that allows the description of filtrations and define an $L_{\omega_1,\omega}(Q)$ τ^*-formula $\theta(P_1, P_2)$ that relies on the properties of the pseudoclosure.

Proof Outline Continued

4. In $\mathcal{N}_1$, using the fact that ψ ‘fails algebraic symmetry’, force with an $\aleph_1$-like dense linear order to get a generic filter G. Conclude that in $\mathcal{N}_1$ for each pair of stationary sets S, T there is a model $M^{S,T}[G]$ such that if $M^{S,T}[G] \models \theta(P_1, P_2)$ then $P_1 \cap P_2 = \emptyset$, $S \subseteq P_1$ and $T \subseteq P_2$
5. Expand $\mathcal{N}_1$ to a vocabulary τ' (including ϵ) by interpreting the symbols of τ^* on the model constructed in step 4. Construct an elementary extension $\mathcal{N}_2$ of $\mathcal{N}_1$ such that ‘stationary’ is absolute between $\mathcal{N}_2$ and V.
6. In $\mathcal{N}_2$ choose $2^{\aleph_1}$ pairs of stationary sets (S^η, T^η) such that the entire set of S^η, T^η are pairwise disjoint modulo the ideal on non-stationary sets. This implies the $M^{S^\eta,T^\eta}[G]$ are pairwise non-isomorphic in $\mathcal{N}_2$. Since stationary is absolute between $\mathcal{N}_2$ and V, in V there are $2^{\aleph_1}$ models of ψ.

The class of models

$\boldsymbol{K}_T$ is the class of atomic models of the countable first order theory T.

Definition

The atomic class $\boldsymbol{K}_T$ is extendible if there is a pair $M \preceq N$ of countable, atomic models, with $N \neq M$.

Equivalently, $\boldsymbol{K}_T$ is extendible if and only if there is an uncountable, atomic model of T.

We assume throughout that $\boldsymbol{K}_T$ is extendible. We work in the monster model of T, which is usually not atomic.

A new notion of closure

Definition

An atomic tuple **c** is in the pseudo-algebraic closure of the finite, atomic set B ($\mathbf{c} \in \mathrm{pcl}(B)$)
if for every atomic model M such that $B \subseteq M$, and $M\mathbf{c}$ is atomic, $\mathbf{c} \subseteq M$.

When this occurs, and **b** is any enumeration of B and $p(\mathbf{x}, \mathbf{y})$ is the complete type of **cb**, we say that $p(\mathbf{x}, \mathbf{b})$ is pseudo-algebraic.

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Example I

Our notion, pcl of algebraic differs from the classical first-order notion of algebraic as the following examples show:

Example

Suppose that an atomic model M consists of two sorts. The *U*-part is countable, but non-extendible (e.g., *U* infinite, and has a successor function *S* on it, in which every element has a unique predecessor). On the other sort, *V* is an infinite set with no structure (hence arbitrarily large atomic models). Then, if an element $x_0 \in U$ is not algebraic over $\emptyset$ in the normal sense but is in $\mathrm{pcl}(\emptyset)$.

Example II

Example

Let $L = A, B, \pi, S$ and T say that A and B partition the universe with B infinite, $\pi : A \to B$ is a total surjective function and S is a successor function on A such that every π-fiber is the union of S-components. K_T is the class of $M \models T$ such that every π-fiber contains exactly one S-component. Now choose elements $a, b \in M$ for such an M such that $a \in A$ and $b \in B$ and $\pi(a) = b$. Clearly, a is not algebraic over b in the classical sense, but $a \in \mathrm{pcl}(\mathrm{b})$.

Definability of pseudo-algebraic closure

Strong ω-homogeneity of the monster model of T yields:

Fact

If $p(\mathbf{x}, \mathbf{y})$ is the complete type of $\mathbf{cb}$, then

$$\mathbf{c} \in \mathrm{pcl}(\mathbf{b}) \quad \text{if and only if} \quad \mathbf{c}' \in \mathrm{pcl}(\mathbf{b}')$$

for any $\mathbf{c}'\mathbf{b}'$ realizing $p(\mathbf{x}, \mathbf{y})$. In particular, the truth of $c \in \mathrm{pcl}(\mathbf{b})$ does not depend on an ambient atomic model.

Further, since a model which atomic over the empty set is also atomic over any finite subset, moving M to N we have:

Fact

If $\mathbf{c} \notin \mathrm{pcl}(\mathrm{B})$, witnessed by M then for every countable, atomic $N \supset B$, there is a realization $\mathbf{c}'$ of $p(\mathbf{x}, B)$ such that $\mathbf{c}' \not\subseteq N$.

Stronger Version

Lemma

Let N be an atomic model containing $\mathbf{b}\boldsymbol{a}$. If $\mathbf{b}$ is not pseudoalgebraic over $\boldsymbol{a}$ then $\mathrm{tp}(\mathbf{b}/\boldsymbol{a})$ is realized in $N - \mathrm{pcl}(\boldsymbol{a}\mathbf{b})$.

