Categoricity

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Contents

1 Introduction

I will of course rewrite the introduction when the monograph is finished so it reflects what I ended up with not what I thought would end up with. But here are the aims now.

The following themes are interweaved.

1. I want to expound Shelah's categoricity theorem for $L_{\omega_1,\omega}$. Very vaguely, I think of this as having two parts. a) Categoricity in enough powers implies excellence; b) excellence implies categoricity transfers.

Most of what is written on this line lies in part b). In order for a) + b) to yield categoricity in a small initial segment of the cardinals yields categoricity in all cardinals, our analysis of b) cannot assume the existence of arbitrarily large models. We motivate the exposition by doing Zilber's special case of quasiminimal excellence and connecting it with his conjectures on complex exponentiation. But the natural axiomatization of Zilber's classes are not in $L_{\omega_1,\omega}$ but in $L_{\omega_1,\omega}(Q)$. So we would like to have the results for $L_{\omega_1,\omega}(Q)$. I don't believe this can be done with out much more sophisticated techniques, e.g. λ-frames of [36, 35, 34]; they are beyond the scope of these notes. But the basic properties of abstract elementary classes are developed so as to prepare for study of e.g. frames. An unwritten chapter will connect $L_{\omega_1,\omega}(Q)$ with AEC's. At the moment we only notice that the first two or three obvious ways to translate are wrong.

Eventually, we will do a).

2. A different perspective is the conjecture that for reasonably well-behaved classes, categoricity should be either eventually true or eventually false. Here we introduce the notion of AEC and prove the presentation theorem and categoricity implies stability Some of the results in this general

framework are applied in the $L_{\omega_1,\omega}$-case. For this part, the assumption of arbitrarily large models is entirely reasonable. Again we expound some simpler versions of this result (using strong amalgamation hypotheses). This too leads to λ-frames.

3. In the midst of this I intend to describe the taxonomy of infinitary categoricity - the relations among various infinitary languages, homogeneous model theory, excellent classes, AEC etc.

I don't organize around techniques (as Keisler did) but there are several techniques that eventually are expounded; combinatorial geometries, EM-models and then the introduction of splitting to get an independence relation generalizing a combinatorial geometry.

The only quoted material is very elementary model theory (say a small part of Marker's book), Morley's omitting types theorem, and two or three theorems from the Keisler book including Lopez-Escobar.

In addition to the fundamental papers of Shelah, this exposition depends heavily on various works by Grossberg and Lessmann.

2
Combinatorial Geometries

Definition 2.1 *A* pregeometry *is a set G together with a dependence relation*

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$
A4. *If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.*
If points are closed the structure is called a geometry.

Definition 2.2 *A geometry is* homogeneous *if for any closed $X \subseteq G$ and $a, b \in G - X$ there is a permutation of G which preserves the closure relation (i.e. an automorphism of the geometry) which fixes X pointwise and takes a to b.*

Exercise 2.3 *If G is a homogeneous geometry, X, Y are maximally independent subsets of G, there is an automorphism of G taking X to Y.*

Definition 2.4

1. *The structure M is* strongly minimal strongly minimal *if every first order definable subset of any elementary extension M' of M is finite or cofinite.*
2. *The theory T is strongly minimal if it is the theory of a strongly minimal structure.*
3. *$a \in$* acl(X) *if there is a first order formula with finitely solutions over X which is satisfied by a.*

Definition 2.5 *Let X, Y be subsets of a structure M. An elementary isomorphism from X to Y is* 1-1 *map from X onto Y such that for every first order formula $\phi(\mathbf{v})$, $M \models \phi(\mathbf{x})$ if and only if $M \models \phi(f\mathbf{x})$.*

Exercise 2.6 *Find X, Y subsets of a structure M such that X and Y are isomorphic but not elementarily isomorphic.*

Exercise 2.7 *Let X, Y be subsets of a structure M. If f takes X to Y is an elementary isomorphism, f extends to an elementary isomorphism from* $\mathrm{acl}(X)$ *to* $\mathrm{acl}(Y)$.

Exercise 2.8 *Show a complete theory T is strongly minimal if and only if it has infinite models and*

1. *algebraic closure induces a pregeometry on models of T;*
2. *any bijection between acl-bases for models of T extends to an isomorphism of the models.*

Exercise 2.9 *A strongly minimal theory is categorical in any uncountable cardinality.*

3
Abstract Quasiminimality

Since I have and will use the term:

Definition 3.1 *A structure M is κ-sequence homogeneous if for any $\boldsymbol{a}, \mathbf{b} \in M$ of length less than κ, if $(M, \boldsymbol{a}) \equiv (M, \mathbf{b})$ then for every c, there exists d such that $(M, \boldsymbol{a}c) \equiv (M, \mathbf{b}d)$.*

An abstract quasiminimal class is a class of structures that satisfy the following five conditions, which we expound leisurely.

Assumption 3.2 (Condition I) *Let $\boldsymbol{K}$ be a class of L-structures which admit a monotone idempotent closure operation cl taking subsets of $M \in \boldsymbol{K}$ to substructures of M such that cl has finite character.*

Strictly speaking, we should write $\mathrm{cl}_M(X)$ rather than $\mathrm{cl}(X)$ but we will omit the M where it is clear from context.

Definition 3.3 *Let A be a subset of $H, H' \in \boldsymbol{K}$. A map from $X \subset H - A$ to $X' \subset H' - A$ is called a partial A-monomorphism if its union with the identity map on A preserves quantifier free formulas.*

Freqently, but not necessarily we will have $A = G$ which is in $\boldsymbol{K}$.

Definition 3.4 *Let $Ab \subset M$ and $M \in \boldsymbol{K}$. The (quantifier-free) type of b over A in M, written $\mathrm{tp}_{qf}(b/A; M)$ ($\mathrm{tp}_{qf}(b/A; M)$), is the set of (quantifier-free) first-order formulas with parameters from A true of b in M.*

Exercise 3.5 *Why is M a parameter in Definition 3.4?*

Exercise 3.6 *Let $Ab \subset M$, $Ab' \subset M'$ with $M, M' \in bK$. Show there is a partial A-monomorphism taking a' to b' if and only if* $\mathrm{tp}_{qf}(b/A; M) = \mathrm{tp}_{qf}(b'/A; M')$.

The next assumption connects the geometry with the structure on members of $\boldsymbol{K}$.

Assumption 3.7 (Condition II) *Let $G \subseteq H, H' \in \boldsymbol{K}$ with G empty or in $\boldsymbol{K}$.*

1. *If f is a bijection between X and X' which are* separately cl-*independent (over G) subsets of H and H' then f is a partial G-monomorphism.*
2. *If f is a partial G-monomorphism from H to H' taking $X \cup \{y\}$ to $X' \cup \{y'\}$ then $y \in \mathrm{cl}(XG)$ iff $y' \in \mathrm{cl}(X'G)$.*

Condition 3.7.2) has an *a priori* unlikely strength: quantifier free formulas determine the closure; in practice, the language is specifically expanded to guarantee this condition. Part 2 of Assumption 3.7 implies that each M with $G \subseteq M \in \boldsymbol{K}$ is finite sequence homogeneous over G.

Assumption 3.8 (Condition III : $\aleph_0$-homogeneity over models) *If f is a partial G-monomorphism from H to H' with finite domain X then for any $y \in H$ there is y' in an extension $H'' \in \boldsymbol{K}$ of H' such that $f \cup \{\langle y, y'\rangle\}$ extends f to a partial G-monomorphism.*

If H and H' have finite dimension, we might need to extend H'. Alternative formulations would be to only make the assumption for $y \in \mathrm{cl}_H(X)$ or to require that H' have infinite dimension.

Question 3.9 *Let a, b be independent over the empty set. Suppose f_a, f_b map* $\mathrm{cl}(a)$ *(*$\mathrm{cl}(b)$*) into a $\boldsymbol{K}$-structure H. Is $f_a \cup f_b$ a monomorphism? (We prove below that the answer is yes assuming exchange; is exchange necessary?*

Definition 3.10 *We say a closure operation satisfies the* countable closure condition *if the closure of a countable set is countable.*

We easily see:

Lemma 3.11 *Suppose Assumptions I, II, and III are satisfied by* cl *on an uncountable structure $M \in \boldsymbol{K}$ that satisfies the countable closure condition.*

1. *For any finite set $X \subset M$, if $a, b \in M - \mathrm{cl}(X)$, a, b realize the same $L_{\omega_1,\omega}$ type over X.*
2. *Every $L_{\omega_1,\omega}$ definable subset of M is countable or cocountable. This implies that $a \in \mathrm{cl}(X)$ iff it satisfies some ϕ over X, which has only countably many solutions.*

Proof. Condition 1) follows directly from Conditions II and III (Assumption 3.7 and Assumption 3.8) by constructing a back and forth. To see condition 2), suppose both ϕ and $\neg\phi$ had uncountably many solutions with ϕ defined over X. Then there are a and b satisfying ϕ and $\neg\phi$ respectively and neither is in $\mathrm{cl}(X)$; this contradicts 1).

The ω-homogeneity yields by an easy induction:

Lemma 3.12 *Suppose Assumptions I II and III hold. Let $G \in \boldsymbol{K}$ be countable and suppose $G \subset H_1, H_2 \in \boldsymbol{K}$.*

1. *If $X \subset H - G$, $X' \subset H - G$ are finite and f is a G-partial monomorphism from X to X' then f extends to a G-partial monomorphism from $\mathrm{cl}_H(GX)$ to $\mathrm{cl}_{H'}(GX')$.*

2. *If X is independent set of cardinality at most $\aleph_1$, and f is a G-partial monomorphism from X to X' then f extends to a G-partial monomorphism from $\mathrm{cl}_H(GX)$ to $\mathrm{cl}_{H'}(GX')$.*

Proof. The first statement is immediate from homogeneity. The second follows by induction from the first (by replacing G by $\mathrm{cl}(GX_0)$ for X_0 a countable initial segment of X). $\square_{3.12}$

For algebraic closure the cardinality restriction on X is unnecessary. We will have to add Assumption 3.8 to remove the restriction in excellent classes.

Assumption 3.13 (Condition IV) cl *satisfies the exchange axiom: $y \in \mathrm{cl}(Xx) - \mathrm{cl}(X)$ implies $x \in \mathrm{cl}(Xy)$.*

Zilber omits exchange in the fundamental definition but it arises in the natural contexts he considers so we make it part of quasiminimal excellence. Note however that the examples of first order theories with finite Morley rank greater than 1 (e.g. [2]) fail exchange.

In the following definition it is essential that $\subset$ be understood as *proper* subset.

Definition 3.14 1. *For any Y, $\mathrm{cl}^-(Y) = \bigcup_{X \subset Y} \mathrm{cl}(X)$.*

2. *We call C (the union of)* an n-dimensional cl-independent system *if $C = \mathrm{cl}^-(Z)$ and Z is an independent set of cardinality n.*

To visualize a 3-dimensional independent system think of a cube with the empty set at one corner A and each of the independent elements z_0, z_1, z_2 at the corners connected to A. Then each of $\mathrm{cl}(z_i, z_j)$ for $i < j < 3$ determines a side of the cube: $\mathrm{cl}^-(Z)$ is the union of these three sides; $\mathrm{cl}(Z)$ is the entire cube.

Assumption 3.15 (Condition V) *Let $G \subseteq H, H' \in \boldsymbol{K}$ with G empty or in $\boldsymbol{K}$. Suppose $Z \subset H - G$ is an n-dimensional independent system,*

$C = \mathrm{cl}^{-}(Z)$, and X is a finite subset of $\mathrm{cl}(Z)$. Then there is a finite C_0 contained in C such that: for every G-partial monomorphism f mapping X into H', for every G-partial monomorphism f_1 mapping C into H', if $f \cup (f_1 \restriction C_0)$ is a G-partial monomorphism, $f \cup f_1$ is also a G-partial monomorphism.

We can rephrase the conclusion as for any $\boldsymbol{a} \in \mathrm{cl}(Z)$ there is a finite $C_{\boldsymbol{a}} \subseteq C$ such that $\mathrm{tp}_{qf}(\boldsymbol{a}/C_{\boldsymbol{a}}; H)$ implies $\mathrm{tp}_{qf}(\boldsymbol{a}/C; H)$.

Thus Condition IV, which is the central point of excellence, asserts (for dimension 3) that the type of any element in the cube over the union of the three given sides is determined by the type over a finite subset of the sides. The 'thumbtack lemma' of Chapter 4 verifies this condition in a specific algebraic context. Here is less syntactic version of Assumption 3.8

Definition 3.16 *We say $M \in \boldsymbol{K}$ is* prime *over the set $X \subset M$ if every partial monomorphism of X into $N \in \boldsymbol{K}$ extends to a monomorphism of M into N.*

Remark 3.17 *Note that in first order logic this corresponds to 'algebraically prime' rather than 'elementarily prime'. In the first order context algebraically prime is a notoriously unstable (in a nontechnical sense) concept.*

Lemma 3.18 *Let $G \subseteq H, H' \in \boldsymbol{K}$ with G empty or in $\boldsymbol{K}$ and countable. Suppose $Z \subset H - G$ is an n-dimensional independent system, $C = \mathrm{cl}^{-}(Z)$, then $\mathrm{cl}(Z) \subseteq H$ is prime over C.*

Proof. Fix an embedding f from C into H' containing G. We must extend f to $\hat{f}$ mapping $\mathrm{cl}(Z)$ into H'. We can enumerate $\mathrm{cl}(X)$ as $a_i : i < \omega$. Let A_n denote $\{a_i : i < n\}$. Now define by induction an increasing sequence of finite sets $B_n : n < \omega$ such that $\mathrm{tp}_{qf}(A_n/B_n; H)$ implies $\mathrm{tp}_{qf}(A_n/C; H)$ and $\bigcup_{n<\omega} B_n = C$. Now, using only part of Lemma 3.12 1), construct an increasing family of maps f_n with the domain of $f_n = A_n \cup B_n$. Then the union of these functions is the required embedding. $\square_{3.18}$

Theorem 3.19 *Let $\boldsymbol{K}$ be a quasiminimal excellent class and suppose $H, H' \in \boldsymbol{K}$ satisfy the countable closure condition. Let $\mathcal{A}, \mathcal{A}'$ be* cl-*independent subsets of H, H' with $\mathrm{cl}(\mathcal{A}) = H$, $\mathrm{cl}(\mathcal{A}') = H'$, respectively, and ψ a bijection between $\mathcal{A}$ and $\mathcal{A}'$. Then ψ extends to an isomorphism of H and H'.*

Suppose further, that some model of $\boldsymbol{K}$ contains an infinite cl-*independent set. Then the class of $\boldsymbol{K}$-structures which satisfy the countable closure condition is categorical in every uncountable cardinality.*

The remainder of this section is devoted to the proof of Theorem 3.19

Notation 3.20 *Fix a countable subset $\mathcal{A}_0$ and write $\mathcal{A}$ as the disjoint union of $\mathcal{A}_0$ and a set $\mathcal{A}_1$; without loss of generality, we can assume ψ*

is the identity on $\mathrm{cl}_H(\mathcal{A}_0)$ *and work over* $G = \mathrm{cl}_H(\mathcal{A}_0)$*. We may write* $\mathrm{cl}^*(X)$ *to abbreviate* $\mathrm{cl}(\mathcal{A}_0 X)$*.*

Lemma 3.21 *Suppose* X, Y *are subsets of* $\mathcal{A}_1$*. Suppose* $\mathbf{b} \in \mathrm{cl}(\mathcal{A}_0 X)$ *and* $\mathbf{c} \in \mathrm{cl}(\mathcal{A}_0 Y)$ *and* $p(\mathbf{b}, \mathbf{c}, \mathbf{g})$ *for some quantifier-free type* q *(over* $\mathbf{g} \in G$*). Then there is a map* π *into* $\mathrm{cl}(\mathcal{A}_0 Y)$ *whose domain includes* $\mathbf{bcg}$*, that fixes* $\mathbf{cg}$*, and such that* $p(\mathbf{b}^\pi \mathbf{cg})$ *holds.*

Proof. Choose finite $A^* \subset \mathcal{A}_0$ such that $\mathbf{g} \in \mathrm{cl}(A^*)$, $\mathbf{b} \in \mathrm{cl}(A^* X)$, and $\mathbf{c} \in \mathrm{cl}(A^* Y)$. Let $G_0 = \mathrm{cl}(A^* Y)$. Extend the identity map on G_0 to π_1 with domain $G_0 X$ by mapping $X - Y$ into $\mathcal{A}_0 - (A^* Y)$. By Assumption 3.7 .1, π_1 is a partial G_0-monomorphism. By Lemma 3.12 π_1 extends to a partial G_0-monomorphism π from $\mathrm{cl}(A^* XY)$ into $\mathrm{cl}^*(Y)$. Clearly π has the required property. $\square_{3.21}$

Lemma 3.22 *Suppose* X, Y *are subsets of* $\mathcal{A}_1$ *and that* ψ_X *and* ψ_Y *are each partial* G*-monomorphisms from* H *into* H' *with* $\mathrm{dom}\,\psi_X = \mathrm{cl}(\mathcal{A}_0 X)$ *and* $\mathrm{dom}\,\psi_Y = \mathrm{cl}(\mathcal{A}_0 Y)$ *that agree on* $\mathrm{cl}(\mathcal{A}_0 X) \cap \mathrm{cl}(\mathcal{A}_0 Y)$*. Then* $\psi_X \cup \psi_Y$ *is a partial* G*-monomorphism.*

Proof. By Lemma 3.12 there is a partial G-monomorphism ψ_{XY} which extends ψ_X and maps $\mathrm{cl}^*(XY)$ into H'. Another partial map on $\mathrm{cl}^*(X) \cup \mathrm{cl}^*(Y)$ is given by $\psi_X \cup \psi_Y$. It suffices to show that for any $\mathbf{b} \in \mathrm{cl}^*(X)$, $\mathbf{g} \in G$ and $\mathbf{c} \in \mathrm{cl}^*(Y)$ and any quantifier free R, $H' \models R(\psi_X(\mathbf{b}), \psi_{XY}(\mathbf{c}), \mathbf{g})$ if and only if $H' \models R(\psi_X(\mathbf{b}), \psi_Y(\mathbf{c}), \mathbf{g})$. So we are finished if we apply the following lemma to H'. To apply the Lemma, note that $\psi_{XY} \circ \psi_Y^{-1}$ is a partial G-monomorphism taking $\psi_Y(\mathbf{c})$ to $\psi_{XY}(\mathbf{c})$.

