The large and small in model theory: What are the amalgamation spectra of infinitary classes?

John T. Baldwin
University of Illinois at Chicago

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Characterizing cardinals by $L_{\omega_1,\omega}$

$L_{\omega_1,\omega}$ satisfies downward Lowenheim Skolem to $\aleph_0$ for sentences.
It does not satisfy upward Lowenheim Skolem.

Definition

The sentence ϕ of $L_{\omega_1,\omega}$ characterizes κ if ϕ has no model of cardinality $> \kappa$.

Theorem Hjorth

For every countable α, there is a complete (i.e. Scott) $L_{\omega_1,\omega}$-sentence ϕ_α that characterizes $\aleph_\alpha$.

Rephrasing Hjorth

Souldatos formulation

Theorem 3.6. (Hjorth) If κ is characterizable then at least one of the following holds:

1. Some complete sentence $\phi_0 \in L_{\omega_1,\omega}$ homogeneously characterizes κ^+, or
2. there is a countable model *M* in a language that contains a unary predicate *P* and a binary predicate $<$ and whose Scott sentence ϕ_1
   1. characterizes κ^+,
   2. in every model of ϕ, $<$ is a dense linear order without endpoints and
   3. in every model N of ϕ of size κ^+, every initial segment of $(P^N, <^N)$ has size at most κ.

From COMPLETE $L_{\omega_1,\omega}$ to ATOMIC 'first order'

From $L_{\omega_1,\omega}$ to 'first order'

1. $\phi \in L_{\omega_1,\omega} \rightarrow (T, \Gamma)$
2. complete $\phi \in L_{\omega_1,\omega} \rightarrow (T, Atomic)$

The translation

Theorem

[Chang/Lopez-Escobar] Let ψ be a sentence in $L_{\omega_1,\omega}$ in a countable vocabulary τ. Then there is a countable vocabulary τ' extending τ, a first order τ'-theory T, and a countable collection of τ'-types Γ such that reduct is a 1-1 map from the models of T which omit Γ onto the models of ψ.

The proof is straightforward. E.g., for any formula ψ of the form $\bigwedge_{i<\omega} \phi_i$, add to the language a new predicate symbol $R_\psi(\mathbf{x})$. Add to T the axioms

$$(\forall \mathbf{x})[R_\psi(\mathbf{x}) \rightarrow \phi_i(\mathbf{x})]$$

for $i < \omega$ and omit the type $p = \{\neg R_\psi(\mathbf{x})\} \cup \{\phi_i : i < \omega\}$.

Δ-complete

Let Δ be a fragment of $L_{\omega_1,\omega}$ that contains ϕ.
ϕ is Δ-complete if for every $\psi \in \Delta$
$\phi \models \psi$ or $\phi \models \neg\psi$.
(If Δ is omitted we mean complete for $L_{\omega_1,\omega}$.)

small=complete

Let Δ be a fragment of $L_{\omega_1,\omega}$ that contains ϕ.

Definition

A τ-structure *M* is Δ-small if *M* realizes only countably many Δ-types (over the empty set).

'small' means $\Delta = L_{\omega_1,\omega}$

Generalized Scott's theorem

A structure satisfies a complete sentence of $L_{\omega_1,\omega}$ if and only if it is small.

Reducing complete to atomic

The models of a complete sentence in $L_{\omega_1,\omega}$ can be represented as:
$\boldsymbol{K}$ is the class of atomic models (realize only principal types) of a complete first order theory (in an expanded language).

The virtues of Disjoint Amalgamation

Methods

1. extending Fraissé style arguments
   1. looking for atomic models
   2. the importance of (strong) disjoint amalgamation
2. absolute indiscernibility
3. excellence
4. combinatorics

Extending Fraïssé

Fix a class $\boldsymbol{K}_0$ of finite models in a countable vocabulary.
$\boldsymbol{K}_0$ may not be closed under substructure.

Theorem

If $\boldsymbol{K}_0$ has amalgamation and joint embedding and contains only finitely many members then
there is a countable generic atomic model M.

Laskowski-Shelah (1992); Hjorth (2002)
$\hat{\boldsymbol{K}}_0$ denotes the class of structures B such that every finite subset $B_0 \subseteq B$ is contained in a $B' \subseteq B$ with $B' \in \boldsymbol{K}_0$.
T is the theory of the generic; ϕ_M is its Scott sentence.

Disjoint AP

B-Friedman-Koerwien-Laskowski

Theorem

If,in addition, $\boldsymbol{K}_0$ has disjoint amalgamation then T has a model in $\aleph_1$.