Proof. Let M_1 be a countable submodel of N containing $\boldsymbol{a}\mathbf{b}$ and M_0 an elementary submodel of M_1 containing $\boldsymbol{a}$ but not $\mathbf{b}$. Note $M_0 \approx M_1$. Let M_2 be the image of M_1 under an automorphism f of the monster taking M_0 to M_1. Then $f(\mathbf{b})$ is not in $\mathrm{pcl}(\boldsymbol{a}\mathbf{b})$ (It's in $M_2 - M_1$.). Since M_2 is atomic over $\boldsymbol{a}\mathbf{b}$, there is an embedding g of M_2 into N realizing $\mathrm{tp}(\mathbf{b}/\boldsymbol{a})$ by $g(f(\mathbf{b}))$ not in $\mathrm{pcl}(\boldsymbol{a}\mathbf{b})$ since pcl is invariant under automorphism.

pcl = qcl

Lemma

$a \in \mathrm{pcl}(\mathbf{b})$ if and only $\mathrm{tp}(a/\mathbf{b})$ is realized only countably many times in any model of T.

Iterating the last result, a type not in the pseudoclosure is realized arbitrarily often.
But if $p(x, \mathbf{b})$ is pseudoalgebraic, all realizations of p must be in any countable model M containing **b**.

Countable closure property

Lemma

For any finite $\boldsymbol{a}$, any $N \in \boldsymbol{K}_T$, $\mathrm{pcl}_\mathrm{N}(\boldsymbol{a}) = \mathrm{N} \cap \mathrm{pcl}(\mathrm{a})$ satisfies $|\mathrm{pcl}_\mathrm{N}(\mathrm{a})| = \aleph_0$.

Proof. By the last Lemma, if **b** is algebraic over $\boldsymbol{a}$ then for any $N \in \boldsymbol{K}_T$ (i.e. N is atomic), $\mathrm{tp}(\mathbf{b}/\boldsymbol{a})$ is realized only countably many times in N. Whether $\mathbf{b} \in \mathrm{pcl}_\mathrm{N}(\boldsymbol{a})$ depends, by the remark after Definition 8, only on $\mathrm{tp}(\boldsymbol{a}\mathbf{b})$.
This type must be atomic and there are only countably many atomic types of finite sequences. So $\mathrm{pcl}_\mathrm{N}(\boldsymbol{a})$ is countable.

quasiclosure not quasiminimal

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Pseudo-minimal sets

Definition

1. A possibly incomplete type q over $\mathbf{b}$ is pseudominimal if for any finite, $\mathbf{b}^* \supseteq \mathbf{b}$, $\boldsymbol{a} \models q$, and $\mathbf{c}$ such that $\mathbf{b}^*\mathbf{c}\boldsymbol{a}$ is atomic, if $\mathbf{c} \subset \mathrm{pcl}(\mathbf{b}^*\boldsymbol{a})$, and $\mathbf{c} \notin \mathrm{pcl}(\mathbf{b}^*)$, then $\boldsymbol{a} \in \mathrm{pcl}(\mathbf{b}^*\mathbf{c})$.
2. M is pseudominimal if $x = x$ is pseudominimal in M.

‘Density’

Definition

$\boldsymbol{K}_T$ satisfies <u>‘density’ of pseudominimal types</u> if for every atomic **e** and atomic type $p(\mathbf{e}, \mathbf{x})$ there is a **b** with **eb** atomic and $q(\mathbf{e}, \mathbf{b}, \mathbf{x})$ extending p such that q is pseudominimal.

Failing 'density'

Lemma

$\boldsymbol{K}_T$ fails 'density' of pseudominimal types if, after naming a finite tuple $\mathbf{e}$, there is a complete 1-type $\tilde{p}(x)$ over $\mathbf{e}$ such that
for any finite, atomic $\mathbf{b}$ containing $\mathbf{e}$ and complete $q(\mathbf{e}, \mathbf{b}, \mathbf{x})$ extending $\tilde{p}$ there are a finite atomic $\mathbf{b}^* \supset \mathbf{b}$, $\boldsymbol{a} \models q$, and $\mathbf{c}$ such that
$\mathbf{b}^*\mathbf{c}\boldsymbol{a}$ is atomic, $\mathbf{c} \subset \mathrm{pcl}(\mathbf{b}^*\boldsymbol{a})$, $\mathbf{c} \not\in \mathrm{pcl}(\mathbf{b}^*)$, and $\boldsymbol{a} \not\in \mathrm{pcl}(\mathbf{b}^*)$, but $\boldsymbol{a} \not\in \mathrm{pcl}(\mathbf{b}^*\mathbf{c})$.

Shelah calls this notion 'failure of algebraic symmetry'.

Finitely at-saturated; striations of models

Definition

1. Given a countable atomic M, a countable, atomic $N \succeq M$ is an at-finitely saturated extension of M if, for every finite $\mathbf{b} \subseteq M$ and every non-algebraic $p(\mathbf{x}, \mathbf{b})$, there is a realization $\mathbf{c}$ in N with $\mathbf{c} \not\subseteq M$.
2. An at-finitely saturated chain is an ω-sequence $M_0 \preceq M_1 \preceq \ldots$ of countable, atomic models with M_{n+1} an at-finitely saturated extension of M_n.
3. A striation of a countable, atomic M is an at-finitely saturated chain $\langle N_n : n \in \omega \rangle$ with $M = \bigcup_{n \in \omega} N_n$.

Striating models

Lemma

1. Every countable atomic M has a countable, atomic, at-finitely saturated extension N:
2. For every countable, atomic model M, there is an at-finitely saturated chain $\langle M_n : n \in \omega \rangle$ with $M_0 = M$;
3. Every countable, atomic model M has a striation.