Lemma 3.23 *Let* $X, Y, Y' \subseteq \mathcal{A}_1$*. Let* $\mathbf{b} \in \mathrm{cl}(\mathcal{A}_0 X)$*,* $\mathbf{c} \in \mathrm{cl}(\mathcal{A}_0 Y)$*, and* $\mathbf{c}' \in \mathrm{cl}(\mathcal{A}_0 Y')$*. Suppose* f *is a partial* G*-monomorphism taking* $\mathbf{c}$ *to* $\mathbf{c}'$*, then* f *is a partial* $\mathrm{cl}^*(X)$ *monomorphism.*

Proof. If not there exists $\mathbf{b} \in \mathrm{cl}^*(X)$ and $\mathbf{g} \in G$ and a quantifier free R such that $R(\mathbf{b}, \mathbf{c}, \mathbf{g}) \wedge \neg R(\mathbf{b}, \mathbf{c}', \mathbf{g})$. Now apply Lemma 3.21, to obtain a map π into $\mathrm{cl}(\mathcal{A}_0 YY')$ which fixes $\mathbf{c}, \mathbf{c}', \mathbf{g}$ and such that $R(\mathbf{b}^\pi, \mathbf{c}, \mathbf{g}) \wedge \neg R(\mathbf{b}^\pi, \mathbf{c}', \mathbf{g})$. This contradicts that $\mathbf{c}, \mathbf{c}'$ are partially isomorphic over G. $\square_{3.23}$

$\square_{3.22}$

We have by straightforward induction.

Corollary 3.24 *Suppose* $\langle X_i : i < m \rangle$ *are subsets of* A *and that each* ψ_{X_i} *is a partial* G*-monomorphisms from* H *into* H' *with* $\mathrm{dom}\,\psi_{X_i} = \mathrm{cl}(\mathcal{A}_0 X_i)$ *and that for any* i, j*,* ψ_{X_i} *and* ψ_{X_j} *agree whenever both are defined. Then* $\psi_X \cup \psi_Y$ *is a partial* G*-monomorphism.*

Proof of Theorem 3.19. Note that $H = \lim_{X \subset \mathcal{A}; |X| < \aleph_0} \mathrm{cl}(X)$. We have the obvious directed system on $\{\mathrm{cl}(X) : X \subset \mathcal{A}; |X| < \aleph_0\}$. So the theorem follows immediately if for each finite X we can choose $\psi_X : \mathrm{cl}(X) \to H'$ so that $X \subset Y$ implies $\psi_X \subset \psi_Y$. We prove this by induction on $|X|$. Suppose $|Y| = n+1$ and we have appropriate ψ_X for $|X| < n+1$. We will prove two statements by induction.

1. $\psi_Y^- : \mathrm{cl}^-(Y) \to H'$ defined by $\psi_Y^- = \bigcup_{X \subset Y} \psi_X$ is a monomorphism.

2. ψ_Y^- extends to ψ_Y defined on $\mathrm{cl}(Y)$.

The first step is done by induction using Corollary 3.24 with the $X \subset Y$ with $|X| = n$ as the X_i. The exchange axiom is used to guarantee that the maps ψ_X agree where more than one is defined. The second step follows by Lemma 3.18.

We have shown that the isomorphism type of a structure in $\boldsymbol{K}$ with countable closures is determined by the cardinality of a basis for the geometry. If M is an uncountable model in $\boldsymbol{K}$ that satisfies the countable closure condition, the size of M is the same as its dimension so there is at most one model in each uncountable cardinality which has countable closures. It remains to show that there is at least one.

Lemma 3.25 *If there is an $H \in \boldsymbol{K}$ which contains an infinite* cl-*independent set, then there are members of $\boldsymbol{K}$ of arbitrary cardinality which satisfy the countable chain condition.*

Proof. Let L^* be a countable fragment of $L_{\omega_1,\omega}$ containing the sentence axiomatizing $\boldsymbol{K}$ and the formulas defining independence. Let X be a countable independent set of H, H_1 the closure of Xa where a is independent from X and let $H_0 = \mathrm{cl}_{H_1}(X)$. Note that $H_0 \prec_{L^*} H_1$ by a back and forth. Since cl is L^*-definable $H_0 = \mathrm{cl}_{H_0}(X)$. So we have a model which is isomorphic to a proper L^*-elementary extension. As in [42] since all members of $\boldsymbol{K}$ with countably infinite dimension are isomorphic, one can construct a continuous L^*-elementary increasing chain of members of $\boldsymbol{K}$ for any $\alpha < \aleph_1$. Thus we get a model of power $\aleph_1$. Note that each $H_{\aleph_1}$ has countable closures since the closure of any countable set is contained in a model isomorphic to H_ω. Continue the same construction, to construct a model of power $\aleph_2$. Now we use the categoricity established above to pass limit ordinals. Since the chain is L^*-elementary, each $H_\alpha \in \boldsymbol{K}$. Now we get arbitrarily large models using the categoricity at κ to construct a model of power κ^+. $\square_{3.19}$

Note that a sentence ψ can define a quasiminimal excellent class without being $\aleph_0$-categorical. But we could extend ψ to ψ' - the Scott sentence of the model with countably infinite cl-dimension and attain $\aleph_0$-categoricity.

The following corollary seems to rely on the categoricity argument. The key is Condition II (**??**) and for countable G it follows from Lemma 3.12. But in general we use Theorem 3.19.

Corollary 3.26 *Let $\boldsymbol{K}$ be a quasiminimal excellent class, with $G \subset H, H'$ all in $\boldsymbol{K}$. If $a \in H, a' \in H'$ realize the same quantifier free type over G (i.e.*

there is a G-monomorphism taking a to a') then there is a $\boldsymbol{K}$-isomorphism from $\mathrm{cl}(Ga)$ *onto* $\mathrm{cl}(Ga')$.

Thus (G, a, H) and (G, a', H') realize the same Galois type.

Exercise 3.27 *Define a notion of almost quasiminimality analogous to almost strong minimality and prove that almost quasiminimal classes are categorical in all powers ([44]).*

Question 3.28 *Zilber's proof of Theorem 3.19 is considerably more complicated. I think this is because he does not assume exchange. How would you modify the argument here to avoid the use of exchange?*

4

Covers of the Multiplicative Group of $\mathbb{Z}$

The first approximation to a quasiminimal axiomatization of complex exponentiation considers short exact sequences of the following form.

$$0 \to Z \to H \to F^* \to 0. \tag{4.1}$$

H is a torsion-free divisible abelian group (written additively), F is an algebraically closed field, and exp is the homomorphism from $(H, +)$ to $(F^*, \cdot)$, the multiplicative group of F. We can code this sequence as a structure for a language L:

$$(H, +, E, S),$$

where $E(h_1, h_2)$ iff $\exp(h_1) = \exp(h_2)$ and we pull back sum by the defining $H \models S(h_1, h_2, h_3)$ iff $F \models \exp(h_1) + \exp(h_2) = \exp(h_3)$. Thus H now represents both the multiplicative and additive structure of F.

To guarantee Assumption 3.7.2 we expand the language further. Let $\exp : H \mapsto F^*$. For each affine variety over $\mathbb{Q}$, $\hat{V}(x_1, \ldots x_n)$, we add a relation symbol V interpreted by $H \models V(h_1, \ldots, h_n)$ iff $F \models V(\exp(h_1), \ldots, \exp(h_n))$. This includes the definition of S mentioned above; we have some fuss to handle the pullback of relations which have 0 in their range.

Lemma 4.1 *There is an $L_{\omega_1,\omega}$-sentence Σ such that there is a* 1-1 *correspondence between models of Σ and sequences (4.1).*

The sentence asserts first that the quotient of H by E with $+$ corresponding to $\times$ and S to $+$ is an algebraically closed field. We use $L_{\omega_1,\omega}$ to guarantee the kernel is 1-generated. This same proviso insures that the relevant closure condition has countable closures.

Definition 4.2 *For $X \subset H \models \Sigma$,*

$$\mathrm{cl}(X) = \exp^{-1}(\mathrm{acl}(\exp(X))$$

where acl *is the field algebraic closure in F.*

Using this definition of closure the key result of [43] asserts:

Theorem 4.3 *Σ is quasiminimal excellent with the countable closure condition and categorical in all uncountable powers.*

Our goal is this section is to prove this result modulo one major algebraic lemma. We will frequently work directly with the sequence (1) rather than the coded model of Σ. Note that (1) includes the field structure on F. That is, two sequences are isomorphic if there are commuting maps H to H' etc. where the first two are group isomorphisms but the third is a field isomorphism.

It is easy to check that Conditions I and IV and countable closures are satisfied: cl gives a combinatorial geometry such that the countable closure of countable sets is countable. We need more notation about the divisible closure (in the multiplicative group of the field to understand the remaining conditions.

Definition 4.4 *By a* divisibly closed multiplicative subgroup *associated with $a \in \mathbb{C}^*$, $a^{\mathbb{Q}}$, we mean a* **choice** *of a multiplicative subgroup containing a and isomorphic to the additive group $\mathbb{Q}$.*

Definition 4.5 *We say $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$,* determine the isomorphism type *of $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$ over the subfield k of $\mathbb{C}$ if given subgroups of the form $c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$ and ϕ_m such that*

$$\phi_m : k(b_1^{\frac{1}{m}} \ldots b_\ell^{\frac{1}{m}}) \to k(c_1^{\frac{1}{m}} \ldots c_\ell^{\frac{1}{m}})$$

is a field isomorphism it extends to

$$\phi_\infty : k(b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}}) \to k(c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}}).$$

To see the difficulty consider the following example.

Example 4.6 *Let a_1 and a_2 be linearly independent over $\mathbb{Q}$ complex numbers such that $(a_1-1)2 = a_2$. Suppose ϕ, which maps $\mathbb{Q}(a_1, a_2)$ to $\mathbb{Q}(c_1, c_2)$, is a field isomorphism. ϕ does not extend to an isomorphism of their divisible hulls, we might have $a_1 - 1 = \sqrt{a_2}$ but $c_1 - 1 = -\sqrt{c_2}$.*

As in Lecture 3, for G a subgroup of H, H' and $H, H' \models \Sigma$, a partial function ϕ on H is called a G-monomorphism if it preserves L-quantifier free formulas with parameters from G.

Fact 4.7 *Suppose $b_1, \ldots b_\ell \in H$ and $c_1, \ldots c_\ell \in H'$ are each linearly independent sequences (from G) over $\mathbb{Q}$. Let $\hat{G}$ be the subfield generated by* $\exp(G)$. *If $\hat{G}(\exp(b_1)^{\mathbb{Q}}, \ldots \exp(b_\ell)^{\mathbb{Q}}) \approx \hat{G}(\exp(c_1)^{\mathbb{Q}}, \ldots \exp(c_\ell)^{\mathbb{Q}})$ as fields, then mapping b_i to c_i is a G-monomomorphism preserving each variety V.*

Proof. Let $G \subset H$ and suppose rational q_i, r_i are rational numbers, $h_i \in H - G, g_i \in G$. Then

$$H \models V(q_1 b_1, \ldots q_\ell b_\ell, r_1 g_1, \ldots r_m g_m)$$

iff

$$\hat{H} \models \hat{V}(\exp(q_1 b_1), \ldots \exp(q_\ell b_\ell), \exp(r_1 g_1), \ldots \exp(r_m g_m))$$

iff

$$\hat{G}(\exp(b_1)^{\mathbb{Q}}, \ldots \exp(b_\ell)^{\mathbb{Q}}, \exp(g_1)^{\mathbb{Q}}, \ldots \exp(g_m)^{\mathbb{Q}})$$
$$\models \hat{V}(\exp(q_1 b_1), \ldots \exp(q_\ell b_\ell), \exp(r_1 g_1), \ldots \exp(r_m g_m)).$$

From this fact, it is straightforward to see that Condition II in the definition of quasiminimal excellence holds. For II.i) we need that there is only one type of a closure-independent sequence. But Fact 4.7 implies that for $\mathbf{b} \in H$ to be closure independent, the associated $\exp(\mathbf{b})$ must be algebraically independent and of course there is a unique type of an algebraically independent sequence. For II.ii) holds since added to language of Σ predicates for the pull-back of all quantifier-free relations on the field F. (Zilber doesn't do this.)

For Condition III and the excellence condition we need an algebraic result. In the following, $\sqrt{1}$ denotes the subgroup of roots of unity. We call this result the thumbtack lemma based on the following visualization of Kitty Holland. The various *nth* roots of $b_1, \ldots b_m$ hang on threads from the b_i. These threads can get tangled; but the theorem asserts that by sticking in a finite number of thumbtacks one can ensure that the rest of strings fall freely. The proof involves the theory of fractional ideals of number fields, Weil divisors, and the normalization theorem. For $a_1, \ldots a_r$ in $\mathbb{C}$, we write $\mathrm{gp}(a_1, \ldots a_r)$ for the multiplicative subgroup generated by $a_1, \ldots a_r$. The following general version of the theorem is applied for various sets of parameters to prove quasiminimal excellence.

In the following Lemma we write $\sqrt{1}$ for the group of roots of unity. If any of the L_i are defined, the reference to $\sqrt{1}$ is redundant. We write $\mathrm{gp}(\boldsymbol{a})$ for the multiplicative subgroup generated by $\boldsymbol{a}$.

Remark 4.8 *Let k be an algebraically closed subfield in $\mathbb{C}$ and let $\boldsymbol{a} \in \mathbb{C} - k$. A field theoretic description of the relation of $\boldsymbol{a}$ to k arises by taking the* *irreducible variety over k realized by $\boldsymbol{a}$. $\boldsymbol{a}$ is a generic realization of variety given by a finite conjunction $\phi(\mathbf{x}, \mathbf{b})$ of polynomials generating the ideal in $k[\mathbf{x}]$ of those polynomials which annihilate $\boldsymbol{a}$. From a model theoretic standpoint we can say, choose $\mathbf{b}$ so that the type of $\boldsymbol{a}/k$ is the unique nonforking extension of $\mathrm{tp}(\boldsymbol{a}/\mathbf{b})$. We use the model theoretic formulation below. See [1], page 39.*

Theorem 4.9 (thumbtack lemma) *[43]*

Let $P \subset \mathbb{C}$ be a finitely generated extension of $\mathbb{Q}$ and $L_1, \ldots L_n$ algebraically closed subfields of the algebraic closure $\hat{P}$ of P. Fix multiplicatively divisible subgroups $a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}}$ with $a_1, \ldots a_r \in \hat{P}$ and $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$. If $b_1 \ldots b_\ell$ are multiplicatively independent over $\mathrm{gp}(a_1, \ldots a_r) \cdot \sqrt{1} \cdot L_1^* \cdot \ldots L_n^*$ then for some m $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$, determine the isomorphism type of $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}}$ over $P(L_1, \ldots L_n, \sqrt{1}, a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}})$.

Lemma 4.10 *Condition III of quasiminimal excellence holds.*

Proof. We must show: If $G \models \Sigma$ and f is a partial G-monomorphism from H to H' with finite domain $X = \{x_1, \ldots, x_r\}$ then for any $y \in H$ there is y' in some H'' with $H' \prec_{\boldsymbol{K}} H''$ such that $f \cup \{\langle y, y' \rangle\}$ extends f to a partial G-monomorphism. Since $G \models \Sigma$, $\exp(G)$ is an algebraically closed field. For each i, let a_i denote $\exp(x_i)$ and similarly for x_i', a_i'. Choose a finite sequence $\mathbf{d} \in \exp(G)$ such that the sequence $(a_1, \ldots a_r)$ is independent (in the forking sense) from $\exp(G)$ over $\mathbf{d}$ and $\mathrm{tp}(a_1, \ldots a_r)/\mathbf{d})$ is stationary. Now we apply the thumbtack lemma. Let P_0 be $\mathbb{Q}(\mathbf{d})$. Let $n = 1$ and L_1 be the algebraic closure of P_0. We set $P_0(\mathbf{d}, a_1, \ldots a_r)$ as P. Take b_1 as $\exp(y)$ and set $\ell = 1$.

Now apply Lemma 4.9 to find m so that $b_1^{\frac{1}{m}}$ determines the algebraic type of $(b_1)^{\mathbb{Q}}$ over $L_1(a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}}) = P_0(L_1, a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}})$. Let $\hat{f}$ denote the map f induces from $\hat{H}$ to $\hat{H}'$ over $\hat{G}$. Choose ${b_1'}^{\frac{1}{m}}$ to satisfy the quantifier free field type of $\hat{f}(\mathrm{tp}(b_1^{\frac{1}{m}}/L_1(a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}})$. Now by Lemma 4.9, $\hat{f}$ extends to field isomorphism between $L_1(a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}}, b_1^{\mathbb{Q}}))$ and $L_1((a_1')^{\mathbb{Q}}, \ldots (a_r')^{\mathbb{Q}}, (b_1')^{\mathbb{Q}})$. Since the sequence $a_1, \ldots a_r)$ is independent (in the forking sense) from $\exp(G)$ over L_1, we can extend this map to take $\exp(G)(a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}}, b_1^{\mathbb{Q}})$ to $\exp(G)((a_1')^{\mathbb{Q}}, \ldots (a_r')^{\mathbb{Q}}, (b_1')^{\mathbb{Q}})$ and pull back to find y'; this suffices by Fact 4.7.

$\square_{4.10}$

Note there is no claim that $y' \in H'$ and there can't be.

One of the key ideas discovered by Shelah in the investigation of nonelementary classes is that in order for types to be well-behaved one may have to make restrictions on the domain. (E.g., we may be able to amalgamate types over models but not arbitrary types.) This principle is illustrated by the following definition and result of Zilber.

Definition 4.11 *$C \subseteq F$ is finitary if C is the union of the divisible closure (in $\mathbb{C}^*$) of a finite set and finitely many algebraically closed fields.*

Now we establish Condition IV, excellence. Note that this is a stronger condition than excellence since there is no independence requirement on the G_i.

