Study Guide for Final exam Math 215

Solutions

The exam will cover the book up to the chapter with the divison algorithm (15?).

Here are some sample questions.

1. Write down the negation of the statement: $(\forall x \in \mathbb{Z})(\forall w \in \mathbb{Z})(\exists y \in \mathbb{Z})(xy < w)$. Is it true?

   solution: $(\exists x \in \mathbb{Z})(\exists w \in \mathbb{Z})(\forall y \in \mathbb{Z})(xy \geq w)$.

   is false since for any x and w multiplying x by a negative y (positive if x is negative) of sufficiently large absolute value will give $xy < w$.

2. Define the following terms

   1. X and Y are equipotent.
   2. $(X, <)$ is a linearly ordered set
   3. $f : X \mapsto Y$ is a surjection.

   solutions:

   1. There is a bijection between X and Y
   2. $<$ is transitive, irreflexive, asymmetric and satisfies the trichotomy property.
   3. For every $y \in Y$ there is an $x \in X$ with $f(x) = y$.

3. Let f map X to Y:

   1. What are the domain and codomain of $\overleftarrow{f}$?
   2. What is the difference between the range of f and the codomain of f.
   3. What is the pigeonhole principle

   solution:

   1. domain: $\mathcal{P}(Y)$; codomain: $\mathcal{P}(X)$

2. The codomain is the set f is given as mapping to; the range is the set of $y \in Y$ such that there is an x with $f(x) = y$.

3. The pigeonhole principle says that if $|X| > |Y|$ and f maps X to Y, then two elements of X map to the same element of Y.

4. Prove each of the following if it is true or give a counterexample.

1. If $|X| = |Y|$ and f is an injection from X to Y then f is surjective. Does it make a difference if X is finite?

2. If A is infinite $|\mathbb{N}| \leq |A|$.

3. If A is countable then A is infinite.

4. Every surjective function is injective.

solutions:

1. Let $f(n) = 2n$. f is an injection from N to N and certainly $|N| = |N|$ but f is not surjective. But it is true if X and Y are finite. See book.

2. If A is infinite $|\mathbb{N}| \leq |A|$.

   true By definition, A is infinite if and only if it is not finite. To show $|\mathbb{N}| \leq |A|$, we must construct an injection of $\mathbb{N}$ into A. For this, define f by induction. Let $f(1)$ be any element of A. If we have defined f on $\mathbb{N}_n$, then $\{f(i) : i \in \mathbb{N}_n\}$ is not A (otherwise A would be finite). Now let $f(n+1)$ be some element of $A - \{f(i) : i \in \mathbb{N}_n\}$.

3. If A is countable then A is infinite. no; 3 is countable

4. Every surjective function is injective. No; e.g. x^2 from $\mathbb{R}$ to $\mathbb{R}^+$.

5. Use calculus to show the function $f(x) = x^5$ is 1-1.

solution: $f'(x) = 5x^4$ is always positive. Therefore f is increasing on all of R and so must be 1-1. (Write this step out!).

6. The dyadic rationals is the set $D = \{\frac{a}{2^n} : a \in \mathbb{Z}, n \in \mathbb{N}\}$. Show D is countable.

Let $f_a(n) = \frac{a}{2^n}$. Then $F : \mathbb{Z} \times N = D$ is onto since for each $d = \frac{a}{2^n}$, $F(\langle a, n \rangle) = d$. We know $\mathbb{Z} \times N$ is countable by Cantor's theorem (14.2.3) so D is countable.

7. Show that if n is odd, 9 divides $8^n + 1$.

recall: $x^{2n+1} = (x+1)(x^{2n} - x^{2n-1} \ldots - x + 1$. Substitute 8 for x.

8. Sketch the proof that there are uncountably many real numbers.

solution: Suppose the reals are listed $r_k : k \in \mathbb{N}$ where as

$$r_k = a_{k0}.a_{k1}, a_{k2} \ldots$$

where a_{k0} is an integer and for $i < 0$, $a_{ki} \in \{0, 1, \ldots, 9\}$.

Now define $r = .b_0, b_1, \ldots$ where $b_i = 0$ if $a_{ii} = 1$ and $b_i = 1$ if $a_{ii} = 0$.

Then r is not equal to any of the r_k.

9. Prove $A \cap B^c = \emptyset$ implies $A \subseteq B$.

If $x \in A$, then $A \cap B^c = \emptyset$ implies $x \notin B^c$ implies $x \in B$.

10. Suppose $a < b$ and $c < d$ are real numbers and the interval (a, b) intersects the interval $[c, d]$. What can you say about the ordering of the 4 numbers.

We can conclude either $a < c$ and $b > c$ or $b > d$ and $a < d$. E.g. suppose $a < c$. Then there is an x in the intersection so $c < x$ and $x < b$. By transitivity $c < b$.