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Theorem (Gao)

If countable structure has a set of absolute indiscernibles, there is an $L_{\omega_1,\omega}$ equivalent model in $\aleph_1$.

Mergers

Mergers

1. Let θ be a complete sentence of $L_{\omega_1,\omega}$ and suppose M is the countable model of θ and $V(M)$ is a set of absolute indiscernibles in M such $M - V(M)$ projects onto $V(M)$. We will say θ is a *receptive* sentence.
2. For any sentence ψ of $L_{\omega_1,\omega}$, the *merger* of ψ and θ is the sentence $\chi = \chi_{\theta,\psi}$ obtained by conjoining with θ, $\psi \restriction N$.
3. For any model M_1 of θ and N_1 of ψ we write $(M_1, N_1) \models \chi$ if there is a model with such a reduct.

Getting receptive models

Suppose $\boldsymbol{K}_0$ is a class of finite τ structures with disjoint amalgamation and θ_0 is the Scott sentence of the generic. Hjorth and B-Friedman-Koerwien-Laskowski

Construction

Add to τ unary predicates U, V and binary P.
Require that the predicates U and V partition the universe and restrict the relations of τ to hold only within the predicate V. We set $\boldsymbol{K}_1$ as the set of finite τ_1-structures (V_0, U_0, P_0) where $V_0 \restriction \tau \in \boldsymbol{K}$ and P_0 is the graph of a partial function from V_0 into U_0.

To amalgamate, use disjoint amalgamation in the V-sort; extend the projection by the union of the projections. If the disjoint amalgamation contains new points, project them arbitrarily to U. Let $\mathcal{M}$ be the generic model for $\boldsymbol{K}_1$.

A back and forth argument shows $U(\mathcal{M})$ is a set of absolute

Applying merger

Theorem: (Hjorth, B-Friedman-Koerwien-Laskowski)

There is a receptive sentence that characterizes (has only maximal models) $\aleph_1$.

Corollary: (B-Friedman-Koerwien-Laskowski)

If there is a counterexample to Vaught's conjecture there is one that has only maximal models in $\aleph_1$.

crux: Disjoint amalgamation

Fraissé style arguments + excellence

Theorem: (B- Koerwien-Laskowski)

There are a family of complete sentences ϕ_r such that ϕ^r:

1. homogeneously characterizes $\aleph_r$.
2. ϕ_r
   1. has ap up to $\aleph_{r-1}$,
   2. fails ap in $\aleph_{r-1}$,
   3. trivially has ap in $\aleph_r$.

crux: ***K*** satisfies $(< \aleph_0, r+1)$ disjoint amalgamation – i.e. $r+1$-excellence in the finite.

ABSTRACT ELEMENTARY CLASSES

A class of L-structures, $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, is said to be an *abstract elementary class: AEC* if both $\boldsymbol{K}$ and the binary relation $\prec_{\boldsymbol{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$;

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Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

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2. Closure under direct limits of $\prec_{\boldsymbol{K}}$-chains;
3. Downward Löwenheim-Skolem.

Examples

First order and $L_{\omega_1,\omega}$-classes
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locally finite abstract elementary class

A class $\boldsymbol{K}$ of structures and a substructure relation $\prec_{\boldsymbol{K}}$ is a *locally finite abstract elementary class* if it satisfies the normal axioms for an AEC except the usual Löwenheim-Skolem condition is replaced by:

If $M \in \boldsymbol{K}$ and $A \subset M$ of M) is finite, there is a finite $N \in \boldsymbol{K}$ with $A \subset N \prec_{\boldsymbol{K}} M$ (read N is a *strong substructure*).

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Excellence

Definition

A set of $\boldsymbol{K}$-structures $\overline{N} = \langle N_u : u \subsetneq k \rangle$ is a $(< \lambda, k)$-system if it is a directed system of $\boldsymbol{K}$-structures with cardinality $< \lambda$ indexed by the proper subsets of k.

Definition

We say that $\boldsymbol{K}$ has disjoint $(< \lambda, k)$-amalgamation (k-weak-excellence) if

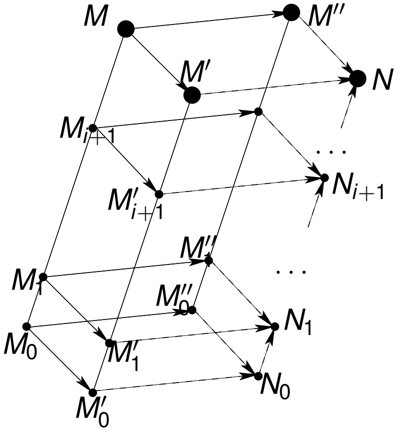
1. $k = 0$ and there is $M \in \boldsymbol{K}$ with $\|M\| = \mu$ for all $\mu < \lambda$.
2. $k = 1$ and for all $\mu < \lambda$, each $M \in \boldsymbol{K}$ with $\|M\| = \mu$ has a proper extension.
3. $k \geq 2$ and for any $(< \lambda, k)$-system $\overline{N}$ there is a model $M \in \boldsymbol{K}$ such that for every $u \subsetneq k$: N_u is a *substructure* of M.