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Proof that Striations Exist

Proof

1. Given a countable, atomic model M, let $\langle p_n(\mathbf{x}, \mathbf{b}) : n < \omega \rangle$ enumerate all complete, non-algebraic types with $\mathbf{b}$ from M. Now form a sequence $\langle N_n : n \in \omega \rangle$ with $N_0 = M$, and, for each n, N_{n+1} contains, by Lemma 11 a realization of p_n that is not contained in N_n (hence not contained in M). Then let $N = \bigcup_{n \in \omega} N_n$.
2. Iterate (1) ω times.
3. Follows from (2) and the fact that any two countable, atomic models are isomorphic.

Striated Sequences

Definition

A striated sequence $\langle \mathbf{b}_k : k < m \rangle$ of length m is a sequence of finite, atomic sequences
$\mathbf{b}_k = \langle b_{k,0}, \ldots b_{k,n_k} \rangle$, where, for each $k < m$,

1. $\mathrm{tp}(b_{k,0}/\bigcup_{i<k} \mathbf{b}_i)$ is non-algebraic and
2. $\mathbf{b}_k \in \mathrm{pcl}(\bigcup_{\mathrm{i}<\mathrm{k}} \mathbf{b}_{\mathrm{i}} \cup \{\mathrm{b}_{\mathrm{k},0}\})$ (where $b_{k,0}$ is the first element of $\mathbf{b}_k$).

A striated type $p(\mathbf{x}_k : k < m)$ is the type of a striated sequence.

Any at-finitely saturated chain $\langle M_n : n \in \omega \rangle$ of length m faithfully realizes every striated type of length m.

Given any finite, atomic set B, it is easy to choose a striated sequence $\langle \mathbf{b}_k : k < m \rangle$ with $B = \bigcup_{k<m} \mathbf{b}_k$. However, this process is not unique (even the m can vary). Return

Main Theorem

Goal Theorem

If $\boldsymbol{K}_T$ fails 'density of pseudominimal types' then $\boldsymbol{K}_T$ has $2^{\aleph_1}$ models of cardinality $\aleph_1$.

We prove this in two steps

1. Force the existence of many models in a model of set theory satisfying Martin's axiom
2. Show that the result is absolute

Goal of the Forcing

We will construct many models in two steps. In the first, we work in the model $\mathcal{N}_1$ of ZFC° + MA and show how to construct for a pair of disjoint stationary sets S, T a model $M^{S,T}$ such that $M^{S,T} \models \theta(S, T)$.

θ will satisfy that if S_1, S_2 are each stationary subsets of $\aleph_1$ and $S_1 - S_2$ is stationary and both M^{S_1,T_1} and M^{S_2,T_2} satisfy $\theta(S_i, T_i)$ then $M^{S_1,T_1} \not\approx M^{S_2,T_2}$.

Properties of $\theta(S, T)$

Given $\psi \in L_{\omega_1,\omega}(\tau)$, the formula $\theta(S, T) \in L_{\omega_1,\omega}(Q)(\tau^*)$ holds of model $M^* \in \tau^*$ if

1. $M^* \restriction \tau \models \psi$
2. M^* admits a filtration as described below.
3. implies a first order τ^*-formula $\theta_1(P_1, P_2)$ which expresses:
    a. If $\alpha \in P_1$ then there is an $a \in M - M_{J_\alpha}$ which catches M_{J_α} but does not strongly catch M_{J_α}.
    b. If $\alpha \in C - (P_1 \cup P_2)$ every $a \in M - M_{J_\alpha}$ which catches M_{J_α} strongly catches M_{J_α}.

More detail

$M^{S,T}$ is actually an expansion of $M[G]$ for a generic set G built by the following forcing. But since $\mathcal{N}_1$ satisfies Martin's axiom $G \in \mathcal{N}$ so S and T remain stationary.
In a model $\mathcal{N}_1$ of ZFC° + MA we construct a pair of disjoint stationary sets S, T and a model $M^{S,T}$ such that $M^{S,T} \models \theta(S, T)$.

This implies such models are not isomorphic when $S \cap T$ is stationary. $M^{S,T}$ is actually an expansion of $M[G]$ for a generic set G built by the following forcing. But since $\mathcal{N}_1$ satisfies Martin's axiom $G \in \mathcal{N}$ so S and T remain stationary.

Relevant Ordered Sets

Definition

Considers linear orders I equipped with a subset P and a binary relation E such that

1. I is $\aleph_1$-like with first element.
2. E is an equivalence relation on I such that
   - a. If t is min(I) or in P, t/E is $\{t\}$
   - b. Otherwise t/E is convex dense subset of L with neither first nor last element.
3. I/E is a dense linear order such that both $\{t/E : t \in P\}$ and $\{t/E : t \notin P\}$ are dense in it,

Setting the stage

For each $\aleph_1$-like dense linear order $(\boldsymbol{I}, E, P)$ with first point and E an equivalence relation as just described, we define a specific quasiorder $\mathbb{Q}_I$. We will force with this quasiorder and obtain a model M_I.

Conditions are atomic formulas in variables $x_{t,n}$ for $t \in \boldsymbol{I}$ and $n < \omega$. Envision constructing a model whose universe is named by the $x_{t,n}$. The variables with fixed t will be contained in the algebraic closure of $\{x_{t,0}\} \cup \{x_{s,n} : s < t, n < n_s\}$.