Lemma 4.12 *Let $G_1, \ldots G_n \subset H$ all be models of Σ and suppose each has finite* cl*-dimension. If $h_1, \ldots h_\ell \in G^- = \mathrm{cl}(G_1 \cup \ldots G_n)$ then there is finite set $A \subset G^-$ such that any ϕ taking $h_1, \ldots, h_\ell$ into H which is an A-monomorphism is also a G^--monomorphism.*

Let $L_i = \exp(G_i)$ for $i = 1, \ldots, n$; $b_j^q = \exp(qh_j)$ for $j = 1, \ldots, \ell$ and $q \in \mathbb{Q}$. We may assume the h_i are linearly independent over the vector space generated by the G_i; this implies the b_i are muliplicatively independent over $L_1^* \cdot L_2^* \cdot \ldots L_n^*$. Now apply the thumbtack lemma with $r = 0$. This gives an m such that the field theoretic type of $b_1^{\frac{1}{m}}, \ldots, b_\ell^{\frac{1}{m}}$ determines the quantifier free type of $(h_1, \ldots, h_\ell)$ over G^-. So we need only finitely many parameters from G^- and we finish.

To prove the following result, apply the thumbtack lemma with the L_i as the fields and the $\boldsymbol{a}_i$ as the finite set.

Corollary 4.13 *Any almost finite n-type over a finitary set is a finite n-type.*

Since we have established all the conditions for quasiminimal excellence, we have proved Theorem 4.3.

Keisler[16] proved Morley's categoricity theorem for sentences in $L_{\omega_1,\omega}$, assuming that the categoricity model was $\aleph_1$-homogeneous. This theorem is the origin of the study of homogeneous model theory which is well expounded in e.g. [4]. We give two examples showing the homogeneity does not follow from categoricity. Marcus [24] showed:

Fact 4.14 *There is a first order theory T with a prime model M such that*

1. *M has no proper elementary submodel.*
2. *M contains an infinite set of indiscernibles.*

Exercise 4.15 *Show that the $L_{\omega_1,\omega}$-sentence satisfied only by atomic models of the theory T in Fact 4.14 has a unique model.*

Example 4.16 *Now construct an $L_{\omega_1,\omega}$-sentence ψ whose models are partitioned into two sets; on one side is an atomic model of T, on the other is an infinite set. Then ψ is categorical in all infinite cardinalities but no model is $\aleph_1$-homogeneous because there is a countably infinite maximal indiscernible set.*

Now we see that the example of this chapter has the same inhomogeneity property.

Consider the basic diagram:

$$0 \to Z \to H \to F^* \to 0. \tag{4.2}$$

Let a be a transcendental number in F^*. Fix h with $\exp(h) = a$ and define $a_n = \exp \frac{h}{n} + 1$ for each n. Now choose h_n so that $\exp(h_n) = a_n$. Let $X_r = \{h_i : i \le r\}$. Note that $a_m = a^{\frac{1}{m}} + 1$ where we have chosen a specific mth root.

Claim 4.17 *$p_r = \mathrm{tp}(h/X_r)$ is a principal type.*

Proof. We make another application of the thumbtack Lemma 4.9 with $\mathbb{Q}(\exp(\mathrm{span}(X_r))$ as P , $a_1, \ldots a_r$ as themselves, all L_i are empty, and a as b_1. By the lemma there is an m such that $a^{\frac{1}{m}}$ determines the isomorphism type of $a^{\mathbb{Q}}$ over $P(a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}})$. That is if ϕ_m is the minimal polynomial of $a^{\frac{1}{m}}$ over P, $(\exists y)\phi_m(y) \wedge y^m = x$ generates $\mathrm{tp}(a/\exp(\mathrm{span}(X_r)))$. Pulling back by Lemma 4.7, we see $\mathrm{tp}(h/X_r)$ is principal and even complete for $L_{\omega_1,\omega}$. In particular, for any $m' \ge m$, any two m'th roots of a have the same type over $\exp(X_r)$. But for sufficiently large r, one of these m'th roots is actually in X_r so $\mathrm{tp}(a/X_r)$ does not imply $p = \mathrm{tp}(a/X)$ for any X. That is, $\mathrm{tp}(a/X)$ is not implied by its restriction to any finite set. And by Lemma 4.7 this implies $\mathrm{tp}_{\omega_1,\omega}(h/X)$ is not implied by its restriction to any finite set.

Now specifically to answer the question of Keisler [16], page 123, we need to show there is a sentence ψ in a countable fragment L^* of $L_{\omega_1,\omega}$ such that ψ is $\aleph_1$-categorical but has a model with is not $(\aleph_1, L^*)$-homogeneous. Fix L^* as a countable fragment containing the categoricity sentence for 'covers'. We have shown no formula of $L_{\omega_1,\omega}$ (let alone L^*) with finitely many parameters from X implies p. By the omitting types theorem for L^*, there is a countable model H_0 of ψ which contains an L^*-equivalent copy X' of X and omits the associated p'. By categoricity, H_0 imbeds into H. But H also omits p'. As, if $h' \in H$, realizes p', then $\exp(h') \in \mathrm{acl}(X') \subseteq H_0$ so since the kernel of exp is standard, $h' \in H_0$, contradiction. Thus the type p' cannot be realized so H is not homogeneous.

❚ USED?

Definition 4.18 *1. Let V be an irreducible variety over $C \subseteq F$. The* sequence associated with V *over C is a sequence*

$$\{V^{\frac{1}{m}} : m \in \omega\}$$

such that $V^1 = V$ *and for any* $m, n \in \omega$, *raising to the mth power maps* $V^{\frac{1}{nm}}$ *to* $V^{\frac{1}{n}}$.

2. *If* $V' \subseteq V$ *are varieties over* C, *the pair*

$$\tau = (V - V', \{V^{\frac{1}{m}} : m \in \omega\})$$

is called an almost finite n-type *over* C.

3. *Zilber calls a principal type given by a difference of varieties* $V - V'$ *a* finite n-type *over* C.

Sketch of Proof of Theorem 4.3. Another application of the thumbtack lemma gives directly the homogeneity conditions of Condition III (Assumption 3.8). Exchange, (Assumption 3.13) , is immediate from the definition of closure (4.2). Finitary sets are more general than the n-dimensional independent systems in the definition of quasiminimal excellence, since the subsets don't have to be independent. So if X is a sequence associated with a variety V over an n-dimensional independent system C, Corollary 4.13 allows us to reduce X to a formula over a finite set yielding Excellence (Assumption 3.15). So we finish by Theorem 3.19; the quasimimimal excellence implies categoricity.

5
Abstract Elementary Classes

'Non-elementary classes' is a general term for any logic other than first order. Some of the most natural extensions of first order logic arise by allowing conjunctions of various infinite lengths or cardinality quantifiers. In this chapter we introduce a precise notion of 'abstract elementary class' AEC which generalizes some of these logics. In this monograph we pursue a dual track of proving certain result for general AEC and some for very specific logics, especially $L_{\omega_1,\omega}$ and $L_{\omega_1,\omega}(Q)$. In this chapter we introduce the semantic notion of an AEC and prove a surprising syntactic representation theorem for such classes.

When Jónsson generalized the Fraïsse construction to uncountable cardinalities [13, 14], he did so by describing a collection of axioms, which might be satisfied by a class of models, that guaranteed the existence of a homogeneous-universal model; the substructure relation was an integral part of this description. Morley and Vaught [26] replaced substructure by elementary submodel and developed the notion of saturated model. Shelah [38, 39] generalized this approach in two ways. He moved the amalgamation property from a basic axiom to a constraint to be considered. (But this was a common practice in universal algebra as well.) He made the *substructure* notion a 'free variable' and introduced the notion of an *Abstract Elementary Class*: a class of structures and a 'strong' substructure relation which satisfied variants on Jonsson's axioms. To be precise

Definition 5.1 *A class of* L*-structures,* $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$*, is said to be an* abstract elementary class: AEC *if both* $\boldsymbol{K}$ *and the binary relation* $\prec_{\boldsymbol{K}}$ *are closed under isomorphism and satisfy the following conditions.*

- **A1**. *If* $M \prec_{\boldsymbol{K}} N$ *then* $M \subseteq N$.
- **A2**. $\prec_{\boldsymbol{K}}$ *is a partial order on* $\boldsymbol{K}$.
- **A3**. *If* $\langle A_i : i < \delta \rangle$ *is* $\prec_{\boldsymbol{K}}$*-increasing chain:*
  1. $\bigcup_{i<\delta} A_i \in \boldsymbol{K}$;
  2. *for each* $j < \delta$, $A_j \leq \bigcup_{i<\delta} A_i$
  3. *if each* $A_i \prec_{\boldsymbol{K}} M \in \boldsymbol{K}$ *then* $\bigcup_{i<\delta} A_i \leq M$.
- **A4**. *If* $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ *and* $A \subseteq B$ *then* $A \prec_{\boldsymbol{K}} B$.
- **A5**. *There is a Löwenheim-Skolem number* $\kappa(\boldsymbol{K})$ *such that if* $A \subseteq B \in K$ *there is a* $A' \in \boldsymbol{K}$ *with* $A \subseteq A' \prec_{\boldsymbol{K}} B$ *and* $|A'| < \kappa(\boldsymbol{K})$.

Property **A5** is sometimes called the coherence property and sometimes 'the funny axiom'. Perhaps best is the Tarski-Vaught property since it easily seen to follow in the first order case as an application the Tarski-Vaught test for elementary submodel. However, Shelah sometimes uses 'Tarski-Vaught' for the union axioms.

Exercise 5.2 *Show the class of well-orderings with* $\prec_{\boldsymbol{K}}$ *taken as end extension satisfies the first four properities of an AEC. Does it have a Löwenheim number?*

Exercise 5.3 *The models of a sentence of first order logic or any countable fragment of* $L_{\omega_1,\omega}$ *with the associated notion of elementary submodel as* $\prec_{\boldsymbol{K}}$ *gives an AEC with Löwenheim number* $\aleph_0$.

The logics $L(Q)$ and $L_{\omega_1,\omega}(Q)$ are not immediately seen as AEC. We discuss the connections in Chapter 6.

Notation 5.4 *For any class of model* $\boldsymbol{K}$, $I(\boldsymbol{K}, \lambda)$ *denotes the number of isomorphism types of members of* $\boldsymbol{K}$ *with cardinality* λ.

We call the next result: the presentation theorem. It allows us to replace the entirely semantic description of an abstract elementary class by a syntactic one. I find it extraordinary that the notion of an AEC which is designed to give a version of the Fraïsse construction and thus saturated models, also turns out to allow the use of the second great model theoretic technique of the 50's: Ehrenfeucht-Mostowski models.

Theorem 5.5 *If* K *is an AEC with Lowenheim number* $\mathrm{LS}(\boldsymbol{K})$ *(in a vocabulary* τ *with* $|\tau| \leq \mathrm{LS}(\boldsymbol{K})$*), there is a vocabulary* τ' *with cardinality* $|\mathrm{LS}(\boldsymbol{K})|$, *a first order* τ'*-theory* T' *and a set of* $2^{\mathrm{LS}(\boldsymbol{K})}$ *types* Γ *such that:*

$$\boldsymbol{K} = \{M' \restriction \tau : M' \models T' \text{ and } M' \text{ omits } \Gamma\}.$$

Moreover, if M' *is a* τ''*-substructure of* N' *where* M', N' *satisfy* T' *and omit* Γ *then* $M' \restriction \tau \prec_{\boldsymbol{K}} N' \restriction \tau$.

Proof. Let τ' contain n-ary function symbols F_i^n for $n < \omega$ and $i < \mathrm{LS}(\boldsymbol{K})$. We take as T' the theory which asserts only that its models are nonempty. For any τ'-structure M' and any $\boldsymbol{a} \in M$, let $M'_{\boldsymbol{a}}$ denote the subset of M' enumerated as $\{F_i^n(\boldsymbol{a}) : i < \mathrm{LS}(\boldsymbol{K})\}$ where $n = \lg(\boldsymbol{a})$; the only requirement on this enumeration is that the first n-elements are $\boldsymbol{a}$. The isomorphism type of $'_{\boldsymbol{a}}$ is determined by the quantifier free τ'-type of $\boldsymbol{a}$. Note that $M'_{\boldsymbol{a}}$ may not be either a τ' or even a τ-structure. Let Γ be the set of quantifier free τ'-types of finite tuples $\boldsymbol{a}$ such that $M'_{\boldsymbol{a}} \restriction \tau \notin \boldsymbol{K}$ or for some $\mathbf{b} \subset \boldsymbol{a}$, $M'_{\mathbf{b}} \restriction \tau \not\prec_{\boldsymbol{K}} M'_{\boldsymbol{a}} \restriction \tau$.

We claim T' and Γ suffice. That is, if $\boldsymbol{K}' = \{M' \restriction \tau : M' \models T'$ and M' omits $\Gamma\}$ then $\boldsymbol{K} = \boldsymbol{K}'$. Let the τ'-structure M' omit Γ; in particular, each $M'_{\boldsymbol{a}}$ is a τ-structure. Write M' as a direct limit of the finitely generated τ-structures $M'_{\boldsymbol{a}}$. (These may not be closed under the operations of τ'.) By the choice of Γ, each $M'_{\boldsymbol{a}} \restriction \tau \in \boldsymbol{K}$ and if $\boldsymbol{a} \subseteq \boldsymbol{a}'$, $M'_{\boldsymbol{a}} \restriction \tau \prec_{\boldsymbol{K}} M'_{\boldsymbol{a}'} \restriction \tau$, and so by the unions of chains axioms $M' \restriction \tau \in \boldsymbol{K}$. Conversely, if $M \in \boldsymbol{K}$ we define by induction on $|\boldsymbol{a}|$, structures $M_{\boldsymbol{a}}$ for each finite subset $\boldsymbol{a}$ of M. Let $M_\emptyset$ be any $\prec_{\boldsymbol{K}}$-substructure of M with cardinality $\mathrm{LS}(\boldsymbol{K})$ and let the $\{F_i^0 : i < \mathrm{LS}(\boldsymbol{K})\}$ be constants enumerating the universe of $M_\emptyset$. Given a sequence $\mathbf{b}$ of length $n+1$, choose $M_{\mathbf{b}} \prec_{\boldsymbol{K}} M$ with cardinality $\mathrm{LS}(\boldsymbol{K})$ containing all the $M_{\boldsymbol{a}}$ for $\boldsymbol{a} \subset \mathbf{b}$ of smaller cardinality. Let $\{F_i^{n+1}(\mathbf{b}) : i < \mathrm{LS}(\boldsymbol{K})\}$ enumerate the universe of $M_{\mathbf{b}}$ (and give the function the same value on any ordering of the range of $\mathbf{b}$). Now each $M_{\boldsymbol{a}} \restriction \tau \in \boldsymbol{K}$ and if $\mathbf{b} \subset \mathbf{c}$, $M_{\mathbf{b}} \prec_{\boldsymbol{K}} M_{\mathbf{c}}$ so M' omits Γ as required.

The moreover holds for the partial τ'-structures $M'_{\boldsymbol{a}}$ directly by the choice of Γ and extends to arbitrary structures by the union of chain axioms on an AEC. In more detail, we have M' is a direct limit of finite structures $M'_{\boldsymbol{a}}$ and N' is a $\prec_{\boldsymbol{K}}$-direct limit of $N'_{\boldsymbol{a}}$ where $M'_{\boldsymbol{a}} = N'_{\boldsymbol{a}}$ for $\boldsymbol{a} \in M$ because $M' \restriction \tau$ is a τ-substructure of $N' \restriction \tau$. Each $M'_{\boldsymbol{a}} \restriction \tau \prec_{\boldsymbol{K}} N' \restriction \tau$ so the direct limit $M' \restriction \tau$ is a strong submodel of $N' \restriction \tau$. $\square_{5.5}$

We have represented $\boldsymbol{K}$ as a $PC\Gamma$ class in the following sense.

Definition 5.6 *A* $PC(T, \Gamma)$ class *is the class of reducts to* $\tau \subset \tau'$ *of models of a first order theory* τ'*-theory which omit all types from the specified collection* Γ *of types in finitely many variables over the empty set.*

We write $PC\Gamma$ *to denote such a class without specifying either* T *or* Γ. *And we write* $\boldsymbol{K}$ *is* $PC(\lambda, \mu)$ *if* $\boldsymbol{K}$ *can be presented as* $PC(T, \Gamma)$ *with* $|T| \leq \lambda$ *and* $|\Gamma| \leq \mu$. *In the simplest case, we say* $\boldsymbol{K}$ *is* λ*-presented if* $\boldsymbol{K}$ *is* $PC(\lambda, \lambda)$.

In this language we have shown any AEC $\boldsymbol{K}$ is $2^{\mathrm{LS(K)}}$-representable.

Keisler [16] proves a number of strong results for $PC\Gamma$-classes (more precisely for what he calls PC_δ-classes which are somewhat more special). In particular, he proves a categoricity transfer theorem between cardinals κ and λ of certain specific forms. (See Theorem 24 of Keislerbook). But the following example of Silver highlights the weakness of $PC\Gamma$-classes and the need to study AEC.

Example 5.7 *Let $\boldsymbol{K}$ be class of all structures (A, U) such that $|A| \leq 2^|U|$. Then $\boldsymbol{K}$ is actually a PC-class. But $\boldsymbol{K}$ is κ categorical if and only if $\kappa = \beth_\alpha$ for a limit ordinal α. (i.e. $\mu < \kappa$ implies $2^\mu < \kappa$.) Thus there are PC-classes for the which both the categoricity spectrum and its complement are cofinal in all cardinals.*

Exercise 5.8 *Show that if $\boldsymbol{K}$ is an AEC in a similarity type of cardinality λ, $\boldsymbol{K}$ can be presented as a $PC\Gamma(\lambda, 2^\lambda)$-class.*

Remark 5.9 *1. There is no use of amalgamation in this theorem.*

2. The only penalty for increasing the size of the language or the Löwenheim number is that the size of L' and the nunber of types omitted; thus θ must be chosen larger.