Larger models from disjoint amalgamation

Claim

For any $s < \omega$, if $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has the disjoint $(< \lambda, s+1)$-amalgamation property, then it has the disjoint $(< \lambda^+, s)$-amalgamation property.

$(\lambda, 3)$-ap implies $(\lambda^2, 2)$-ap



The Amalgamation Spectrum
B- Koerwien-Laskowski

Focus today

We work with three classes of models $\boldsymbol{K}_0, \hat{\boldsymbol{K}}$ and $\mathbf{At} = \mathbf{At}(\boldsymbol{K}_0)$.

$\boldsymbol{K}_0$ is a collection of finite structures. $\hat{\boldsymbol{K}}$ contains exactly those structures that are locally in $\boldsymbol{K}_0$; this is what is meant by a *locally finite AEC*.

If $(\boldsymbol{K}_0, \subseteq)$ satisfies the amalgamation property then there is countable generic model M and **At** is the collection of all structures satisfying the Scott sentence ϕ_M of M.

Now our principal results go in two directions: constructing larger models and bounding the possible cardinality.

Bounding the cardinality

Usual algebraic notion of closure:
$\mathrm{cl}_M(A)$ is the smallest substructure of M containing A and closed under the function symbols in the vocabulary.
A set B is *independent* if, for every $b \in B$, $b \notin \mathrm{cl}(B \setminus \{b\})$.

Lemma

For every $k \in \omega$, if cl is a locally finite closure relation on a set X of size $\aleph_k$, then there is an independent subset of size $k+1$.

BKL example

For a fixed $r \geq 1$, let τ_r be the (countable) vocabulary consisting of countably many $(r+1)$-ary functions f_n and countably many $(r+1)$-ary relations R_n.

Consider the class $\boldsymbol{K}^r$ of finite τ_r-structures (including the empty structure) that satisfy the following three sentences of $L_{\omega_1,\omega}$:

- The relations $\{R_n : n \in \omega\}$ partition the $(r+1)$-tuples;
- For every $(r+1)$-tuple $\boldsymbol{a} = (a_0, \ldots, a_r)$, if $R_n(\boldsymbol{a})$ holds, then $f_m(\boldsymbol{a}) = a_0$ for every $m \geq n$;
- There is no independent subset of size $r+2$.

The construction in the finite

John T. Baldwin
University of Illinois at Chicago

We will verify here a slightly stronger notion: *strong disjoint amalgamation*: replace in Definition 19.3 'N_u is a *substructure* of M', by 'the universe of M is $\bigcup_{u \subsetneq k} |N_u|$'.

Theorem

For each $r \geq 1$, $\boldsymbol{K}^r$ has *strong disjoint* $(<\aleph_0, r+1)$*-amalgamation*. Further, $\boldsymbol{K}^r$ does not have *disjoint* $(<\aleph_0, r+2)$*-amalgamation*.

Consequences

Theorem: (B- Koerwien-Laskowski)

There are a family of complete sentences ϕ_r such that ϕ^r:

1. homogeneously characterizes $\aleph_r$.
2. ϕ_r
   1. has ap up to $\aleph_{r-1}$,
   2. fails ap in $\aleph_{r-1}$,
   3. trivially has ap in $\aleph_r$.

Why characterize?

Suppose each model of $\hat{\boldsymbol{K}}$ admits a locally finite closure relation cl such that there is no cl-independent subset of size $r+2$. Then $\hat{\boldsymbol{K}}$ has only maximal models in $\aleph_r$ and so (disjoint) 2-amalgamation is trivially true in $\aleph_r$;

Contrasts

Excellence is sufficient

If $\boldsymbol{K}$ is excellent then it has arbitrarily large models and the amalgamation property.

Excellence is not necessary

(B-Kolesnikov) Non-excellent classes with arbitrarily large models, ap (and much more).

B-Laskowski-Koerwien measures the strength of excellence as a sufficent condition for model existence (and ap).

Question

Is there an AEC that is categorical up to $\aleph_n$ and has no larger models?

