Two problem solutions

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Page 184, number 9. Prove (11.1.4) that if there is an injection $f : X \mapsto N_n$ then X is finite and the cardinality of X is at most n.

Proof. We work by induction on n. If $n = 1$, then an injection into N_1 must be onto. So f is invertible and X is a finite set with cardinality n.

Induction Hypothesis: Suppose that for any X if there is an injection f from X into N_k then X is finite and the cardinality of X is at most k.

Induction step: We must prove for any X if there is an injection f from X into N_{k+1} then X is finite and the cardinality of X is at most $k + 1$.

Case 1: $k + 1$ is not in the range of f. Then f is an injection into N_k and the result is immediate from the induction hypothesis.

Case 2: $k + 1$ is in the range of f. Say $f(a) = k + 1$. Now let g be the restriction of f to $X - \{a\}$. Then g is an injection of $X - \{a\}$ into N_k. So again by induction, $X - \{a\}$ is finite and $|X - \{a\}|$ is some $m \leq k$. Then by 10.2.1 (the addition principle), $X = X \cup \{a\}$ is a disjoint union of finite sets, so X is finite and $|X| = m + 1 \leq k + 1$.

Page 184 number 10. Prove (11.1.6) that if X and Y are non-empty finite sets with $|X| < |Y|$, there is no surjection from X onto Y.

Proof. Suppose for contradiction that such f exists. By the definition of finite there exists an $m < n$ and functions g_1, g_2 such that g_1 is a bijection from N_m onto X and g_2 is a bijection from N_m onto Y. But then $h = g_2^{-1} \circ f \circ g_1$ is a surjection from N_m onto N_n. Now we can find an injection $h' : N_n \mapsto N_m$; $h'(y)$ is defined to be the least $k < m$ such that $h(k) = y$. Now by 11.1.1 the existence of h' implies $m \leq n$. This contradiction completes the proof.