3. We can (and Shelah does) observe that the class of pairs (M, N) with $M \prec_{\boldsymbol{K}} N$ forms a $PC\Gamma(\aleph_0, 2^{\aleph_0})$ but it seems that the moreover clause of Theorem 5.5 is a more useful version. See Theorem 11.9 and its applications. The moreover clause appears in Grossberg's account: [7] and in Makowsky's [23].

We will see many problems can be reduced to classes of structures of the following sort.

Definition 5.10 *1. A* finite diagram *or $EC(T, \Gamma)$* class *is the class of models of first order theory which omit all types from a specified collection Γ of types in finitely many variables over the empty set.*

2. $EC(T, Atomic)$ denotes the class of atomic *models of T.*

The last definition abuses the $EC(T, \Gamma)$ notation, since for consistency, we really should write nonatomic. But atomic is shorter and emphasizes that we are restricting to the atomic models of T.

Some authors attach the require that $\boldsymbol{K}$ satisfy amalgamation over sets to the definition of finite diagram. We stick with the original definition from [27] and reserve the more common term, *homogeneous model theory* for the classes with set amalgamation.

Exercise 5.11 *The models of an $EC(T, \Gamma)$ with the ordinary first order notion of elementary submodel as $\prec_{\boldsymbol{K}}$ gives an AEC with Löwenheim number $\aleph_0$.*

Restricting an AEC to models of bounded cardinality or even to a single cardinal provides an important tool for studying the entire class. We introduce here two notions of this sort. In [33], the notion of λ-frame is a strengthening of what we call here a weak AEC by introducing an abstract notion of dependence.

Definition 5.12 *1. For any AEC, $\boldsymbol{K}$ we write, e.g. $\boldsymbol{K}_{\leq\mu}$ for the associated class of structures in $\boldsymbol{K}$ of cardinality at most $\boldsymbol{K}$.*

2. *Note that if $\mu \geq \mathrm{LS}(\boldsymbol{K})$, ($\boldsymbol{K}_{<\mu}$) and $\boldsymbol{K}_{\leq\mu}$ have all properties of an AEC except the union of chain axioms apply only to chains of length of length $\leq \mu$ ($< \mu$).*

3. *We call such an object a weak AEC.*

Exercise 5.13 *If $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ is an abstract elementary class then the restriction of $\boldsymbol{K}$ and $\prec_{\boldsymbol{K}}$ to models of cardinality λ gives a weak abstract elementary class.*

The next two exercises are worked out in detail in [33].

Exercise 5.14 *If $\boldsymbol{K}_\lambda$ is a weak abstract elementary class show $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ is an AEC with Löwenheim number λ if $\boldsymbol{K}$ and $\prec_{\boldsymbol{K}}$ are all direct limits of $\boldsymbol{K}_\lambda$ and $\prec_{\boldsymbol{K}_\lambda}$ respectively.*

Exercise 5.15 *Show that if the AEC's $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ have Lówenheim number λ and the same restriction to models of size λ they are identical above λ.*

Jónsson's axioms included the amalgamation property and the joint embedding property. Here we consider them as additional properties; establishing amalgamation from hypotheses on the spectrum of $\boldsymbol{K}$ will be a major theme.

We say $\boldsymbol{K}$ has the amalgamation property if $M \leq N_1$ and $M \leq N_2 \in K$ with all three in $\boldsymbol{K}$ implies there is a common strong extension N_3 completing the diagram. Joint embedding means any two members of $\boldsymbol{K}$ have a common strong extension. Crucially, we amalgamate only over members of $\boldsymbol{K}$; this distinguishes this context from the context of homogeneous structures.

Lemma 5.16 *If $\boldsymbol{K}_{<\kappa}$ has the amalgamation property, then $\boldsymbol{K}_{<\kappa}$ is partitioned into a family of weak-AEC's that each have the joint embedding property.*

Proof. Define $M \simeq N$ if they have a common strong extension. Since $\boldsymbol{K}_{<\kappa}$ has the amalgamation property, $\simeq$ is an equivalence relation. It is not hard to check that each class is closed under short unions and so is a weak-AEC. $\square_{5.16}$

6
Non-definability of Well-order in $L_{\omega_1,\omega}(Q)$

In this chapter we extend the Lopez-Escobar/ Morley theorem [22, 25] on the nondefinability of well-order from $L_{\omega_1,\omega}$ to $L_{\omega_1,\omega}(Q)$. This extension seems to be well-known to cognoscenti but I was unable to find it in the literature. We begin with some background on $L_{\omega_1,\omega}(Q)$ and with attempts to regard the models of an $L_{\omega_1,\omega}(Q)$ as an AEC.

Definition 6.1 *The logic $L(Q)$ adds to first order logic the expression $(Qx)\phi(x)$ which holds if there are uncountably many solutions of ϕ. The analogous expansion of $L_{\omega_1,\omega}$ is called $L_{\omega_1,\omega}(Q)$.*

Exercise 6.2 *The models of a sentence of $L(Q)$ with the associated notion of elementary submodel as $\prec_{\boldsymbol{K}}$ does* not *give an AEC.*

It is easy to verify the following statement.

Definition 6.3 *Let ψ be a sentence in $L_{\omega_1,\omega}(Q)$ and let L^* be the smallest countable fragment of $L_{\omega_1,\omega}(Q)$ containing ψ. Define a class $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ by letting $\boldsymbol{K}$ be the class of models of ψ in the standard interpretation. We consider several notions of strong submodel.*

1. *$M \prec^* N$ if*
   (a) *$M \prec_{L^*} N$ and*
   (b) *$M \models \neg(Qx)\theta(x, \boldsymbol{a})$ then $\{b \in N : N \models \theta(b, \boldsymbol{a}) = \{b \in M : N \models \theta(b, \boldsymbol{a})$.*
2. *$M \prec^{**} N$ if*

(a) $M \prec_{L^{**}} N$,

(b) $M \models \neg(Qx)\theta(x, \boldsymbol{a})$ *then* $\{b \in N : N \models \theta(b, \boldsymbol{a}) = \{b \in M : N \models \theta(b, \boldsymbol{a})$, *and*

(c) $M \models (Qx)\theta(x, \boldsymbol{a})$ *implies* $\{b \in N : N \models \theta(b, \boldsymbol{a})$ *properly contains* $\{b \in M : N \models \theta(b, \boldsymbol{a})$.

The following exercises are easy but informative.

Exercise 6.4 $(\boldsymbol{K}, \prec^*)$ *is an AEC.*

Exercise 6.5 $(\boldsymbol{K}, \prec^{**})$ *is* not *an AEC. (Hint: Consider the second union axiom in Definition 5.1 and model with a definable uncountable set.)*

Remark 6.6 *The Löwenhheim number of the AEC* $(\boldsymbol{K}, \prec^*)$ *defined in Definition 6.3 is* $\aleph_1$. *We would like to translate an* $L_{\omega_1,\omega}(Q)$*-sentence to an AEC with Löwenhheim number* $\aleph_0$ *and which has at least approximately the same number of models in each uncountable cardinality. This isn't quite possible but a suitable substitute can be found. This translation will require several steps. We begin here with a fundamental result about* $L_{\omega_1,\omega}(Q)$*; in Chapter 7, we will complete the translation.*

Here are the background results in $L_{\omega_1,\omega}$. They are proved as Theorem 12 and Theorem 28 from [16].

Theorem 6.7 (Lopez-Escobar, Morley) *Let* ψ *be an* $L_{\omega_1,\omega}(\tau)$*-sentence and suppose* $P, <$ *are a unary and a binary relation in* τ. *Suppose that for each* $\alpha < \omega_1$, *there is a model* M_α *of* ψ *such that* $<$ *linear orders* $P(M_\alpha)$ *and* α *imbeds into* $(P(M_\alpha), <)$. *Then there is a (countable) model* M *of* ψ *such that* $(P(M), <)$ *contains a copy of the rationals.*

If N is linearly ordered, N is an *end extension* of M if every element of M comes before every element of $N - M$.

Theorem 6.8 *Let* L^* *be a countable fragment of* $L_{\omega_1,\omega}$. *If a countable model* M *has a proper* L^**-elementary end extension, then it has one with cardinality* $\aleph_1$.

These two results can be combined to show that if a sentence in $L_{\omega_1,\omega}$ has a model that linearly orders a set in order type ω_1 then it has a model of cardinality $\aleph_1$ where the order is not well-founded. We imbed that argument in proving the same result for $L_{\omega_1,\omega}(Q)$.

Theorem 6.9 *Let* τ *be a similarity type which includes a binary relation symbol* $<$ *and a predicate symbol* P. *Suppose* ψ *is a sentence of* $L_{\omega_1,\omega}(Q)$, $M \models \psi$, *and the order type of* $(P(M), <)$ *imbeds* ω_1. *There is a model* N *of* ψ *with cardinality* $\aleph_1$ *such that the order type of* $(P(N), <)$ *imbeds* $\mathbb{Q}$.

Proof. Extend the vocabulary τ to τ' by adding a function symbol $f_\phi(\mathbf{x}, y)$ for each formula $(Qy)\phi(y, \mathbf{x})$ in $L_{\omega_1,\omega}(Q)$. Expand M to a τ'-structure M' by interpreting f_ϕ as follows:

1. If $M \models (Qy)\phi(y, \boldsymbol{a})$, $(\lambda y)f_\phi(y, \boldsymbol{a})$ is a partial function with domain the solution set of $\phi(y, \boldsymbol{a})$ *onto* M.
2. If $M \models \neg(Qy)\phi(y, \boldsymbol{a})$, $(\lambda y)f_\phi(y, \boldsymbol{a})$ is a partial function with domain the solution set of $\phi(y, \boldsymbol{a})$ into the first ω elements of the order.

Now let L^* be a countable fragment of $L_{\omega_1,\omega}$ which contains every subformula of ψ which is in $L_{\omega_1,\omega}$ and formulas expressing the properties of the Skolem functions for $L_{\omega_1,\omega}(Q)$ that we have just defined. Let ψ^* be an L^*-sentence which asserts that 'ω is standard' and a translation of ψ obtained by replacing each subformula of ψ of the form $(Qy)\phi(y, \mathbf{z})$ $(\neg(Qy)\phi(y, \mathbf{z}))$ by the formula $f_\phi(y, \mathbf{z})$ is onto ($f_\phi(y, \mathbf{z})$ maps into ω). Then for any τ'-structure N of cardinality $\aleph_1$ which satisfies ψ^*, $N \restriction \tau$ is a model of ψ. Let the sentence χ assert M is an *end* extension of $P(M)$. For every $\alpha < \omega_1$ there is a model M_α of $\psi^* \wedge \chi$ with order type of $(P(M), <)$ greater than α. (Start with P as α and alternately take an L^*-elementary submodel and close down under $<$. After ω steps we have the P for M_α.) Now by Theorem 6.7 there is countable structure $(N_0, P(N_0))$ such that $P(N_0)$ contains a copy of $(Q, <)$ and N_0 is an end extension of $P(N_0)$. By Theorem 6.8, N_0 has an L^*-elementary extension N of cardinality $\aleph_1$. Clearly, $P(N)$ contains a copy of $(Q, <)$ and, as observed, $N \models \psi$. $\square_{6.9}$

7
Categoricity implies Completeness

The logic $L_{\omega_1,\omega}$ is obtained by extending the formation rules of first order logic to allow countable conjunctions and disjunctions. A *fragment* of $L_{\omega_1,\omega}$ is a set of formulas closed under subformula and the finitary operations (i.e. finite conjunction, negation and quantification).

7.1 Completeness

Definition 7.1 *A sentence ψ in $L_{\omega_1,\omega}$ is called complete if for every sentence ϕ in $L_{\omega_1,\omega}$, either $\psi \models \phi$ or $\psi \models \neg\phi$.*

In first order logic, the theory of a structure is a well-defined object; here such a theory is not so clearly specified. An infinite conjunction of first order sentences behaves very much like a single sentence; in particular it satisfies both the upward and downward Löwenheim Skolem theorems. In contrast, the conjunction of all $L_{\omega_1,\omega}$ true in an uncountable model may not have a countable model. In its strongest form Morley's theorem asserts: Let T be a first order theory having only infinite models. If T is categorical in some uncountable cardinal then T is complete and categorical in every uncountable cardinal. This strong form does not generalize to $L_{\omega_1,\omega}$; take the disjunction of a sentence which is categorical in all cardinalities with one that has models only up to, say, $\beth_2$. Since $L_{\omega_1,\omega}$ fails the upwards Löwenheim-Skolem theorem, the categoricity implies completeness argument that holds for first order sentences fails. However, if the $L_{\omega_1,\omega}$-sentence ψ is categorical in κ, then, applying the downwards Löwenheim-Skolem theorem, for every sentence ϕ either $\psi \rightarrow \phi$ or all models of ϕ have cardinality less than κ. So if ϕ and ψ are κ-categorical sentences with a common model of power κ they are equivalent. Such a sentence in necessarily $\aleph_0$-categorical (using downward Löwenheim-Skolem). Moreover, every countable structure is characterized by a complete sentence – its Scott sentence. So if a model satisfies a complete sentence, it is $L_{\infty,\omega}$-equivalent to a countable model.

For purposes of this chapter, one can think of $\tau = \tau'$ in the following. The greater generality will be used a bit later.

Definition 7.2 *Let $\tau \subseteq \tau'$.*

1. *A τ'-structure M is L^*-small for L^* a countable fragment of $L_{\omega_1,\omega}(\tau)$ if M realizes only countably many L^*-types.*

2. *A τ'-structure M is τ-small if realizes only countably many $L_{\omega_1,\omega}(\tau)$-types.*

Let M be the only model of power κ of an $L_{\omega_1,\omega}$-sentence ψ. We want to find sufficient conditions so that there is a complete sentence ψ' which implies ψ and is true in M. We will two such conditions: ψ has arbitrarily large models; ψ has few models of $\aleph_1$. One key tool for this analysis is a different representation of $L_{\omega_1,\omega}$-sentences.

It is quite easy to see:

Exercise 7.3 *If ψ is a complete sentence in $L_{\omega_1,\omega}$ in a countable language L then every model M of ψ realizes only countably many $L_{\omega_1,\omega}$-types.*

In general, an $L_{\omega_1,\omega}$-type may contain uncountably many formulas. But,

Exercise 7.4 *If the structure M realizes only countably many $L_{\omega_1,\omega}$-types, then for every tuple $\boldsymbol{a}$ in M there is a formula $\phi(\mathbf{x}) \in L_{\omega_1,\omega}$ such $M \models \phi(\mathbf{x}) \rightarrow \psi(\mathbf{x})$ for each $L_{\omega_1,\omega}$-formula true of $\boldsymbol{a}$.*

But we will give the short argument for the converse: small models have Scott sentences. A Scott sentence for a countable model M is a complete sentence satisfied by M; it characterizes M up to isomorphism among countable models. The Scott sentence for an uncountable small model is the Scott sentence for a countable L^*-submodel of M, where L^* is the smallest fragment containing a formula for each type realized in M.

Lemma 7.5 *Let M be a τ structure for some countable τ. If ψ is a sentence in $L_{\omega_1,\omega}$ and M is a model of ψ that realizes only countably many $L_{\omega_1,\omega}$-types then there is a complete $L_{\omega_1,\omega}$-sentence ψ' so that*

1. *$\psi' \models \psi$;*

2. *$M \models \psi'$.*

Proof. Let L^* be the smallest fragment of $L_{\omega_1,\omega}$ containing ψ and the conjunction of each countable type in $L_{\omega_1,\omega}$ type realized in M. Let N be a countable L^*-elementary submodel of M and let ψ' be a Scott sentence for N. Clearly ψ' is complete. By the choice of L^*, ψ' is in L^*; so $M \models \psi'$. $\square_{7.5}$

Theorem 7.6 *Let ψ be a sentence in $L_{\omega_1,\omega}$ in a countable vocabulary τ. Then there is a countable vocabulary τ' extending τ, a first order τ'-theory T, and a collection of τ'-types Γ such that reduct is a* 1-1 *map from the models of T which omit Γ onto the models of ψ.*

Proof. Expand τ to τ' by inductively adding a predicate $P_\phi(\mathbf{x})$ for each L^*-formula ϕ. Fix a model of ψ and expand it to a τ'-structure by interpreting the new predicates so that they represent each finite Boolean connective and quantification faithfully: E.g.

$$P_{\neg\phi(\mathbf{x})} \leftrightarrow \neg P_\phi(\mathbf{x}),$$

and

$$P_{(\forall \mathbf{x})\phi(\mathbf{x})} \leftrightarrow (\forall \mathbf{x}) P_\phi(\mathbf{x}),$$

and that, as far as first order logic can, the P_ϕ preserve the infinitary operations: for each i,

$$P_{\bigwedge_i \phi_i(\mathbf{x})} \rightarrow P_{\phi_i(\mathbf{x})}.$$

Let T be the first order theory of any such model and consider the set Γ of types

$$p_{\bigwedge_i \phi_i(\mathbf{x})} = \{\neg P_{\bigwedge_i \phi_i(\mathbf{x})}\} \cup \{P_{\phi_i(\mathbf{x})} : i < \omega\}.$$

Now if M is a model of T which omits all the types in Γ, $M|\tau \models \psi$ and each model of ψ has a *unique* expansion to a model of T which omits the types in Γ (since this is an expansion by definitions in $L_{\omega_1,\omega}$). $\square_{7.6}$

Since all the new predicates in the reduction described above are $L_{\omega_1,\omega}$-definable this is a natural extension of Morley's procedure of replacing each first order formula ϕ by a predicate symbol P_ϕ, thus guaranteeing amalgamation over sets for first order categorical T; the amalgamation does *not* follow in this case. In general, finite diagrams do not satisfy the upper Löwenheim-Skolem theorem.

Since there is a 1-1 correspondence between models of ψ and models of T that omit Γ, we can reduce spectrum considerations for sentences with arbitrarily large models to the study of $EC(T,\Gamma)$-classes (Definition 5.10). In addition, we have represented the models of ψ as a $PC\Gamma$ class in the following sense.