The forcing conditions

Suppose $\boldsymbol{K}_T$ fails 'density' of pseudominimal types, witnessed by $\tilde{p}$.

Definition

$\mathbb{Q}_{\boldsymbol{I}}$ **defined:** Let I be an $\aleph_1$-like dense linear order with minimal element $\min(I)$. $p \in \mathbb{Q}_I$ if and only if the following conditions hold.

1. The variables of p are $\mathbf{x_n} = \{x_{t,i} : i < n_t, t \in u\}$ where $u = u_p$ is a finite subset of $\boldsymbol{I}$ and $\mathbf{n}_p = \langle n_t : t \in u\rangle$ gives the number of variables of p at each level t. Sometimes we write $\mathbf{x}_p$ to denote the variables appearing in the condition p.)

The forcing conditions continued

2. $p(\mathbf{x_n})$ is a principal type in T over $\emptyset$.
3. If $t \in P$ and $t \in u_p$, $\tilde{p}(x_{t,0}) \in p$.
4. p 'says' $x_{t,0}$ is not algebraic over $\{x_{s,\ell} : s <_I t, \ell < n_s\}$.
5. p 'says' $x_{t,i}$ is algebraic over $\{x_{s,\ell} : s <_I t, \ell < n_s\} \cup \{x_{t,0}\}$ for $i < n_t$.

Basic Properties to get atomic models

Claim

For each dense $\aleph_1$-like linear order $\boldsymbol{I}$, $\mathbb{Q}_{\boldsymbol{I}}$ is a ccc partial order.

Now we list the crucial ‘constraints’. These ‘constraints’ are collections of conditions, which we will prove to be dense and open in $\mathbb{Q}$. In stating the constraints we will use the linear order $\boldsymbol{I}$ and the predicates P and E.

Henkin Constraints

Henkin Witness Constraints

For any $\mathbf{n} = \langle n_t : t \in u \rangle$ for u a finite subset of I, and any formula $\phi(y, \mathbf{x_n})$ where $\mathbf{x_n} = \{x_{t,n} : t \in u, n < n_t\}$, we define the following sets of constraints.

i: Henkin witnesses For any $s \in I$, the following is a constraint:

$\mathcal{I}_{\phi,s}$ is the set of $p \in \mathbb{Q}$ such that:

1. $\mathrm{dom}(\mathbf{n}) = u \subseteq u_p$.
2. $t \in u$ implies $n_t \leq n_{p,t}$ so $\mathbf{x_n} \subset \mathbf{x}_p$.
3. For some (t_1, n_1),

$$p(\mathbf{x}_p) \vdash (\exists y)\phi(y, \mathbf{x}_n) \rightarrow \phi(x_{t_1,n_1}, \mathbf{x}_n).$$

Henkin Constraints continued

The location of t_1 in the order I depends on how ϕ affects the relative algebraicity of $\mathbf{x_n}$.

4. $\phi(y, \mathbf{x_n})$ implies y is algebraic in some $\mathbf{x'_n} \subset \mathbf{x_n}$, $\mathbf{x'_n}$ is minimal such and r is maximal so that some $x_{r,m}$ occurs in $\mathbf{x'_n}$. Then $t_1 = r$.
5. $\phi(y, \mathbf{x_n})$ implies y is not algebraic in any $\mathbf{x_n}$. The t_1 is above u and $t_1 < s$.

Fullness constraints

Obtaining a full model in following sense motivates one family of constraints.

Definition

A model M with uncountable cardinality is said to be λ-full if for every $\mathbf{a} \in M$ every non-algebraic $p \in S_{at}(\mathbf{a})$ is realized at least λ-times in M.

fullness $\mathcal{I}^1_{p,s} = \{q :$
q is incompatible with $p \restriction I_{<s}$ or there is $p_1 \leq_I q$ with p, p_1 isomorphic over $I_{<s}\}$.

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Prescribed Uncountable models exist

Theorem

If T has an uncountable atomic model, the 'Henkin constant' constraints and the 'fullness' constraints are dense-open. Thus there is an uncountable full model in $\boldsymbol{K}_T$.
Proof. The Henkin witness constraints are dense open. Open is immediate since the ordering is provability.

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The 'prescribed' refers to the skeleton of I.

Density of Henkin Constraints

For density, consider $\mathcal{I}_{\phi,s}$. Let $q(\mathbf{x_n}) \in \mathbb{Q}_I$ and suppose $(\exists y)\phi(y, \mathbf{x_n}) \in q$. Suppose $\phi(y, \mathbf{x_n})$ implies y is algebraic in some $\mathbf{x}'_\mathbf{n} \subset \mathbf{x_n}$, $\mathbf{x}'_\mathbf{n}$ is minimal such and r is maximal so that some $x_{r,m}$ occurs in $\mathbf{x}'_\mathbf{n}$. Let $u_p = u_q$; add x_{r,n_r+1} to the variables of $\mathbf{x_n}$, and let p be any completion of $q \cup \{\phi(x_{r,n_r+1}, \mathbf{x_n})\}$. Note that x_{r,n_r+1} is algebraic in $x_{r,0}$ and points indexed below r; since q is a condition, so is p.

Suppose $\phi(y, \mathbf{x_n})$ implies y is not algebraic in $\mathbf{x_n}$. Choose t_1 above u with $t_1 < s$ and $t_1 \notin P$. Now form p by adding: $\phi(x_{t_1,0}, \mathbf{x_n})$ to q.

