Study Guide to Second Midterm

March 11, 2007

Name:

Several of these were review problems for the first midterm. If you did well on the midterm, you may not have thought about them. There are more problems here than will be on the exam. The exam is cumulative; you are responsible for concepts from the first exam. I will post solutions to those problems in which the class show sufficient interest.

1. Make sure you know and can use the definitions of such notions as symmetric difference, union, power set, injection, surjection, bijection, finite ....

   Look these up.

2. Show explicitly using the axioms of order that if $a < b$, $c < d$ are real numbers then $(a, b) \cup (c, d) = (c, d)$ iff $c \leq a$ and $d \geq b$.

   Suppose $(a, b) \cup (c, d) = (c, d)$. Now let $x \in (a, b) \cup (c, d)$ so $x > a$ and $x < b$ or $x > c$ and $x < d$. But then $(a, b) \cup (c, d) = (c, d)$ implies $x > c$. So every element greater than a is greater than c. This implies $c \leq a$; otherwise we would have $c > c$. Similarly any $x \in (a, b) \cup (c, d)$ is less than d. In particular, so $d \geq b$; as otherwise we would have $d < b$, and that implies $d < d$. These contradictions show $c \leq a$ and $d \geq b$.

   I leave the other direction for now.

3 Prove from the axioms on ordered fields that there is no greatest real number a such that $a^2 < 2$. (Hint: Use that $\sqrt{2}$ is a real number. This does *not* need any complicated algebra.)

   Remark: Note first that for any positive real numbers, $a < b$ implies $a^2 < b^2$. To see this multiply the inequality $a < b$ by the positive number a to get $a^2 < ab$ and by the positive number b to get $ab < b^2$. By transitivity of $<$, $a^2 < b^2$.

   To solve the problem at hand, suppose for contradiction that a is the greatest real number with $a^2 < 2$. Then $\sqrt{2} - a > 0$, by the contrapositive of the first remark, and if $b = a + \frac{\sqrt{2}-a}{2}$, $\sqrt{2} > b > a$. So $b^2 < 2$ by the first remark, contradiction.

4 You may assume basic algebraic properties without explicit reference to the axioms in this problem. Recall that for integers d, s, n, 'd divides n means there is an s such that $n = ds$. Let d, x and y be integers. Prove

(a) If d divides x and d divides y then d divides $x + y$.

(b) If d divides $x + y$ and d divides y then d divides x.

(c) Explain the connection between the two results.

We work entirely with integers. If d divides x and d divides y, then for some m and n $x = dm$, $y = dn$ so $x + y = dm + dn = d(m + n)$ so d divides $x + y$.

Ask me if you try the other way and are not sure.

connection: Consider the three statements: d divides x, d divides y, d divides $x + y$. If any two of them are true, so is the third.

3. True or false: if f is a surjection from X to Y then f has an inverse.

   false. Let $f : R \mapsto R$ by taking x to x^2. There there is no inverse mapping R to R. (of course we can restrict to x^2 to positive reals and get an inverse to x^2 with domain and codomain $\Re^{\geq 0}$.)

   To check your understanding, make a finite example.

4. Draw Venn diagrams and use an element proof to show: $A \cup (A \cap B) = A$. (There was a typo here.)

   proof. If $x \in A \cup (A \cap B)$ then $x \in A$ or both $x \in A$ and $x \in B$. So x is certainly in A. Conversely, every element of A is in the larger set $A \cup (A \cap B)$. so we finish.

5. Let $f : X \to Y$ and $g : Y \to Z$. Suppose $g \circ f$ is surjective.

   (a) Show g is surjective.
   Since $g \circ f$ is surjective, for every $z \in Z$ there is an x with $g(f(x)) = z$. So g maps $f(x)$ to z showing g is also surjective.

   (b) Show by example that f does not need to be surjective.

Let $X = \{c\}$, $Y = \{a, b\}$, $Z = \{d\}$:

Let $f = \{\langle c, a\rangle\}$ and $g = \{\langle a, d\rangle\}$. Then $g \circ f(c) = d$ so $g \circ f$ is surjective. But f clearly isn't.

6. Does the function $e^x : \Re \rightarrow \Re$ have an inverse $\ln : \Re \rightarrow \Re$?

   No, it does not map onto $\Re$. More precisely, e^x is not surjective as a map with codomain $\Re$ since no number that is not strictly positive is in the range.

   If not choose an appropriate domain and codomain so a restriction of e^x does have an inverse.

   Let the new domain remain $\Re$ but with codomain $\Re^+$ (the strictly positive reals. Then ln maps $\Re^+$ onto $\Re$

   where $\ln(x) = y$ iff $y = e^x$. This is well defined for positive x.

   Why did you make the specific choice of domain and codomain.

   We choose the domain and codomain to make sure the restriction of e^x is surjective and remains injective.