Definition 7.7 *A* $PC(T,\Gamma)$ class *is the class of reducts to $\tau \subset \tau'$ of models of a first order theory τ'-theory which omit all types from the specified collection Γ of types in finitely many variables over the empty set.*

We write $PC\Gamma$ *to denote such a class without specifying either* T *or* Γ. *And we write* $\boldsymbol{K}$ *is* $PC(\lambda,\mu)$ *if* $\boldsymbol{K}$ *can be presented as* $PC(T,\Gamma)$ *with* $|T| \leq \lambda$ *and* $|\Gamma| \leq \mu$. *In the simplest case, we say* $\boldsymbol{K}$ *is* λ*-presented if* $\boldsymbol{K}$ *is* $PC(\lambda,\lambda)$.

We have shown every complete $L_{\omega_1,\omega}$-sentence in a countable language is ω presented.

Exercise 7.8 *Show that* ψ *is a sentence in* $L_{\lambda^+,\omega}$ *in a language of cardinality* κ, ψ *is* μ*-presented where* μ *is the larger of* κ *and* λ.

Exercise 7.9 *In general a* $PC\Gamma$ *class will not be an AEC class of* τ *structures. Why?*

Now, modify the proof of Theorem 7.6 to show:

Exercise 7.10 *Let* ψ *be a complete sentence in* $L_{\omega_1,\omega}$ *in a countable language* L. *Then there is a countable language* L' *extending* L *and a first order* L'*-theory* T *such that reduct is a* 1-1 *map from the atomic models of* T *onto the models of* ψ. *So in particular,* any complete sentence of $L_{\omega_1,\omega}$ can be replaced (for spectrum purposes) by considering the atomic models of a first order theory.

7.2 Arbitrarily Large Models

To show a categorical sentence with arbitrarily large models extends to a complete sentence we need the method of Ehrenfeucht-Mostowski models. 'Morley's method' (Section 7.2 of [6]) is a fundamental technique in first order model theory. It is essential for the foundations of simplicity theory and for the construction of indiscernibles in infinitary logic. We quote the first order version here; in Lemma 12.4, we prove the analog for abstract elementary classes.

Notation 7.11
1. *For any linearly ordered set* $X \subseteq M$ *where* M *is a* τ'*-structure and* $\tau' \supseteq \tau$, *we write* $\mathbf{D}_\tau(X)$ *(diagram) for the set of* τ*-types of finite sequences (in the given order) from* X. *We will omit* τ *if it is clear from context.*
2. *Such a diagram of an order indiscernible set,* $\mathbf{D}_\tau(X) = \Phi$, *is called 'proper for linear orders'* .
3. *If* X *is a sequence of* τ*-indiscernibles with diagram* $\Phi = \mathbf{D}_\tau(X)$ *and any* τ *model of* Φ *has built in Skolem functions, then for any linear ordering* I, $EM(I,\Phi)$ *denotes the* τ*-hull of a sequence of order indiscernibles realizing* Φ.
4. *If* $\tau_0 \subset \tau$, *the reduct of* $EM(I,\Phi)$ *to* τ_0 *is denoted* $EM_{\tau_0}(I,\Phi)$.

Exercise 7.12 *Suppose τ 'contains Skolem functions' and $X \subset M$ is sequence of order indiscernibles with diagram Φ. Show that for any linearly ordered set Z, $EM(Z, \Phi)$ is a model that is τ-elementarily equivalent to M.*

Lemma 7.13 *If $(X, <)$ is a sufficiently long linearly ordered subset of a τ-structure M, for any τ' extending τ (the length needed for X depends on $|\tau'|$) there is a countable set Y of τ'-indiscernibles (and hence one of arbitrary order type) such that $\mathbf{D}_\tau(Y) \subseteq \mathbf{D}_\tau(X)$. This implies that the only (first order) τ-types realized in $EM(X, \mathbf{D}_{\tau'}(Y))$ were realized in M.*

We need a little background on orderings.

Definition 7.14 *A linear ordering $(X, <)$ is k-transitive if every map between increasing k-tuples extends to an order automorphism of $(X, <)$.*

Exercise 7.15 *Show any 2-transitive linear order is k-transitive for all finite k.*

Exercise 7.16 *Show there exist 2-transitive linear orders in every cardinal; hint: take the order type of an ordered field.*

Exercise 7.17 *If $\Phi(Y)$ is the diagram of a sequence of τ-order indiscernibles, show any order isomorphism of Y extends to an automorphism of the τ-structure $EM(Y, \Phi)$.*

Definition 7.18 *For any model M and a, B contained in M, the* Galois-type *of a over B in M is the orbit of a under the automorphisms of M which fix B.*

This notion of Galois type requires an ambient model M. We will speak indiscriminately of the number of Galois types in M as an upper bound on the number of Galois n-types over any finite n.

Exercise 7.19 *If Y is a 2-transitive linear ordering and then for any τ and Φ is proper for linear orders, $EM(Y, \Phi)$ has $|\tau|$ Galois types.*

Exercise 7.20 *For any reasonable logic $\mathcal{L}$ (i.e. a logic such that truth is preserved under isomorphism) and any model M the number of Galois types over the empty set in M is at most the number of $\mathcal{L}$-types over the empty set in M.*

Now we can make our first application of the omitting types theorem.

Corollary 7.21
1. *If an $L_{\omega_1,\omega}(\tau)$-sentence ψ has arbitrarily large models then in every infinite cardinality ψ has a model which realizes only countably many $L_{\omega_1,\omega}(\tau)$-types over the empty set.*
2. *Thus, if ψ is categorical in some cardinal, ψ is implied by a consistent complete sentence ψ'.*

Proof. By Theorem 7.6, we can extend τ to τ' and choose a first order theory T and a countable set of types Γ such $\text{mod}(\psi) = PC_\tau(T, \Gamma)$. Since ψ has arbitrarily large models we can apply Theorem 12.1 to find τ''-indiscernibles for a Skolemization of T in an extended language τ''. Now take an Ehrenfeucht-Mostowski τ''-model M for the Skolemization of T over a set of indiscernibles ordered by a 2-transitive dense linear order. Then for every n, M has only countably many orbits of n-tuples and so realizes only countably many types in any logic where truth is preserved by automorphism – in particular in $L_{\omega_1,\omega}$. So the τ-reduct of M realizes only countably many $L_{\omega_1,\omega}(\tau)$-types. If ψ is κ-categorical, let ψ' be the Scott sentence of this Ehrenfeucht-Mostowski model with cardinality κ. $\square_{7.21}$

The countability of the language is crucial for this result.

7.3 Few models in small cardinals

For the second case, $I(\aleph_1, \psi) < 2^{\aleph_1}$, we must quote some hard results. In particular, we rely on the undefinability of well-order in $L_{\omega_1,\omega}(Q)$, which we treated in Chapter 6

Theorem 7.22 *If the τ $L_{\omega_1,\omega}(Q)$-sentence ψ has a model of cardinality $\aleph_1$ which is L^*-small for every countable τ-fragment L^* of τ-small model of cardinality $\aleph_1$.*

Proof. Add to τ a binary relation $<$, interpreted as a linear order of M with order type ω_1. Using that M realizes only countably many types in any τ-fragment, write $L_{\omega_1,\omega}(Q)(\tau)$ as a continuous increasing chain of fragments L_α such that each type in L_α realized in M is a formula in $L_{\alpha+1}$. Extend the similarity type to τ'' by adding new $2n+1$-ary predicates $E_n(x, \mathbf{y}, \mathbf{z})$ and $n+1$-ary functions f_n. Let M satisfy $E_n(\alpha, \boldsymbol{a}, \mathbf{b})$ if and only if $\boldsymbol{a}$ and $\mathbf{b}$ realize the same L_α-type and let f_n map M^{n+1} into the initial ω elements of the order, so that $E_n(\alpha, \boldsymbol{a}, \mathbf{b})$ implies $f_n(\alpha, \boldsymbol{a}) = f_n(\alpha, \mathbf{b})$. Note:

1. $E_n(\beta, \mathbf{y}, \mathbf{z})$ refines $E_n(\alpha, \mathbf{y}, \mathbf{z})$ if $\beta > \alpha$;
2. $E_n(0, \boldsymbol{a}, \mathbf{b})$ implies $\boldsymbol{a}$ and $\mathbf{b}$ satisfy the same quantifier free τ-formulas;
3. If $\beta > \alpha$ and $E_n(\beta, \boldsymbol{a}, \mathbf{b})$, then for every c_1 there exists c_2 such that
   (a) $E_{n+1}(\alpha, c_1\boldsymbol{a}, c_2\mathbf{b})$ and
   (b) if there are uncountably many c such that $E_{n+1}(\alpha, c\boldsymbol{a}, c_1\boldsymbol{a})$ then there are uncountably many c such that $E_{n+1}(\alpha, c\mathbf{b}, c_2\mathbf{b})$.
4. f_n witnesses that for any $a \in M$ each equivalence relation $E_n(a, \mathbf{y}, \mathbf{z})$ has only countably many classes.

All these assertions can be expressed by an $L_{\omega_1,\omega}(Q)(\tau'')$ sentence ϕ. Let L^* be the smallest τ''-fragment containing $\phi \wedge \psi$. Now add a unary predicate symbol P and a sentence χ which asserts M is an *end* extension of $P(M)$. For every $\alpha < \omega_1$ there is a model M_α of $\phi \wedge \psi \wedge \chi$ with order type of $(P(M), <)$ greater than α. (Start with P as α and alternately take an L^*-elementary submodel and close down under $<$. After ω steps we have the P for M_α.) Now by Theorem 6.9 there is a structure N_1 of cardinality $\aleph_1$ satisfying $\phi \wedge \psi \wedge \chi$ such $P(N_1)$ is not well-founded. Fix an infinite decreasing sequence $d_0 > d_1 > \ldots$ in N_1. For each n, define $E_n^+(\mathbf{x}, \mathbf{y})$ if for some i, $E_n(d_i, \mathbf{x}, \mathbf{y})$. Now using i), ii) and iii) prove by induction on the quantifier rank of ϕ that $N \models E_n^+(\boldsymbol{a}, \mathbf{b})$ implies $N \models \phi(\boldsymbol{a})$ if and only if $N \models \phi(\mathbf{b})$ for every $L_{\omega_1,\omega}(Q)(\tau)$-formula ϕ. For each n, $E_n(d_0, \mathbf{x}, \mathbf{y})$ refines $E_n^+(\mathbf{x}, \mathbf{y})$ and by iv) $E_n(d_0, \mathbf{x}, \mathbf{y})$ has only countably many classes; so N is small. $\square_{7.22}$

Now we show that sentences of $L_{\omega_1,\omega}(Q)$ that have few models can be extended to complete sentences. We rely on the following result of Keisler [Theorem 45 of [16] for $L_{\omega_1,\omega}$ and Corollary 5.10 of [15].

Theorem 7.23 *For any $L_{\omega_1,\omega}(Q)$-sentence ψ and any fragment L^* containing ψ, if ψ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then for any $M \models \psi$ of cardinality $\aleph_1$, M realizes only countably many L^*-types over the empty set.*

Theorem 7.24 *If an $L_{\omega_1,\omega}$-sentence ψ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then there is a complete $L_{\omega_1,\omega}$-sentence ψ' that implies ψ and has a model of cardinality $\aleph_1$.*

Proof. By Theorem 7.23 every model of ψ of cardinality $\aleph_1$ is L^*-small for every countable fragment L^*. By Theorem 7.22 ψ has a model of cardinality $\aleph_1$ which is small. By Lemma 7.5, we finish. $\square_{7.24}$

So to study categoricity of $L_{\omega_1,\omega}$-sentence ψ, we have established the following reduction. If ψ has arbitrarily large models, without loss of generality, ψ is complete. If ψ has few models of power $\aleph_1$, we can study a subclass of the models of ψ defined by a complete $L_{\omega_1,\omega}$-sentence ψ'. We will in fact prove sufficiently strong results about ψ' to deduce a nice theorem for ψ. Note that since ψ' is complete, the models of ψ' form an $EC(T, \mathit{Atomic})$-class in an extended similarity type τ'.

Now we see how to phrase this result for $L_{\omega_1,\omega}(Q)$. By Theorem 7.23 and Theorem 7.22 we replace any $L_{\omega_1,\omega}(Q)$-sentence with few models in $\aleph_1$ with a ψ as in following definition.

Definition 7.25 *Let ψ be a small $L_{\omega_1,\omega}(Q)$-complete sentence with vocabulary τ in the countable fragment L^* of $L_{\omega_1,\omega}(Q)$. Form τ' by adding predicates for formulas as in Theorem 7.6 but also add for each formula $(Qx)\phi(x, \mathbf{y})$ a predicate $R_{(Qx)\phi(x,\mathbf{y})}$ and add the axiom*

$$(\forall x)(Qx)\phi(x, \mathbf{y}) \leftrightarrow R_{(Qx)\phi(x,\mathbf{y})}.$$

Let ψ' be the conjunction of the τ' $L_{\omega_1,\omega}(Q)$-axioms encoding this expansion. By $T(\psi)$, we mean the first order τ' theory containing all first order consequences of ψ'. Let K_1 be the class of atomic models of ψ'.

Notation 7.26 *1. Let $\leq^*$ be the relation on $\boldsymbol{K}_1$: $M \leq^* N$ if $M \prec_{\tau'} N$ and for each formula $\phi(x, \mathbf{y})$ and $\mathbf{m} \in M$, if $M \models \neg R_{(Qx)\phi(x,\mathbf{m})}$ then $R_{\phi(x,\mathbf{m})}$ has the same solutions in M and N.*

2. Let $\leq^{**}$ be the relation on $\boldsymbol{K}_1$: $M \leq^* N$ if $M \prec_{L'} N$ and for each formula $\phi(x, \mathbf{y})$ and $\mathbf{m} \in M$, if $M \models \neg R_{(Qx)\phi(x,\mathbf{m})}$ if and only if $R_{\phi(x,\mathbf{m})}$ has the same solutions in M and N.

It is easy to check that $(\boldsymbol{K}_1, \leq^*)$ is an AEC. but $(\boldsymbol{K}_1, \leq^{**})$ is not an AEC. It can easily happen that each of a family of models $M_i \leq^{**} M$ but $\bigcup_i M_i \not\leq^{**} M$. In the Chapter 8, we will explore some of the complications this causes.

8
A model in $\aleph_2$

A first order sentence with an infinite model has models in all cardinalities; in particular no sentence is *categorical*, has exactly one model. Sentences of $L_{\omega_1,\omega}$ may be categorical but only in cardinality $\aleph_0$. In the early 70's I asked whether a sentence of $L(Q)$ could have exactly one model of cardinality $\aleph_1$. Shelah [28] showed the answer was no with some additional set theoretic hypotheses that he removed in [38]. In this chapter we introduce methods of getting structural properties on the models in an AEC that have cardinality λ by restricting the number of models of cardinality λ^+. And from these conditions on models of cardinality λ we show the existence of a model of power λ^+. Most strikingly, we present Shelah's proof that if a sentence of $L_{\omega_1,\omega}(Q)$ is categorical in $\aleph_1$ then it has a model of cardinality $\aleph_2$.

The general setting here will be an AEC. We show first that if an AEC is categorical in λ and λ^+ and has no 'maximal triple' in power λ then it has a model in power λ^+. Then we show in $L_{\omega_1,\omega}$ there are no maximal triples in $\aleph_0$; we finish by massaging the proof to handle $L_{\omega_1,\omega}(Q)$.

Definition 8.1 *We say (M,N) is a* proper pair *in λ, witnessed by a, if we mean $M \prec_{\boldsymbol{K}} N$ and $a \in N - M$ and $|M| = |N| = \lambda$.*

The fixed a is not used in the next Lemma but plays a central role in the proof of Lemma 8.4.

Lemma 8.2 *If an AEC $\boldsymbol{K}$ is categorical in λ and has a proper pair (M,N) in λ then there is a model in $\boldsymbol{K}$ with cardinality λ^+.*

Proof. Let $M_0 = M$. For any α, given M_α, choose $M_{\alpha+1}$ so that $(M, N) \approx (M_\alpha, M_{\alpha+1})$ and take unions at limits. The union of M_α for $\alpha < \lambda^+$ is as required. $\square_{S8.2}$

Definition 8.3 *A* maximal triple *is a triple (M, a, N) such that a witnesses that (M, N) is a proper pair and if (M', N') satisfies $M \prec_{\boldsymbol{K}} M'$, $N \prec_{\boldsymbol{K}} N'$ and $M' \prec_{\boldsymbol{K}} N'$ then $a \in M'$.*

Lemma 8.4 *Suppose $\boldsymbol{K}$ is categorical in λ and λ^+. If there are no maximal triples of cardinality λ and there is a proper pair of cardinality λ then there is a proper pair of cardinality λ^+.*

Proof. Let a witness that (M_0, N_0) is a proper pair in λ. Since there are no maximal triples, we can construct proper pairs (M_i, N_i) such that M_{i+1} is a proper $\prec_{\boldsymbol{K}}$ extension of M_i and N_{i+1} is a $\prec_{\boldsymbol{K}}$ extension of N_i but no M_i contains a; that is, the properness of each (M_i, N_i) is witnessed by the same a. So $(\bigcup_{i<\lambda^+} M_i, \bigcup_{i<\lambda^+} N_i)$ is the required proper pair. $\square_{8.4}$

So we have shown the required result if we can show there are no maximal triples in λ. For this, we need two further definitions.

Definition 8.5

1. *$M \prec_{\boldsymbol{K}} N$ is a* cut-pair *if there exist models N_i for $i < \omega$ such that $M \prec_{\boldsymbol{K}} N_{i+1} \prec_{\boldsymbol{K}} N_i \prec_{\boldsymbol{K}} N$ and $\bigcap_{i<\omega} N_i = M$.*
2. *A proper pair (M, N) has a* first element *if there is an $a \in N$ such that for every N' such that M is a proper $\prec_{\boldsymbol{K}}$-submodel of N' and $N' \prec_{\boldsymbol{K}} N$, $a \in N' - M$.*

Note that if (M, a, N) is a maximal triple then (M, N) has a first element (though not conversely).