Blocking Strong Catching Constraints

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blocking strong catching

$\mathcal{I}^1_{t,P,s_0}$ is the set of $q \in \mathbb{Q}$ such that:
There exists $s_1 \in u_q$ with $s_0 < s_1 < t$ and $\neg E(s_0, s_1)$ such that q says $x_{s_1,0} \in \mathrm{acl}(\{x_{t,0} \cup \{x_{s,n} : s \leq s_0, s \in u_q, n < n_{q,s}\}$

Return

The ‘blocking strong catching’ conditions are dense.
Consider $\mathcal{I}_{t,P,s_0}$.

Blocking Strong Catching Constraints are dense

The relevant p are those such that $t, s_0 \in u_p$, $P(t)$ holds in the structure on I and $t > s_0 > u_p \cap I_{<t}$. Choose s_1 with $s_0 < s_1 < t$, $s_1 \notin P$, $\neg E(s_0, s_1)$ and $\neg E(r, s_1)$ for any $r \in u_p$.

Failure of density of pseudominimal types can be written: There is a $\tilde{p}(x)$ such that for any consistent complete atomic type $\tilde{q}(\mathbf{y}, \mathbf{x})$ extending $\tilde{p}$ there is an $\tilde{r}(\mathbf{y}, \mathbf{z}, u, x)$ that implies: $p(x)$, $q(\mathbf{y}, x)$, $x \notin \mathrm{pcl}(\mathbf{yz})$ and

$$u \in \mathrm{pcl}(\mathbf{yz}\mathrm{x}) \text{ but } \mathrm{x} \notin \mathrm{pcl}(\mathbf{yz}\mathrm{u}).$$

Take x as the singleton $x_{t,0}$ and $\mathbf{y}$ as the variables $x_{s,m}$ with $s \in u_p \cap I_{s_0}$ and $\tilde{q}$ as the condition p restricted to these variables and find $\tilde{r}$ Then take u as $x_{s_1,0}$ and assign the variables in $\mathbf{z}$ to $x_{r,i}$ for $r < s_1$. Now let the extension q of the condition p be $p \cup \tilde{r}(\mathbf{y}, \mathbf{z}, u, x)$. Since $\neg P(s_1)$ it doesn't matter what type $x_{s_1,0}$ realizes over the empty set.

Coding $\aleph_1$-like linear orders

Definition

Let I be an $\aleph_1$-like dense linear order.

1. $\overline{J} = \langle J_\alpha : \alpha < \aleph_1 \rangle$ is a <u>decomposition</u> of I if it is a $\subset$-increasing continuous sequence of countable initial segments of I without last element whose union is I.
2. If the linear order I is equipped with an equivalence relation E whose classes are countable convex subsets of I, the initial segments of I in the decompostion must respect the equivalence relation. (For every $a \in I$ and every α, either J_α is disjoint from a/E or contains a/E.) We call this an <u>E-decomposition</u>.
3. For $\alpha < \aleph_1$, let t_α be the least upper bound of the J_α (if there is one).
4. Then let $\mathrm{stat}(\overline{J}) = \{\alpha : t_\alpha$ is well-defined $\}$

Coding of stationary sets in linear order: result

Lemma

Suppose I is an $\aleph_1$-like dense linear order.

1. If $\overline{J}^1, \overline{J}^2$ are two decompositions of I then $\{\alpha : J^1_\alpha = J^2_\alpha\}$ is closed and unbounded.
2. We can set $\mathrm{stat}(I)$ as $\mathrm{stat}(\overline{J})$ for some (any) decomposition $\overline{J}$ and the value is the same up to the filter of closed unbounded sets.
3. For any stationary S, there is an $\aleph_1$-like dense linear order with $\mathrm{stat}(I) = S$.
4. If $\mathrm{stat}(I^1) = \mathrm{stat}(I^2)$ (mod cub filter) then I^1 and I^2 are order-isomorphic.

Coding of stationary sets in models

Definition

Suppose ***I*** is an $\aleph_1$-like dense linear order. *M* has an ***I***-filtration if *M* is a model of cardinality $\aleph_1$ and for some decomposition $\overline{J}$ of ***I***, for each $\alpha < \aleph_1$, there is a model $M_{J_\alpha} \prec M$ such that *M* is a continuous increasing union of the M_{J_α}.

We will build models with ***I***-filtrations unfortunately can't quite recover the stationary set which determines the linear order.

Coding by Catching and Strong Catching

Definition

Let $M \prec N \in \boldsymbol{K}_T$ and $a \in N - M$.

1. We say that a catches M in N if $b \in \mathrm{pcl}(\mathrm{Ma}, \mathrm{N}) - \mathrm{M}$ implies $a \in \mathrm{pcl}(\mathrm{Mb}, \mathrm{N})$.
2. If M has an $\boldsymbol{I}$ filtration and J is an initial segment of I, we say that a strongly catches M_J in M if $a \in M$ catches M_J in M and for every large enough $s \in J$,

$$\mathrm{pcl}(\mathrm{M}_{<\mathrm{s}}\mathrm{a}) \cap \mathrm{M}_{\mathrm{J}} = \mathrm{M}_{<\mathrm{s}}.$$

Coding by Catching and Strong Catching: limit points catch but don't strongly catch

Lemma: Catch not strongly catch

Suppose $M = M_G$. If J is an initial segment of $\boldsymbol{I}$ which has a least upper bound in $M - M_J$, there is an $a \in M - M_J$ such that a catches M_J but a does not strongly catch M_J.