Joint embedding and Amalgamation B-Koerwien-Souldatos

Joint embedding vrs amalgamation

A Contrast

1. (Shelah) If an AEC has AP(κ) for every κ, then it has the (full-) Amalgamation Property.
2. The full-Joint Embedding Property is *not* equivalent to the conjunction of JEP(κ), for all infinite κ.

Spectrum of disjoint amalgamation in AEC

Kolesnikov and Lambie-Hanson have given a family of AEC's (of coloring classes) in a countable vocabulary which satisfy the amalgamation property but have no models above $\beth_{\omega_1}$.

Specific classes fail dap for the first time arbitrarily close to $\beth_{\omega_1}$.

Bipartite graphs

Let $\tau_0 = \{A, B, C, E\}$ where A, B, C are unary predicates and E is a ternary relation. Let σ_0 be the conjunction of the following statements:

- A, B, C are non-empty and partition the universe.
- $E(a, b, c)$ means there is an edge colored c between a and b.

Let σ_1 be the conjunction of σ_0 and
There are no monochromatic $K_{2,2}$ subgraphs.

Key Combinatorial fact

In any model of σ_1, if $|A| > |C|^+$ then $|B| \leq |C|$.

By symmetry, the same is true if we switch the roles of A and B.

Constructing a (κ^+, κ^+)-model

Lemma: For any κ, there is a (κ^+, κ^+) model $\mathcal{M} \models \sigma_1$ such that $C^{\mathcal{M}} = \kappa$.

Let $A^{\mathcal{M}}$ and $B^{\mathcal{M}}$ be two copies of κ^+. Fix a function F from $\kappa^+ \times \kappa^+$ to κ such that:

1. for all α, $F(\alpha, \alpha) = 0$ and
2. for all $\alpha \in A$, $F(\alpha, \cdot)$

is a one-to-one function when restricted to the set $\{\beta \in B | \beta \leq \alpha\}$.

Symmetrically, demand that for all $\beta \in B$, $F(\cdot, \beta)$ is a one-to-one function when restricted to $\{\alpha \in A | \alpha \leq \beta\}$.

Both conditions are possible because all initial segments have size $\kappa = |C^{\mathcal{M}}|$.

Now link $\alpha \in A$ to $\beta \in B$ by the color $F(\alpha, \beta)$.

The Construction works!

Towards contradiction, assume that there are distinct α_1, α_2 in A and β_1, β_2 in B with all four values $F(\alpha_i, \beta_j)$ $(i, j \in \{1, 2\})$ identical.
Without loss of generality assume that $\max\{\alpha_1, \alpha_2, \beta_1, \beta_2\} = \alpha_1$.
By the choice of F, $F(\alpha_1, \beta_1)$ must be different than $F(\alpha_1, \beta_2)$. Contradiction.

From Combinatorics to model theory

Given a class $\boldsymbol{K}_0$ of finite structures and associated $\hat{\boldsymbol{K}}$ that homogenously characterizes κ, merge this sentence with σ_1 -taking the set of absolute indiscernibles as the colors. Call this sentence σ_κ.

Theorem

σ_κ has

1. 2^κ maximal models in κ^+.
2. arbitrarily large models.

Spectrum of disjoint amalgamation in AEC

B-Koerwien-Souldatos

For any countable family of characterizable cardinals λ_i, there is an AEC that has $2^{\lambda_i^+}$ maximal models in λ_i, fails AP everywhere and has arbitrarily large models.

So maximal models can be arbitrarily close to $\beth_{\omega_1}$ and then no more maximal models.

Open Questions

Open Question

Is there an $L_{\omega_1,\omega}$-sentence that has maximal models in uncountably many cardinals but arbitrarily large models?

Open Question

Is there a complete- $L_{\omega_1,\omega}$-sentence that has maximal models in two (consecutive) cardinals (but arbitrarily large models?)

Hanf Numbers for JEP, AP etc

Lower bounds

The previous results show the Hanf number for JEP and DAP is at least $\beth_{\omega_1}$.

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Upper bounds: B-Boney

Let κ be strongly compact and ***K*** be an AEC with Löwenheim-Skolem number less than κ.

- If ***K*** satisfies *JEP*$(< \kappa)$ then $\boldsymbol{K}_{\geq\kappa}$ satisfies *JEP*.
- If ***K*** satisfies *AP*$(< \kappa)$ then ***K*** satisfies *AP*.

crux: strongly compact cardinals. Direct proof is by ultraproducts. Proof using modification of first order arguments and compactness of $L_{\kappa,\kappa}$ leads to interesting issues about the presentation theorem.

The big gap

Let κ be a strongly compact cardinal
Some Hanf numbers are at most κ:
jep, dap, ap

In ZFC, those Hanf numbers are at least $\beth_{\aleph_1}$.