Example 8.6 *Let $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ be the collection of dense linear orders with elementary submodel. $(\mathbb{Q}, <)$ be the rational order.*

1. *$((-\infty, 1), [1, \infty))$ has a first element but $(-\infty, 1), 1, [1, \infty)$ is not a mazimal triple.*
2. *$(-\infty, \sqrt{}2)(\sqrt{}2, \infty)$ is a cut-pair.*

We give the idea of the following proof; the details are clear in both [38] and [9].

Lemma 8.7 *Suppose $\boldsymbol{K}$ is λ-categorical. If $\boldsymbol{K}$ has a cut-pair in cardinality λ and it has a pair with first element in λ, then $I(\lambda^+, \boldsymbol{K}) = 2^{\lambda^+}$.*

Proof. Let (M, N) be a cut-pair. For S a stationary subset of λ^+, define M_i^S for $i < \lambda^+$ so that (M_i, M_{i+1}) is isomorphic to (M, N) if i is 0 or a successor ordinal. But if i is a limit ordinal, let (M_i, M_{i+1}) be a cut-pair if $i \in S$ and for some a, let (M_i, a, M_{i+1}) have a first element a if $i \notin S$. Then, let $M^S = \bigcup_{i<\lambda^+} M_i^S$. Now, if $S_1 - S_2$ is stationary, $M^{S_1} \not\approx M^{S_2}$.

If f were an isomorphism between them by intersecting $S_1 - S_2$ with the cub of i such that f maps $M_i^{S_1}$ to $M_i^{S_2}$, we find a δ such $(M_\delta^{S_1}, M_{\delta+1}^{S_1})$ is a cut pair and $(M_\delta^{S_2}, a_\delta, M_{\delta+1}^{S_2})$ has a first element. But then f cannot be an isomorphism from $M_{\delta+1}^{S_2}$ onto $M_{\delta+1}^{S_2}$ since there is no place for a_δ to go.$\square_{8.7}$

Now we need the following result, which depends heavily on our restricting λ to be $\aleph_0$ and also requiring the AEC to be a $PC\Gamma(\aleph_0, \aleph_0)$ class. Some extensions to other cardinalities are mentioned in [38].

Lemma 8.8 *If $\boldsymbol{K}$ is a $\aleph_0$-categorical $PC\Gamma(\aleph_0, \aleph_0)$ class which is also an AEC and that has a model of power $\aleph_1$, then there is a cut pair in $\aleph_0$.*

Proof. Recall that $\boldsymbol{K}$ is the class of τ-reducts of models of a first order theory T, which omit a countable set Γ of types. Let $M \in \boldsymbol{K}$ be a model with universe $\aleph_1$; write M as $\bigcup_{i<\aleph_1} M_i$. For simplicity, assume the universe of M_0 is $\aleph_0$. Expand M to a τ^*-structure M^* by adding the order $\aleph_1$ and a binary function g such that $g(i, x)$ is a τ-isomorphism from M_0 to M_i. Note that a unary predicate P naming M_0 and a binary relation $R(x, y)$ such that $R(a, i)$ if and only $a \in M_i$ are easily definable from g. Moreover, for each i, $\{x : R(x, i)\}$ is closed under the functions of τ^*.

Let ψ be a sentence in $L_{\omega_1,\omega}(\tau^*)$ describing this situation. By Theorem 6.7 (Theorem 12 of [16]), there is a model N^* of ψ with cardinality $\aleph_0$ in which $<$ is not well-founded. For any $b \in N^*$, let

$$N_b = \{x \in N^* : R(x, b)\}.$$

Let a_i for $i < \omega$ be a properly descending chain. Then if $N_i = N_{a_i}$, which has universe $\{x \in N^* : R(x, a_i)\}$,

$$N_i \restriction \tau \prec_{\boldsymbol{K}} N^* \restriction \tau$$

and because of g, each N_i is τ-isomorphic to $P(N^*)$. Let I be the set of $b \in N^*$ such that for some i, $b < a_i$. Then

$$N_I = \bigcup_{b \in I} N_b \restriction \tau \prec_{\boldsymbol{K}} N^* \restriction \tau$$

by the union axiom, Definition 5.1 A3.3. Our required cut-pair is (N_I, N_0). $\square_{8.8}$

Theorem 8.9 *If $\boldsymbol{K}$ is a $\aleph_0$-categorical $PC\Gamma(\aleph_0, \aleph_0)$ class which is also an AEC and with a unique model of power $\aleph_1$, then there is a model of power $\aleph_2$.*

Proof. By Lemma 8.8, there is a cut-pair in $\aleph_0$. Since ψ is $\aleph_1$-categorical, Lemma 8.7 implies there is no pair with first element and hence no maximal triple in $\aleph_0$. So by Lemma 8.4 there is a proper pair in $\aleph_1$ and then by Lemma 8.2, there is a model of power $\aleph_2$. $\square_{8.9}$

Corollary 8.10 *An $\aleph_1$-categorical sentence ψ in $L_{\omega_1,\omega}$ has a model of power $\aleph_2$.*

Proof. By Theorem 7.24, we may assume ψ has the form of Theorem 8.9. $\square_{8.10}$

We want to extend Corollary 8.10 from $L_{\omega_1,\omega}$ to $L_{\omega_1,\omega}(Q)$. The difficulty is to find an appropriate AEC. By the arguments of Chapter 7, we can find an $L_{\omega_1,\omega}(Q)$-complete sentence ψ' which is satisfied by the model of cardinality $\aleph_1$.

Recall the associated classes $(\boldsymbol{K}_1, \leq^*)$ and $(\boldsymbol{K}_1, \leq^{**})$ from Chapter 7. Note that $\boldsymbol{K}_1$ may have $2^{\aleph_1}$ models of cardinality $\aleph_1$. So we will have to work with $(\boldsymbol{K}_1, \leq^{**})$ and we have to redo some of the previous arguments. The crucial difficulty is that $(\boldsymbol{K}_1, \leq^{**})$ does not satisfy Definition 5.1 A3.3

We need to check a variant on Lemma 8.8.

Lemma 8.11 $(\boldsymbol{K}_1, \leq^{**})$ *has a cut pair in* $\aleph_0$.

Proof. The argument is identical to Lemma 8.8 except for one key point. The penultimate sentence of the proof read: Then $N_I = \bigcup_{b\in I} N_b \restriction \tau \prec_{\boldsymbol{K}} N^* \restriction \tau$ by the union axiom Definition 5.1 A3.3. This is precisely the union axiom that fails for $\leq^{**}$. But in this situation, for any $i < \omega$ we have $N_I \leq^* N_{i+1} \leq^{**} N_i$ so $N_I \leq^{**} N_i$, which is exactly what we need. $\square_{8.11}$

Corollary 8.12 *An $\aleph_1$-categorical sentence ψ in $L_{\omega_1,\omega}(Q)$ has a model of power $\aleph_2$.*

Proof. Since Lemma 8.2 uses only A3.1 of Definition 5.1, which holds of $(\boldsymbol{K}_1, \leq^{**})$, it suffices to show there is a $(\boldsymbol{K}_1, \leq^{**})$ proper pair of cardinality $\aleph_1$ that are standard models of ψ. By Lemma 8.11, there is a $(\boldsymbol{K}_1, \leq^{**})$ cut-pair in $\aleph_0$. Again, Lemma 8.7 does not depend on A3.3. So applying it, we get $2^{\aleph_1}$ non-isomorphic models of $\boldsymbol{K}_1$ and since we took $\leq^{**}$-extensions $\aleph_1$-times, each is actually a standard model of ψ. But this contradicts the categoricity of ψ so there must no maximal triple in $(\boldsymbol{K}_1, \leq^{**})$. By Lemma 8.4 (which again does not depend on A3.3) we have a standard $(\boldsymbol{K}_1, \leq^{**})$ proper pair of cardinality $\aleph_1$ and we finish. $\square_{8.12}$

9
Galois types and Saturation

We work in this section under the following strong assumption.

Assumption 9.1 ***K*** *is an abstract elementary class.*

1. ***K*** *has arbitrarily large models.*

2. ***K*** *satisfies the amalgamation property and the joint embedding property.*

3. *The Lowenheim-number of* ***K****,* LS(***K***)*, is* $\aleph_0$*.*

▌next paragraph repeats earlier definition in aec chpa.

We say ***K*** has the amalgamation property if $M \leq N_1$ and $M \leq N_2 \in K$ with all three in ***K*** implies there is a common strong extension N_3 completing the diagram. Joint embedding means any two members of ***K*** have a common strong extension. Crucially, we amalgamate only over members of ***K***; this distinguishes this context from the context of homogeneous structures.

In this section we take advantage of joint embedding and amalgamation to find a monster model. We then define types in terms of orbits of stabilizers of submodels. This allows an identification of 'model-homogeneous' with 'saturated'. That is, we give an abstract account of Morley-Vaught [26].

Definition 9.2 *M is μ-model homogenous if for every $N \prec_{\boldsymbol{K}} M$ and every $N' \in \boldsymbol{K}$ with $|N'| < \mu$ and $N \prec_{\boldsymbol{K}} N'$ there is a $\boldsymbol{K}$-embedding of N' into M over N.*

To emphasize, this differs from the homogeneous context because the N must be *in* $\boldsymbol{K}$. It is easy to show:

Lemma 9.3 *If M_1 and M_2 are μ-model homogeneous of cardinality $\mu > \mathrm{LS}(\boldsymbol{K})$ then $M_1 \approx M_2$.*

Proof. If M_1 and M_2 have a common submodel N of cardinality $< \mu$, this is an easy back and forth. Now suppose N_1, (N_2) is a small model of M_1, (M_2) respectively. By the joint embedding property there is a small common extension N of N_1, N_2 and by model homogeneity N is embedded in both M_1 and M_2. $\square_{9.3}$

Note that in the absence of joint embedding to get uniqueness, we would (as in [38]) have to add to the definition of 'M is model homogeneous' that all models of cardinality $< \mu$ are embedded in M.

Exercise 9.4 *Suppose M is μ-model homogeneous with cardinality μ, $N_0, N_1, N_2 \in \boldsymbol{K}$ with $N_0 \prec N_1, N_2 \prec M$, and f is isomorphism between N_1 and N_2 over N_0. Then f extends to an automorphism of M.*

Theorem 9.5 *If $\mu *^{<\mu *} = \mu *$ and $\mu * \geq 2^{\mathrm{LS}(\boldsymbol{K})}$ then there is a model $\mathbb{M}$ of cardinality $\mu *$ which is model homogeneous.*

We call the model constructed in Theorem 9.5, the monster model. From now on all, structures considered are substructures of $\mathbb{M}$ with cardinality $< \mu *$. The standard arguments for the use of a monster model in first order model theory ([12, 5] apply here.

Definition 9.6 *Let $M \in \boldsymbol{K}$, $M \prec_{\boldsymbol{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The Galois type of a over M ($\in \mathbb{M}$) is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.*

We freely use the phrase, 'Galois type of a over M', dropping the ($\in \mathbb{M}$) since $\mathbb{M}$ is fixed. Note that *a priori* this notion depends on the embedding of Ma into an $N \in \boldsymbol{K}$ and the embedding of N into $\mathcal{M}$. Since we have assumed amalgamation, our usage is justified as long as the base is an $M \in \boldsymbol{K}$. In more general situations, the Galois type is an equivalence class of an equivalence relation on triples (M, a, N). This is an equivalence relation on the class of M that are amalgamations for extensions in the same cardinality. (See [40, 41].) Since we have amalgamation and have fixed $\mathcal{M}$, we don't need the extra notation. The following definition and exercise show the connection of the situation as described here with the more complicated description elsewhere. They are needed only to link with the literature.

Definition 9.7 *For $M \prec_{\boldsymbol{K}} N_1 \in \boldsymbol{K}$, $M \prec_{\boldsymbol{K}} N_2 \in \boldsymbol{K}$ and $a \in N_1 - M$, $b \in N_2 - M$, write $(M, a, N_1) \sim (M, b, N_2)$ if there exist strong embeddings f_1, f_2 of N_1, N_2 into some N^* which agree on M and with $f_1(a) = f_2(b)$.*

Exercise 9.8 *If $\boldsymbol{K}$ has amalgamation, $\sim$ is an equivalence relation.*

Exercise 9.9 *Suppose $\boldsymbol{K}$ has amalgamation and joint embedding. Show $(M, a, N_1) \sim (M, b, N_2)$ if and only if there are embeddings g_1 and g_2 of N_1, N_2 into $\mathbb{M}$ that agree on M and such that $g_1(a)$ and $g_2(b)$ have the same Galois type over $g_1(M)$ in $\mathbb{M}$.*

Definition 9.10 *The set of Galois types over M is denoted* $\mathrm{ga} - \mathrm{S(M)}$.

We say a Galois type p over M is realized in N with $M \prec_{\boldsymbol{K}} N \prec_{\boldsymbol{K}} \mathbb{M}$ if $p \cap N \neq \emptyset$.

Definition 9.11 *The model M is μ-Galois saturated if for every $N \prec_{\boldsymbol{K}} M$ with $|N| < \mu$ and every Galois type p over N, p is realized in M.*

Again, *a priori* this notion depend on the embedding of M into $\mathcal{M}$; but with amalgamation it is well-defined.

The following model-homogeneity=saturativity theorem was announced with an incomplete proof in [39]. Full proofs are given in Theorem 6.7 of [9] and .26 of [36]. Here, we give a simpler argument making full use of the amalgamation hypothesis. In Chapter **??**, we discuss what can be done with weaker amalgamation hypotheses.

Theorem 9.12 *For $\lambda > \mathrm{LS}(\boldsymbol{K})$, The model M is λ-Galois saturated if and only if it is λ-model homogeneous.*

Proof. It is obvious that λ-model homogeneous implies λ-Galois saturated. Let $M \prec_{\boldsymbol{K}} \mathbb{M}$ be λ-saturated. We want to show M is λ-model homogeneous. So fix $M_0 \prec_{\boldsymbol{K}} M$ and N with $M \prec_{\boldsymbol{K}} N \prec_{\boldsymbol{K}} \mathbb{M}$. Say, $|N| = \mu < \lambda$. We must construct an embedding of N into M. Enumerate $N - M$ as $\langle a_i : i < \mu \rangle$. We will define f_i for $i < \mu$ an increasing continuous sequence of maps with domain N_i and range M_i so that $M_0 \prec_{\boldsymbol{K}} N_i \prec_{\boldsymbol{K}} \mathbb{M}$, $M_0 \prec_{\boldsymbol{K}} M_i \prec_{\boldsymbol{K}} M$ and $a_i \in N_{i+1}$. The restriction of $\bigcup_{i<\mu} f_i$ to N is required embedding. Let $N_0 = M_0$ and f_0 the identity. Suppose f_i has been defined. Choose the least j such that $a_j \in N - N_i$. By the model homogeneity of $\mathbb{M}$, f_i extends to an automorphism $\hat{f}_i$ of $\mathbb{M}$. Using the saturation, let $b_j \in M$ realize the Galois type of $\hat{f}_i(a_j)$ over M_i. So there is an $\alpha \in \mathrm{aut}\,\mathbb{M}$ which fixes M_i and takes b_j to $\hat{f}_i(a_j)$. Choose $M_{i+1} \prec_{\boldsymbol{K}} M$ with cardinality μ and containing $M_i b_j$. Now $\hat{f}_i^{-1} \circ \alpha$ maps M_i to N_i and b_j to a_j. Let $N_{i+1} = \hat{f}_i^{-1} \circ \alpha(M_{i+1})$ and define f_{i+1} as the restriction of $\alpha^{-1} \circ \hat{f}_i$ to N_{i+1}. Then f_{i+1} is as required. $\square_{9.12}$

In the remainder of this section we discuss some important ways in which Galois types behave differently from 'syntactic types'.

Note that if $M \prec_{\boldsymbol{K}} N \prec_{\boldsymbol{K}} \mathbb{M}$, then $p \in$ ga$-$S(N) extends $p' \in$ ga$-$S(M) if for some (any) a realizing p and some (any) b realizing p' there is an automorphism α fixing M and taking a to b.

Lemma 9.13 *If $M = \bigcup_{i<\omega} M_i$ in an increasing chain of members of $\boldsymbol{K}$ and $\{p_i : i < \omega\}$ satisfies $p_{i+1} \restriction M_i = p_i$, there is a $p_\omega \in$ ga $-$ S(M) with $p_\omega \restriction M_i = p_i$ for each i.*

Proof. Let a_i realize p_i. By hypothesis, for each $i < \omega$, there exists f_i which fixes M_{i-1} and maps a_i to a_{i-1}. Let g_i be the composition $f_0 \circ f_1 \circ \ldots f_i$. Then g_i maps a_i to a_0, fixes M_0 and $g_i \restriction M_{i-1} = g_{i-1} \restriction M_{i-1}$. Let M'_i denote $g_i(M_i)$ and M' their union. Then $\bigcup_{i<\omega} g_i$ is an isomorphism between M and M'. So by model-homogeneity there exists an automorphism h of $\mathbb{M}$ with $h \restriction M_i = g_i \restriction M_i$ for each i. Now $g_i^{-1} \circ h$ fixes M_i and maps a_ω to a_i for each i. This completes the proof. $\square_{9.13}$

Now suppose we wanted to prove Lemma 9.13 for chains of length $\delta > \omega$. The difficulty can be seen at stage ω. In addition to the assumptions of Lemma 9.13, we are given $\{a_i : i \leq \omega\}$ and $f_{\omega,i}$ which fixes M_i and maps a_ω to a_i. We can construct g_i as in the original proof. The difficulty is to find g_ω which extends all the g_i and maps a_ω to a_0. In the argument for Lemma 9.13, we found a map h and an element (which we will now call a'_ω such that h takes a'_ω to a_0 while h extends all the g_i. We would be done if a_ω and a'_ω realized the same galois type over $M = M_\omega$. In fact, a_ω and a'_ω realized the same galois type over each M_i. So the following *locality* condition (for chains of length ω) would suffice for this special case. Moreover, by a further induction locality would give Lemma 9.13 for chains of arbitrary length. Unfortunately, locality probably does not hold for all AEC with amalgamation.