Coding by Catching but not Strong Catching: Proof

Level Constraints

Suppose $t = \sup J$. We claim that $a_{t,0}$ catches M_J but does not strongly catch M_J. For catches, it suffices to show each $a_{t,n} \notin M_J$ satisfies $a_{t,0} \in \mathrm{acl}(M_J a_{t,n})$. If not, by the 'level constraint', $a_{t,n} = a_{s,m}$ for some $s \in J$; but $a_{t,n} \notin M_J$.

Block Strong Catching Constraints

To show $a_{t,0}$ does not strongly catch M_J, choose any $s_0 < t$. By the blocking strong catching condition $\mathcal{I}_{t,P,s_0}$, there is a condition $q \in G$ and there exists $s_1 \in u_q$ with $s_0 < s_1 < t$ and $\neg E(s_0, s_1)$ and such that q says

$x_{s_1,0} \in \mathrm{acl}(\{x_{t,0}\} \cup \{x_{s,n} : s \leq s_0, s \in u_q, n < n_{q,s}\})$.

Since the decomposition respects E, $s_1 \notin M_J$. Thus for arbitrarily large $s_0 < t$, $\mathrm{acl}(a_{t,0} M_{<s_0} \cap M_J) \not\subseteq M_{<s_0}$. So $a_{t,0}$ does not strongly catch M_J.

Coding by Catching and Strong Catching: (almost) if no sup, catch implies strongly catch

Lemma: Catch implies strongly catch

If J is an initial segment of $\boldsymbol{I}$ with no least upper bound and with no least E- class above J and $b \in M - M_J$ catches M_J then b strongly catches M_J.

Suppose $b \in N - M_J$ catches M_J; we will show b strongly catches M_J. For some t and n, b instantiates $x_{t,n}$ so for some p, p forces that $b = a_{t,n}$ and b catches M_{J_α} in $M = M[G]$. So $t \in I \setminus J_\alpha$, $p \Vdash a_{t,n} \neq a_{r,m}$ if $r \in J_\alpha$ and $m \in \mathbb{N}$.

Since b does not strongly catch M_J there is $s \in J_\alpha$ with s above $u_p \cap J_\alpha$ but

$$p \not\Vdash \text{`}\mathrm{acl}(bM_{<s}[G], M_G) \cap M_{J_\alpha}[G] \subseteq M_{<s}[G].\text{'}$$

Some $p_1 \in \mathbb{Q}_{\boldsymbol{I}}$ above p forces the strong catching to fail.

Catch implies Strong Catch: proof continued

Let

$$A = \{a_{s_1,n} : s_1 \in \mathrm{dom}(p_1) \cap I_{<s}, n < n_{p_1,s_1}\}.$$

Without loss of generality p_1 forces '$\mathrm{acl}(bA, M_G) \cap M_{J_\alpha}[G] \not\subseteq M_{<s}[G]$'.

Choose s' with $J < s' < t$ and $\neg E(s', t)$ and by the density of P in I/E, a t' with $J < t' < t$ and $P(t')$.

There is an automorphism π of $\boldsymbol{I}$ such that π fixes P, and each of $I_{\geq t}$, u_p and $\mathrm{dom}(p_1) \cap (J)_{<}s$ setwise but $\pi(s) = s' \notin J$. But now $\pi(p_1)$ forces that $b = a_{t,n}$ does not catch M_J in $M = M[G]$. To see this note that $p \leq \pi(p_1)$.

Defining $I^{S,T}$

Fix a partition of $\aleph_1$ into stationary sets S, T, W of $\aleph_1$. Define a linear order $I = I_{S,T}$ as the sum of ordered sets I_α so that:

Fact

Note that in $I^{S,T}$

1. if $\alpha \in S$ then $J_\alpha = \cup_{\beta<\alpha} I_\beta$ has a least upper bound.
2. if $\alpha \in T$ then $J_\alpha = \cup_{\beta<\alpha} I_\beta$ has no least upper bound and there is no least E equivalence class above J_α.
3. if $\alpha \in W$ then $J_\alpha = \cup_{\beta<\alpha} I_\beta$ has no least upper bound but there is a least E equivalence class above J_α.

$\{$

Defining the sentence θ

Notation

Let ψ be a sentence in $L_{\omega_1,\omega}(\tau)$.
In an expanded language τ^+, the sentence $\theta_0 \in L_{\omega_1,\omega}(Q)(\tau^+)$ describes a decomposition of M as we have described.
In a still further expansion to τ^* by adding predicates P_1, P_2, there is a first order τ^*-formula $\theta_1(P_1, P_2)$ which expresses:

a. If $\alpha \in P_1$ then there is an $a \in M - M_{J_\alpha}$ which catches M_{J_α} but does not strongly catch M_{J_α}.
b. If $\alpha \in C - (P_1 \cup P_2)$ every $a \in M - M_{J_\alpha}$ which catches M_{J_α} strongly catches M_{J_α}.

The Crux

Lemma

[$M^{S,T}$]: Forcing with respect to the order $\mathbb{Q}_{I^{S,T}}$ in a model of set theory which satisfies MA yields a model $M^{S,T}[G]$ such $M^{S,T}[G] \models \theta(S, T)$.

Proof. The conditions in $\mathbb{Q}$ determine the τ-diagram of $M^{S,T}[G]$. We must expand to a τ^* structure satisfying $\theta(S, T)$. We interpret L as the set $\{a_{t,0} : t \in I\}$. We define C by choosing one t from each I_α and put $a_{t,0}$ in C. Define R_1 so that $J_\alpha = \bigcup_{\beta<\alpha} I_\beta$ and $R_2(s, a_{t,j})$ if there exists p and $s_1 \leq s$ such that for some m, n, $p \Vdash a_{s_1,m} = a_{t,n}$. Now given disjoint stationary subsets S, T of $\aleph_1$ interpret P_1, P_2 as S, T.

The Crux: Proof cont

We now have a τ^*-structure. Compare with the definition of $\theta(P_1, P_2)$. The initial segments J_α where $\alpha \in S$ have least upper bounds and so there is an element which catches but does not strongly catch M_{J_α}.

But for $\alpha \in T$, J_α has no least upper bounds and no least upper bound in I/E so every element which catches M_{J_α} also strongly catches M_{J_α}.

Thus interpreting P_1 as S and P_2 as T, we obtain τ^*-structure and

$$M^{S,T}[G] \models \theta(S, T).$$

Getting to ZFC

Start with a countable transitive model $\mathcal{N}_0$ of ZFC° . Force to get a model $\mathcal{N}_1$ of *MA*. Now force in $\mathcal{N}_1$ with the forcing condition $\mathbb{Q}_{I^{S,T}}$ (from Definition 28) for a pair of disjoint stationary sets S, T and the associated $\aleph_1$-dense linear ordering $I^{S,T}$. Since $\mathcal{N}_1$ satisfies *MA* and *P* is ccc, there is a generic *G* in $\mathcal{N}_1$.
Expand $\mathcal{N}_1$ to include a predicate *M* and τ^* as well as ϵ; call this vocabulary τ'. Interpret *M* as $M^{S,T}$, the symbols of τ^+ to code the decomposition of $M^{S,T}$, and P_1, P_2 as S, T as in the proof of Lemma 48. We chose θ so $M^{S,T}[G] \models \theta(S, T)$.

The iteration

Now construct an $\aleph_1$-sequence of τ'-elementary extensions, $\mathcal{N}'_\alpha$ by the ultralimit construction or by Hutchinson's methods from the 70's. The sequence can be chosen so that $\mathcal{N}_2 = \mathcal{N}'_{\aleph_1}$ is correct about stationarity.

Now to code the pairs of stationary sets. First partition $\aleph_1$ into two sets V and X. Now in the standard way obtain a set of $\aleph_1$ disjoint subsets S_α of X so that if $\alpha \neq \beta$, $S_\alpha - S_\beta$ is stationary. Since the S_α are pairwise disjoint modulo the non-stationary ideal (in V), the following lemma completes the proof.

Finale

Lemma

If S_1, S_2 are each stationary subsets of X and $S_1 - S_2$ is stationary and both M^{S_1,T_1} and M^{S_2,T_2} satisfy $\theta(S_i, T_i)$ then $M^{S_1,T_1} \not\approx M^{S_2,T_2}$.

Proof. Suppose for contradiction that $f : M^{S_1,T_1} \mapsto M^{S_2,T_2}[G]$ is an isomorphism. On a cub $f \restriction M^{S_1,T_1}_{J_\alpha}$ is an isomorphism onto $M^{S_2,T_2}_{J_\alpha}$. Choose such an $\alpha \in S_1 - S_2$ and therefore in $S_1 \cap (X - T_2)$, let $t \in I^{S_1,T_1}$ be the least upper bound. We have shown the coding by $M^{S_1,T_1}[G]$. That is, $a_{t,0}$ catches M_{J_α} in $M^{S_1,T_1}[G]$ but does not strongly catch M_{J_α} in $M^{S_1,T_1}[G]$. Any possible image of $a_{t,0}$ in M^{S_2,T_2} that catches the image of M_{J_α} in $M^{S_2,T_2}[G]$ also strongly catches the image of M_{J_α} in $M^{S_2,T_2}[G]$. This establishes the non-isomorphism.

Constraints that help determine I

Constraint: Determining level

The variables at the same level (same first subscript) split into two types; a) those that are 'really' on that level are interalgebraic with the first element of the level (over the lower levels) and b) those which are renamings of variables on a lower level.

$\mathcal{I}_{t,n} = \mathcal{I}^1_{t,n} \cup \mathcal{I}^2_{t,n}$ where

1. If $t \neq \min(I)$ and $n < \omega$, $\mathcal{I}^1_{t,n}$= $\{q : t \in u_q, n < n_{q,t},$ and q 'says' $x_{t,0}$ is algebraic over $\{x_{s,\ell} : s \in u_q, s < t, \ell < n_{q,s}\} \cup \{x_{t,n}\}\}$.
2. $\mathcal{I}^2_{t,n}$= $\{q : t \in u_q, n < n_{q,t},$ and q 'says' $x_{t,n} = x_{s,m}$ for some $s \in u_q, s < t, m < n_{q,s}\}$.