Definition 9.14 *$\boldsymbol{K}$ has κ-local galois types if for every continuous increasing chain $M = \bigcup_{i<\kappa} M_i$ of members of $\boldsymbol{K}$ and for any $p, q \in$ ga $-$ S(M): if $p \restriction M_i = q \restriction M_i$ for every i then $p = q$.*

We have sketched the proof of:

Lemma 9.15 *Suppose $\boldsymbol{K}$ has κ-local Galois types. If $M = \bigcup_{i<\kappa} M_i$ in an increasing chain of members of $\boldsymbol{K}$ and $\{p_i : i < \kappa\}$ satisfies $p_{i+1} \restriction M_i = p_i$, there is a $p_\kappa \in$ ga $-$ S(M) with $p_\kappa \restriction M_i = p_i$ for each i.*

Locality provides a key distinction between the general AEC case and homogenous structures. In homogeneous structures, types are syntactic objects and locality is trivial. Thus, as pointed out by Shelah, Hyttinen, and Buechler-Lessmann, Lemma 9.15 applies in the homogeneous context.

10
Homogeneity and Saturation

Assumption 10.1 $\boldsymbol{K}$ *is an abstract elementary class.*

In Chapter 9, we assumed the amalgamation property and developed the notion of Galois type. Here, we show these notions make sense without any amalgamation hypotheses. Of course the notions defined here yield the previous concepts when amalgamation holds.

The goal is to derive properties on embedding models from the realization of Galois types. We want to show that if M^1 realizes 'enough' types over M then any small extension N of M can be embedded into M^1. The idea is first published as 'saturation = model-homogeneity' in 3.10 of [38] (Theorem 10.8 below), where the proof is incomplete. Successive expositions in [36, 9], and by Baldwin led to this version, where the key lemma was isolated by Kolesnikov. In contrast to various of the expositions and like Shelah, we make no amalgamation hypothesis.

Whether we really gain anything by not assuming amalgamation is unclear. I know of no example where either λ-saturated or λ-model homogeneous structures are proved to exist without using amalgamation, at least in λ.

The key idea of the construction is that to embed N into M^1; we construct a $M^2 \prec_{\boldsymbol{K}} M^1$ and a $\boldsymbol{K}$-isomorphism f from M_2 onto an $N_2 \prec_{\boldsymbol{K}} N_3$ where $N \prec_{\boldsymbol{K}} N_3$. Then the coherence axiom tells us restricting f^{-1} to N, gives the required embedding. We isolate the induction step of the construction in the following lemma. We will apply the lemma in two settings. In one case $\overline{M}$ has the same cardinality as M and is presented with a filtration M_i. Then $\hat{M}$ will be one of the M_i. In the second, $\overline{M}$ is a larger saturated model and $\hat{M}$ will be chosen as a small model witnessing the realization of a type.

We work in the most general context with *no* amalgamation hypothesis. We state several definitions to indicate the exact context we are working in. The most appropriate background in Shelah in [36], not [32]. We use our own notation but the relation to his should be clear.

Definition 10.2 *1. For $M \prec_{\boldsymbol{K}} N_1 \in \boldsymbol{K}$, $M \prec_{\boldsymbol{K}} N_2 \in \boldsymbol{K}$ and $a \in N_1 - M$, $b \in N_2 - M$, write $(M, a, N_1) \sim_{At} (M, b, N_2)$ if there exist strong embeddings f_1, f_2 of N_1, N_2 into some N^* which agree on M and with $f_1(a) = f_2(b)$.*

2. Let $\sim$ be the transitive closure of $\sim_{AT}$ (as a binary relation on triples).

3. We say the Galois type a over M in N_1 is the same as the Galois type a over M in N_2 if $(M, a, N_1) \sim (M, b, N_2)$

Exercise 10.3 *If $\boldsymbol{K}$ has amalgamation, $\sim_{AT}$ is an equivalence relation and $\sim = \sim_{AT}$.*

But we do *not* assume amalgamation.

Notation 10.4 *The set of Galois types over M is denoted* $\mathrm{ga} - \mathrm{S(M)}$.

Definition 10.5 *1. We say the Galois type of a over M in N_1 is* strongly realized *in N with $M \prec_{\boldsymbol{K}} N$ if for some $b \in N$, $(M, a, N_1) \sim_{AT} (M, b, N)$.*

2. We say the Galois type of a over M in N_1 is realized *in N with $M \prec_{\boldsymbol{K}} N$ if for some $b \in N$, $(M, a, N_1) \sim (M, b, N)$.*

Now we need a crucial form of the definition of saturated from [36]

Definition 10.6 *The model M is μ-*Galois saturated *if for every $N \prec_{\boldsymbol{K}} M$ with $|N| < \mu$ and every Galois type p over N, p is* strongly *realized in M.*

Under amalgamation we could define saturation using realization and we would have an equivalent notion. Without amalgamation, the notion we have selected is obviously more restricted. For the moment we rely on the assertion in Definition 22 of [36] that in all 'interesting situations' we can use the strong form of saturation.

We use in this construction without further comment two basic observations. If f is a $\boldsymbol{K}$-isomorphism from M onto N and $N \prec_{\boldsymbol{K}} N_1$ there is an M_1 with $M \prec_{\boldsymbol{K}} M_1$ and an isomorphism f_1 (extending f) from M_1 onto N_1. (The dual holds with extensions of M.) Secondly, whenever $f_1 \circ f_2 : N \mapsto M$ and $g_1 \circ g_2 : N \mapsto M$ are maps in a commutative diagram, there is no loss of generality in assuming $N \prec_{\boldsymbol{K}} M$ and $f_1 \circ f_2$ is the identity.

Of course, under amalgamation of models of size $|M|$, we can delete the strongly in following hypothesis.

Lemma 10.7 *Suppose $M \prec_{\boldsymbol{K}} \overline{M}$ and $\overline{M}$* strongly realizes *all Galois-types over M. Let $f : M \mapsto N$ be a $\boldsymbol{K}$-isomorphism and $\tilde{N}$ a $\boldsymbol{K}$-extension of N. For any $a \in \tilde{N} - N$ there is a $b \in \overline{M}$ such that for any $\hat{M}$ with $Mb \subseteq \hat{M} \prec_{\boldsymbol{K}} \overline{M}$ and $|M| = |\hat{M}| = \lambda$, there is an N^* with $N \prec_{\boldsymbol{K}} N^*$ and an isomorphism $\hat{f}$ extending f and mapping $\hat{M}$ onto $\hat{N} \prec_{\boldsymbol{K}} N^*$ with $\hat{f}(b) = a$.*

Proof. Choose $\tilde{M}$ with $M \prec_{\boldsymbol{K}} \tilde{M}$ and extend f to an isomorphism $\tilde{f}$ of $\tilde{M}$ and $\tilde{N}$. Let $\tilde{a}$ denote $\tilde{f}^{-1}(a)$. Choose $b \in \overline{M}$ to strongly realize the Galois type of $\tilde{a}$ over M in $\tilde{M}$. Fix any $\hat{M}$ with $Mb \subseteq \hat{M} \prec_{\boldsymbol{K}} \overline{M}$ and $|M| = |\hat{M}| = \lambda$. By the definition of strongly realize, we can choose an extension M^* of $\tilde{M}$ and $h : \hat{M} \mapsto M^*$ with $h(b) = \tilde{a}$. Lift $\tilde{f}$ to an isomorphism f^* from M^* to an extension N^* of $\tilde{N}$. Then $\hat{f} = (f^* \circ h) \restriction \hat{M}$ and $\hat{N}$ is the image of $\hat{f}$. $\square_{10.7}$

A key point in both of the following arguments is that while the N_i eventually exhaust N, they are not required to be submodels (or even subsets) of N.

Here is the first application.

Theorem 10.8 *Assume $\lambda > \mathrm{LS}(\boldsymbol{K})$. A model M^2 is λ-Galois saturated if and only if it is λ-model homogeneous.*

Proof. It is obvious that λ-model homogeneous implies λ-Galois saturated. Let M^2 be λ-saturated. We want to show M^2 is λ-model homogeneous. So fix $M_0 \prec_{\boldsymbol{K}} M^2$ and N with $M_0 \prec_{\boldsymbol{K}} N$. Say, $|N| = \mu < \lambda$. We construct M^1 as a union of strong submodels M_i of M^2. At the same time we construct N^1 as the union of N'_i which are strong extensions of N and f_i mapping M_i onto N_i. Enumerate $N - M_0$ as $\langle a_i : i < \mu \rangle$. Let $N_0 = M_0$, $N'_0 = N$ and f_0 be the identity. At stage i, f_i, N_i, M_i, N'_i, are defined; we will construct N'_{i+1}, f_{i+1}, N_{i+1}, M_{i+1}. Apply Lemma 10.7 with a_j as a for the least j with $a_j \notin N'_i$; take M_i for M; M_{i+1} is any submodel of M^2 with cardinality μ that witnesses the Galois type of b over M_i in M^2 and plays the role $\hat{M}$ in the lemma; N'_i is $\tilde{N}$ and N_i is N. The role of $\overline{M}$ is taken by M^2 at all stages of the induction. We obtain f_{i+1} as $\hat{f}$, N_{i+1} as $\hat{N}$ and N'_{i+1} as N^*. Finally f is the union of the f_i and N^1 is the union of the N'_i. $\square_{10.8}$

Just how general is Theorem 10.8? It asserts the equivalence of 'M is λ-model homogeneous' with 'M is λ-saturated' and we claim to have proved this without assuming amalgamation. But the existence of either kind of model is near to implying amalgamation on $\boldsymbol{K}_{<\lambda}$. But it is only close. Let

ψ be a sentence of $L_{\omega_1,\omega}$ which has saturated models of all cardinalities and ϕ be a sentence of $L_{\omega_1,\omega}$ which does not have the amalgamation property over models. Now let $\boldsymbol{K}$ be the AEC defined by $\psi \vee \phi$ (where we insist that on each model either the $\tau(\psi)$-relations or the $\tau(\phi)$-relations are trivial but not both). Then $\boldsymbol{K}$ has λ-model homogeneous models of every cardinality (which are saturated) but does not have either the joint embedding or the amalgamation property (or any restriction thereof). However, with some mild restrictions we see the intuition is correct. First an easy back and forth gives us:

Lemma 10.9 *If $\boldsymbol{K}$ has the joint embedding property and $\lambda > \mathrm{LS}(\boldsymbol{K})$ then any two λ-model homogeneous models M_1, M_2 of power λ are isomorphic.*

Proof. It suffices to find a common strong elementary submodel of M_1 and M_2 with cardinality $< \lambda$ but this is guaranteed by joint embedding and $\lambda > \mathrm{LS}(\boldsymbol{K})$. $\square_{10.9}$

Definition 10.10 *For any AEC $\boldsymbol{K}$, and $M \in \boldsymbol{K}$ let $\boldsymbol{K}^M$ be the AEC consisting of all direct limits of strong substructures of M.*

Lemma 10.11 *Suppose M is a λ-model homogeneous member of $\boldsymbol{K}$.*

1. *$\boldsymbol{K}^M_{<\lambda}$ has the amalgamation property.*
2. *If $\boldsymbol{K}$ has the joint embedding property $\boldsymbol{K}_{<\lambda}$ has the amalgamation property.*

Proof. The first statement is immediate and the second follows since then by Lemma 10.9 we have $\boldsymbol{K}^M_{<\lambda} = \boldsymbol{K}_{<\lambda}$. $\square_{10.11}$

Now by Lemma 10.11 and Theorem 10.8 we have:

Corollary 10.12 *If $\boldsymbol{K}$ has a λ-saturated model and has the joint embedding property then $\boldsymbol{K}_{<\lambda}$ has the amalgamation property.*

The corollary, which is Remark 30 of [36], confirms formally the intuition that under mild hypotheses we need amalgamation on $\boldsymbol{K}_{<\lambda}$ to get saturated models of cardinality λ. But we rely on the basic equivalence, proved without amalgamation to establish this result.

Now we have a second application of the Lemma 10.7. This requires an amalgamation hypothesis. Theorem 10.14 is asserted without proof in 1.15 of [33]; another exposition of the argument is in [8].

Definition 10.13 *M_2 is σ-universal over M_1 if $M_1 \prec_{\boldsymbol{K}} M_2$ and whenever $M_1 \prec_{\boldsymbol{K}} M_2'$, with $|M_2'| \prec_{\boldsymbol{K}} \sigma$, there is a (partial isomorphism) fixing M_1 and taking M_2' into M_2.*

This is Definition 1.12 1) from [33]. Note that it does not require that all smaller models $\boldsymbol{K}$ imbed into M_2.

Theorem 10.14 *If $\boldsymbol{K}$ is λ-Galois stable and $\boldsymbol{K}_\lambda$ has the amalgamation property, then for every $M \in \boldsymbol{K}_\lambda$ there is an M^1 with cardinality λ that is λ-universal over M.*

Proof. Construct M^1 as a continuous union for $i < \lambda$ of M_i with $M_0 = M$, and each M_{i+1} realizes all Galois types over M_i. (The existence of the M_{i+1} is guaranteed by the amalgamation hypothesis.) Now fix any strong extension N of M. We will construct a $\boldsymbol{K}$-isomorphism f from M^1 into an extension N^1 of N with $N \subset \overline{N} \prec_{\boldsymbol{K}} N^1$, where $\overline{N}$ denotes the range of f. By the coherence axiom $f^{-1} \upharpoonright N$ is the required map.

To construct f, enumerate $N - M$ as $\langle a_i : i < \lambda \rangle$. We construct a continuous increasing sequence of maps f_i. Let $f_0 = 1_M$. Suppose we have defined f_i, N_i and N'_i with f_i taking M_i onto $N_i \prec_{\boldsymbol{K}} N'_i$. Now apply Lemma 10.7 with a_j as a for the least j with $a_j \notin N'_i$; take M_i for M; M_{i+1} plays the role of both $\overline{M}$ and $\hat{M}$ in the lemma; N'_i is $\tilde{N}$ and N_i is N. We obtain f_{i+1} as $\hat{f}$, N_{i+1} as $\hat{N}$ and N'_{i+1} as N^*. Finally f is the union of the f_i and N^1 is the union of the N'_i. $\square_{10.14}$

The formulation of these results and arguments followed extensive discussions with Rami Grossberg, and Monica Van Dieren. Alexei Kolesnikov singled out Lemma 10.7.

11
Galois Stability

In this section we show that a λ-categorical AEC is μ-stable for μ above the Löwenheim number and below λ. For convenience, we continue to assume the amalgamation property and thus the monster model. The key idea is that for a linear order I and model $EM(I, \Phi)$ automorphisms of I induce automorphisms of $EM(I, \Phi)$. And, automorphisms of $EM(I, \Phi)$ preserve types in *any* reasonable logic; in particular, automorphisms of $EM(I, \Phi)$ preserve Galois types. Note that a model N is (defined to be) stable if few types are realized *in* N. So if N is a brimful model (Definition 11.2) then the model N is σ-stable for every $\sigma < |N|$.

Since we deal with reducts, we will consider several structures with the same universe; it is crucial to keep the vocabulary of the structure in mind. The AEC under consideration has vocabulary τ; it is presented as reducts of models of theory T' (which omit certain types) in a vocabulary τ'. In addition, we have the class of linear orderings (LO) in the background.

We really have three AEC's: $(LO, \subset)$, $\boldsymbol{K}'$ which is $Mod(T')$ with submodel as τ'-closed subset, and $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$. We are describing the properties of the EM-functor between $(LO, \subset)$ and $\boldsymbol{K}'$ or $\boldsymbol{K}$. $\boldsymbol{K}'$ is only a tool that we are singling out to see the steps in the argument. The following definitions hold for any of the three classes and I write $\leq$ for the notion of substructure. In this section of the paper I am careful to use $\leq$ when discussing all three cases versus $\prec_{\boldsymbol{K}}$ for the AEC.

Definition 11.1 *M_2 is σ-universal over M_1 in N if $M_1 \leq M_2 \leq N$ and whenever $M_1 \leq M_2' \leq N$, with $|M_1| \leq |M_2'| \leq \sigma$, there is a $\leq$-embedding fixing M_1 and taking M_2' into M_2.*

I introduce one term for shorthand. It is related to Shelah's notion of *brimmed* in [33] but here the brimful model is bigger than the models it is universal over while brimmed models may have the same cardinality.

Definition 11.2 *M is* brimful *if for every $\sigma < |M|$, and every $M_1 \leq M$ with $|M_1| = \sigma$, there is an $M_2 \leq M$ with cardinality σ that is σ-universal over M_1 in M.*

The next notion just makes it easier to write the proof of the following Lemma.

Notation 11.3 *Let $I \subset J$ be linear orders. We say a and b in J realize the same cut over I and write $a \sim_I b$ if for every $j \in J$, $a < j$ if and only if $b < j$.*

Claim 11.4 (Lemma 3.7 of [37]) *The lexicographic linear order on $I = \lambda^{<\omega}$ is brimful.*

Proof. Let $J \subset I$ have cardinality $\theta < \lambda$. Since we can increase J without harm, we can assume $J = A^{<\omega}$ for some $A \subset \lambda$. Note that $\sigma \sim_J \tau$ if and only if for the least n such that $\sigma \restriction n = \tau \restriction n \in J$, $\sigma(n) \sim_A \tau(n)$. Thus there are only θ cuts over J realized in I. For each cut C_α, $\alpha < \theta$, we choose a representative $\sigma_\alpha \in I - J$ of length n such that $\sigma_\alpha \restriction n - 1 \in J$, so a cut has the form $\{\sigma_\alpha \widehat{\ } \tau : \tau \in \lambda^{<\omega}, \alpha < \theta\}$. We can assume any J^* extending J has the form $J^* = B^{<\omega}$ for some $B \subset \lambda$, say with $\mathrm{otp}(B) = \gamma$. Thus, the intersection of J^* with a cut in J is isomorphic to a subset of $\gamma^{<\omega}$. We finish by noting for any ordinal $|\gamma| = \theta$, $\gamma^{<\omega}$ can be embedded in $\theta^{<\omega}$. Thus, the required θ-universal set over J is $J \cup \{\sigma_\alpha \widehat{\ } \tau : \tau \in \theta^{<\omega}, \alpha < \theta\}$.