Depends on failure of 'density'. Return

Striated Sequences and forcing

Any striation of a model M forms a connection between striated types and forcing conditions in $\mathbb{Q}_I$. Sriated Types

Lemma

Given any forcing condition $p \in \mathbb{Q}_I$, and a striation $\langle N_n : n \in \omega \rangle$ of an at-finitely saturated model M, there is a striated sequence $\langle \mathbf{b}_k : k < m \rangle$ of length m in M realizing p.

Proof. Let $m = |u_p|$ and let $f : m \to u_p$ be the unique order-preserving map. Recursively construct a striated sequence $\langle \mathbf{b}_k : k < m \rangle$ satisfying:

- $\mathbf{b}_k \subseteq N_k$;
- For each k, the first element of $\mathbf{b}_k \not\in N_{k-1}$;
- $\mathrm{tp}(\mathbf{b}_k / B_k) = \mathrm{tp}(\mathbf{x}_{f(k)} / \{x_{s,i} : s \in u_p, s < f(k), i < n_{p,s}\})$, where $B_k = \bigcup \{\mathbf{b}_j : j < k\}$.

That this construction is possible follows immediately from the fact that $\langle N_n : n \in \omega \rangle$ forms a striation of M.

Density of level constraints

Theorem

The ‘level’ constraints are dense.

Consider $\mathcal{I}_{t,n}$.
Let $p(\mathbf{x}_n) \in \mathbb{Q}$. If $t \notin u_p$, let q be complete and say $x_{t,0}$ is not algebraic over $\{x_{s,\ell} : s \in u_p, s < t, \ell < n_{p,s}\}$ and for $j \leq n$, $x_{t,n} = x_{t,j}$. Thus $q \in \mathcal{I}^1_{t,n}$. Suppose $x_{t,0}$ appears in p and $x_{t,n}$ does not, extend p to p' which says for $j \leq n$ such that $x_{t,j}$ does not appear in p, $x_{t,n} = x_{t,j}$. If $p' \in \mathcal{I}^1_{t,n}$ let $q = p'$. If not, to ensure all variables $x_{t,j}$ with $j \leq n$ appear in q at level t, for each $j \leq n$ such that $x_{t,j}$ does not appear in p, put $x_{t,n} = x_{s,0}$ in q for some $s < t$ such that $s \notin P$ and $s \notin u_p$.
Now, $q \in \mathcal{I}^2_{t,n}$.

Proof of density of level constraints continued

Now we consider the case when both $x_{t,0}$ and $x_{t,n}$ appear in p. If p says $x_{t,0}$ is algebraic over $\{x_{s,\ell} : s \in u_p, s < t, \ell < n_{p,s}\} \cup \{x_{t,n}\}$ then $p \in \mathcal{I}^1_{t,n}$ and $q = p$. If p says $x_{t,0}$ is not algebraic over $\{x_{s,\ell} : s \in u_p, s < t, \ell < n_{p,s}\} \cup \{x_{t,n}\}$ and $x_{t,n} = x_{s,\ell}$ for some $s \in u_q, s < t, \ell < n_{q,s}$ then $p = q$ is in $\mathcal{I}^2_{t,n}$ If $x_{t,n}$ has not been assigned a lower level, choose s with $t > s > u_p \cap \{v : v < t\}$. Fix the unique bijection f from u_p into $|u_p|$ and take a striated sequence $\{\mathbf{b}_i : i \leq f(t)\}$ faithfully realizing $p \restriction t$ in $\langle M_{f(i)} : i \leq t\rangle$. Then $b_{f(t),0} \notin M_{f(t)-1}$. Choose r (necessarily less than n) maximal so that $b_{f(t),0}$ and $b_{f(t),r}$ are interalgebraic over $\langle \mathbf{b}_0, \ldots \mathbf{b}_{f(t)-1}\rangle$.

Proof of density of level constraints continued again

Let q be a complete extension of p (in additional variables $x_{s,0}, \ldots, x_{s,n}$) which says that $\{x_{s,0}, \ldots, x_{s,n}\}$ satisfy the same type over $\{x_{v,i} : v < t, i < n_{p,v}, v \in u_p\}$ as $\{x_{t,0}, \ldots, x_{t,n}\}$ and satisfy $x_{t,i} = x_{s,i}$ for $r < i \leq n_{p,t}$. By finite at-saturation, choose $\langle b'_{f(t),r+1}, \ldots b'_{f(t),n_{p,t}} \rangle$ in $M_{f(t)}$ realizing the same type over $\langle \mathbf{b}_0, \ldots \mathbf{b}_{f(t)-1} \rangle$ as $\langle b_{f(t),r+1}, \ldots b_{f(t),n_{p,t}} \rangle$. Then by Fact 11, we can realize the type of $\langle \mathbf{b}_{f(t),0} \ldots \mathbf{b}_{f(t),r} \rangle$ over $\langle \mathbf{b}_0, \ldots \mathbf{b}_{f(t)-1} \rangle \cup \langle b'_{f(t),r+1}, \ldots b'_{f(t),n_{p,t}} \rangle$ as $\langle \mathbf{b}'_{f(t),0} \ldots \mathbf{b}'_{f(t),r} \rangle \in M_{f(t)} - M_{f(t)-1}$ so $\langle \mathbf{b}_0, \ldots \mathbf{b}_{f(t)-1}, \mathbf{b}'_{f(t)} \rangle$ witnesses q. Again, $q \in \mathcal{I}^2_{t,n}$.