Qing Zhang has provided the following elegant argument for the last claim. First show by induction on γ there is a map g embedding γ in $\theta^{<\omega}$. (E.g. if $\gamma = \lim_{i<\theta} \gamma_i$, and g_i maps γ_i into $\theta^{<\omega}$, let for $\beta < \gamma$, $g(\beta) = i \widehat{\ } g_i(\beta)$ where $\gamma_i \leq \beta < \gamma_{i+1}$.) Then let h map $\gamma^{<\omega}$ into $\theta^{<\omega}$ by, for $\sigma \in \gamma^{<\omega}$ of length n, setting $h(\sigma) = \langle g(\sigma(0)), \ldots, g(\sigma(n-1)) \rangle$. $\square_{11.4}$

The argument for Claim 11.4 yields:

Corollary 11.5 *Suppose $\mu < \lambda$ are cardinals. Then for any $X \subset \mu^{<\omega}$ and any Y withX $\subseteq Y \subset \lambda^{<\omega}$ and $|X| = |Y| < \mu$, there is an order embedding of Y into $\mu^{<\omega}$ over X.*

Exercise 11.6 *For an ordinal γ, let $\gamma^{\omega *}$ denote the functions from ω to γ with only finitely many non-zero values. Show $\gamma^{\omega *}$ is a dense linear order and so is not isomorphic to $\gamma^{<\omega}$. Vary the proof above to show $\gamma^{\omega *}$ is brimful.*

Since every τ'-substructure N of $EM(I, \Phi)$ is contained in a substructure $EM(I_0, \Phi)$ for some subset I_0 of I with $|I_0| = |N|$, we have immediately:

Claim 11.7 *If I is brimful as a linear order, $EM(I, \Phi)$ is brimful as an τ'-structure.*

Recall, Morley's omitting types theorem.

▌ repeating earlier statement

Lemma 11.8 *If $(X, <)$ is a sufficiently long linearly ordered subset of a τ-structure M, for any τ' extending τ (the length needed for X depends on $|\tau'|$) there is a countable set Y of τ'-indiscernibles (and hence one of arbitrary order type) such that $\mathbf{D}_\tau(Y) \subseteq \mathbf{D}_\tau(X)$. This implies that the only (first order) τ-types realized in $EM(X, \mathbf{D}_{\tau'}(Y))$ were realized in M.*

Using this result, we can find Skolem models over sets of indiscernibles in an AEC.

Theorem 11.9 *If $\boldsymbol{K}$ is an abstract elementary class in the vocabulary τ, which is represented as a $PC\Gamma$ class witnessed by τ', T', Γ that has arbitrarily large models, there is a τ'-diagram Φ such that for every linear order $(I, <)$ there is a τ'-structure $M = EM(I, \Phi)$ such that:*

1. *$M \models T'$.*
2. *The τ'-structure $M = EM(I, \Phi)$ is the Skolem hull of I.*
3. *I is a set of τ'-indiscernibles in M.*
4. *$M \restriction \tau$ is in $\boldsymbol{K}$.*
5. *If $I' \subset I$ then $EM_\tau(I', \Phi) \prec_{\boldsymbol{K}} EM_\tau(I, \Phi)$.*

Proof. The first four clauses are a direct application of Lemma 12.1, Morley's theorem on omitting types. See also problem 7.2.5 of Chang-Keisler [6]. It is automatic that $EM(I', \Phi)$ is an L' substructure of $EM(I, \Phi)$. The moreover clause of Theorem 5.5 allows us to extend this to $EM_\tau(I', \Phi) \prec_{\boldsymbol{K}} EM_\tau(I, \Phi)$. $\square_{11.9}$

Now using amalgamation and categoricity, we move to the AEC $\boldsymbol{K}$. There are some subtle uses here of the 'coherence axiom': $M \subseteq N \prec_{\boldsymbol{K}} N_1$ and $M \prec_{\boldsymbol{K}} N_1$ implies $M \prec_{\boldsymbol{K}} N$.

Claim 11.10 *If I is brimful as linear order, $EM_\tau(I, \Phi)$ is brimful as a member of $\boldsymbol{K}$.*

Proof. Let $M = EM(I, \Phi)$; we must show $M \restriction \tau$ is brimful as a member of $\boldsymbol{K}$. Suppose $M_1 \prec_{\boldsymbol{K}} M \restriction \tau$ with $|M_1| = \sigma < |M|$. Then there is $N_1 = EM(I', \Phi)$ with $|I'| = \sigma$ and $M_1 \subseteq N_1 \leq M$. By Lemma 11.9.5, $N_1 \restriction \tau \prec_{\boldsymbol{K}} M \restriction \tau$. So $M_1 \prec_{\boldsymbol{K}} N_1 \restriction \tau$ by the coherence axiom. Let M_2 have cardinality σ and $M_1 \prec_{\boldsymbol{K}} M_2 \prec_{\boldsymbol{K}} M \restriction \tau$. Choose a τ'-substructure N_2 of M with cardinality σ containing N_1 and M_2. Now, N_2 can be embedded by a map f into the σ-universal τ'-structure N_3 containing N_1 which is guaranteed by Claim 11.7. But $f(N_2) \restriction \tau \prec_{\boldsymbol{K}} N_3 \restriction \tau$ by the coherence axiom so $N_3 \restriction \tau$ is the required σ-universal extension of M_1. $\square_{11.10}$

Definition 11.11 *1. Let $N \subset \mathbb{M}$. N is λ-Galois-stable if for every $M \subset N$ with cardinality λ, only λ Galois types over M are realized in N.*

2. $\boldsymbol{K}$ is λ-Galois-stable if $\mathbb{M}$ is. That is $\mathrm{aut}_{\mathrm{M}}(\mathbb{M})$ has only λ orbits for every $M \subset \mathbb{M}$ with cardinality λ.

Since we are usually working in an AEC, we will frequently abuse notation and write stable rather than Galois-stable.

Since each Galois type over M_0 realized in M is represented by an M_1 with $M_0 \prec_{\boldsymbol{K}} M_1 \prec_{\boldsymbol{K}} M$, $M = EM(I, \phi)$ brimful, and $|M_1| = |M_0|$, Claim 11.10 implies immediately:

Claim 11.12 *If $\boldsymbol{K}$ is λ-categorical, the model M with $|M| = \lambda$ is σ-Galois stable for every $\sigma < \lambda$.*

Theorem 11.13 *If $\boldsymbol{K}$ is categorical in λ, then $\boldsymbol{K}$ is σ-Galois-stable for every $\sigma < \lambda$.*

Proof. Suppose $\boldsymbol{K}$ is not σ-stable for some $\sigma < \lambda$. Then by Löwenheim-Skolem, there is a model N of cardinality σ^+ which is not σ-stable. Let M be the σ-stable model with cardinality λ constructed in Claim 11.12. Categoricity and joint embedding imply N can be embedded in M. The resulting contradiction proves the result. $\square_{11.13}$

Remark 11.14 *Again, the assumption that $\boldsymbol{K}$ has amalgamation isn't needed here; instead of using Löwenheim-Skolem from the monster, one can use amalgamation on $\boldsymbol{K}_{<\lambda}$ and get joint embedding by restricting to the equivalence class of the categoricity model.*

Corollary 11.15 *Suppose $\boldsymbol{K}$ is categorical in λ and λ is regular. The model of power λ is saturated and so model homogeneous.*

Proof. Choose in $M_i \prec_{\boldsymbol{K}} \mathbb{M}$ using $< \lambda$-stability and Löwenheim-Skolem, for $i < \lambda$ so that each M_i has cardinality $< \lambda$ and M_{i+1} realizes all types over M_i. By regularity, it is easy to check that M_λ is saturated. $\square_{11.15}$

The same argument gives saturated models in smaller regular cardinals; more strongly we can demand that the saturated model be an Ehrenfreuht-Mostowski model.

Corollary 11.16 *Suppose $\boldsymbol{K}$ is an AEC with vocabulary τ that is categorical in λ and λ is regular. Then for every regular μ, $\mathrm{LS}(\boldsymbol{K}) < \mu < \lambda$ there is a model $M_\mu = EM_\tau(I_\mu, \Phi)$ which is saturated. In particular, it is μ-model homogeneous.*

Proof. For any ordered set J of cardinality λ, let $N = EM_\tau(J, \phi)$ be the model of cardinality λ. We construct an alternating chain of $\boldsymbol{K}$-submodels of length μ. $M_0 \prec_{\boldsymbol{K}} M$ is arbitrary with cardinality μ. $M_{2\alpha+1}$ has cardinality μ and realizes all types over $M_{2\alpha}$ (possible by Corollary 11.15). $M_{2\alpha+2}$ has cardinality μ, $M_{2\alpha+1} \prec_{\boldsymbol{K}} M_{2\alpha+2}$ and $M_{2\alpha+2}$ is $EM_\tau(I_{\alpha+1}, \Phi)$ where $I_\alpha \subset I_{\alpha+1} \subset J$ and all I_α have cardinality μ. Then $EM_\tau(I_\mu, \Phi) = \bigcup_{\alpha<\mu} EM_\tau(I_\alpha, \Phi)$ is saturated by regularity. $\square_{11.16}$

Now using stability we can get a still stronger result, eliminating the hypothesis that μ is regular. We show the proofs of both Corollary 11.16 and Corollary 11.17 since in the first case we constructed a saturated model directly and in the second a model homogeneous structure.

Corollary 11.17 *Suppose $\boldsymbol{K}$ is categorical in λ and λ is regular. Then for every μ, $\mathrm{LS}(\boldsymbol{K}) < \mu < \lambda$ there is a model $M_\mu = EM(\mu^{<\omega}, \Phi)$ which is μ-model homogeneous.*

Proof. Represent the categoricity model as $M^* = EM_\tau(\lambda^{<\omega}, \Phi)$. We show $M_\mu = EM_\tau(\mu^{<\omega}, \Phi)$ is model homogenous. Suppose $M_1 \prec_{\boldsymbol{K}} M_\mu \restriction \tau$ with $|M_1| = \sigma < |M_\mu|$. Then there is $N_1 = EM_\tau(I_1, \Phi)$ with $|I_1| = \sigma$, $M_1 \subset N_1$ and $I_1 \subset \mu^{<\omega}$. Let M_2 have cardinality σ and $M_1 \prec_{\boldsymbol{K}} M_2$. By amalgamation, choose $N_2 \in \boldsymbol{K}$ which is an amalgam of N_1 and M_2 over M_1. By the λ-model homogeneity of M^*, there is an embedding of N_2 into M^* over N_1 say with image N_2'. Then $N_2' \subset EM(J, \Phi)$ for some J with $I_1 \subset J \subset \lambda^{<\omega}$ and $|J| = \sigma$. Now by Corollary 11.5 and an argument like that in Claim 11.10, there is an embedding of $EM_\tau(J, \Phi)$ into $M = EM_\tau(\mu^{<\omega}, \Phi)$ over N_1, and *a fortiori* over M_1 and we finish. $\square_{11.17}$

Exercise 11.18 *Show Corollary 11.17 can be marginally strengthened by dropping the hypothesis that λ is regular but requiring that μ be less than the cofinality of λ.*

12
Omitting types and Downward Categoricity

We begin by stating a general version of Morley's omitting types theorem and formalizing the phrase 'sufficiently long' used in the last chapter. Then we deduce the analogous result for abstract elementary classes.

> We include some results on downward categoricity that may eventually form another chapter or disappear.

We quote without proof the combined form Morley's omitting types theorem and his 'two cardinal theorem for cardinals far apart. The exact formulation is from VII.5.3 of [29].

Lemma 12.1 *Suppose Γ is a set of τ-types, T a τ-theory, and P is a one-place τ-predicate. Suppose M is a τ-structure such that*

1. $|M| \geq \beth_{(2^{|\tau|})^+})$
2. $|M| \geq \beth_{(2^{|\tau|})^+}(|P(M)|)$

For every $\mu \geq |\tau|$, there is a model N of T such that

1. $|N| = \mu$;
2. *N omits Γ;*
3. $|P(N)| = |\tau|$.

Definition 12.2 1. *Let $\mu(\kappa)$ be the Hanf number for omitting 2^κ types for a first order theory with vocabulary of size κ.*

2. We write $\mu(\tau)$ for $\mu(|\tau|)$.

We can restate the last result as follows.

Corollary 12.3 *$\mu(\kappa)$ is $\beth_{(2^{|\tau|)})^+})$.*

Now we prove 'Morley's method' for Galois types.

Lemma 12.4 *[II.1.5 of 394] Let $\boldsymbol{K}$ be an AEC in vocabulary of cardinality κ. If $M_0 \leq M$ and $|M| \geq \mu(\kappa)$, we can find an EM-set Φ such that the following hold.*

1. *The τ-reduct of the Skolem closure of the empty set is M_0.*
2. *For every I, $M_0 \leq EM(I, \Phi)$.*
3. *If I is finite, $EM_\tau(I, \Phi)$ can be embedded in M.*
4. *$EM_\tau(I, \Phi)$ omits every galois type over N which is omitted in M.*

Proof. Let τ_1 be the Skolem language given by the presentation theorem and consider M as the reduct of τ_1 structure M^1. Add constants for M_0 to form τ_1'. Now apply Lemma 12.1 to find an EM-diagram Φ (in τ_1') with all τ-types of finite subsets of the indiscernible sequence realized in M. Now 1) and 2) are immediate. 3) is easy (using clause 5) of Theorem 11.9 since we chose Φ so all finite subsets of the indiscernible set (and so their Skolem closures) are realized in M.

The omission of Galois types is more tricky. Consider both M and $N = EM_\tau(I, \Phi)$ embedded in $\mathbb{M}$. Let N^1 denote the τ_1'-structure $EM(I, \Phi)$. We need to show that if $a \in N$, $p = \mathrm{ga-tp(a/M_0)}$ is realized in M. For some $\mathbf{e} \in I$, a is in the τ_1-Skolem hull $N_{\mathbf{e}}$ of $\mathbf{e}$. (Recall the notation from the presentation theorem.) By 3) there is an embedding α of $N_{\mathbf{e}}$ into M^1 over M_0. α is also an isomorphism of $N_{\mathbf{e}} \restriction \tau$ into M. Now, by the model homogeneity, α extends to an automorphism of $\mathbb{M}$ fixing M_0 and $\alpha(a) \in M$ realizes p. $\square_{12.4}$

Now we can rephrase this result as

Corollary 12.5 *The Hanf number for omitting Galois types in any AEC with a vocabulary of size κ is $\mu(\kappa)$.*

This result has immediate applications in the direction of transferring categoricity.

Theorem 12.6 *Suppose $M \in \boldsymbol{K}$ omits a Galois type p over a submodel M_0 with $|M| \geq \mu(|M_0|)$. Then there is no regular cardinal $\lambda \geq |M|$ in which $\boldsymbol{K}$ is categorical.*

Proof. By Lemma 12.4, there is a model $N \in \boldsymbol{K}$ with cardinality λ which omits p. But by Lemma 11.15, the unique model of power λ is saturated. $\square_{12.6}$

Key point for application: If $|P(M)| < \beth_{(2^{|\tau|})^+}$ then $\beth_{(2^{|\tau|})^+}(|P(M)|) = \beth_{(2^{|\tau|})^+}$.

(If $\kappa < \beth_\mu$, there is an $\alpha < \mu$ with $\beth_\alpha \geq \kappa$. So for every $\beta < \mu$,

$$\beth_\beta(\kappa) \leq \beth_{\alpha+\beta} \leq \beth_\mu.$$

So

$$\beth_\mu(\kappa) = \beth_\mu.$$

In [32] Shelah asserts the following result:

Theorem 12.7 *If $\boldsymbol{K}$ is categorical in a regular cardinal λ and $\lambda > \mu(\mu(|\tau|))$ then $\boldsymbol{K}$ is categorical in every θ with $\mu(|\tau|) \leq \theta \leq \lambda$.*

Here is a sketch of the argument. We have shown that there are saturated models of power θ for every $\theta < \lambda$. The obstacle to deducing downward categoricity is that Theorem 12.4 only allows us to transfer the omission of types when the model omitting the type is much bigger than the domain of the type. The first step in remedying this problem is to show that all types are determined by 'relatively small' subtypes. More precisely, we need the notion that Grossberg and Van Dieren [8] have called χ-tame and Shelah [32] refers to has 'having χ-character'. We add an extra parameter to be careful.

Definition 12.8 *We say $\boldsymbol{K}$ is (χ, μ)-tame if for any saturated $N \in \boldsymbol{K}$ with $|N| = \mu < \lambda$ if $p, q, \in \mathrm{ga} - \mathrm{S(N)}$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ then $q = p$.*

Shelah asserts the following in Sections II.1 and II.2.3 of the published version of [32]. The published proof is incomplete; I haven't yet seen the corrections. But it seems to use only Ehrenfeucht-Mostowski type methods.

Theorem 12.9 *Suppose $\boldsymbol{K}$ is λ-categorical for $\lambda \geq \mu(\tau)$ and λ is regular. Then $\boldsymbol{K}$ is (χ, χ_1)-tame for some $\chi < \mu(\tau)$ and any χ_1 with $\chi < \chi_1 \leq \lambda$.*

The naive argument would give $\chi = \mu(\tau)$ since one is omitting types. But omitting in every cardinal below $\mu(\tau)$ is as good as in $\mu(\tau)$ so the conclusion becomes for some χ with $\chi < \mu(\tau)$.

13
Atomic AEC

In this brief section, we describe the setting of the rest of the book.

The following assertion is an exercise in Lecture 4.

Let ψ be a complete sentence in $L_{\omega_1,\omega}$ in a countable language L. Then there is a countable language L' extending L and a first order L'-theory T such that reduct is a 1-1 map from the atomic models of T onto the models of ψ. So in particular, *any complete sentence of $L_{\omega_1,\omega}$ can be replaced (for spectrum purposes) by considering the atomic models of a first order theory.*

This section is indirectly based on [28, 30, 31], where most of the results were originally proved. But our exposition owes a great deal to [20, 19, 17, 10].

Definition 13.1 *A model M is* atomic *if every finite sequence in M realizes a principal type over the* empty set.

Thus if T is $\aleph_0$-categorical every model of T is atomic.

Notation 13.2 *We say an AEC, $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ is if $\boldsymbol{K}$ is the class of atomic models of a first order theory and $\prec_{\boldsymbol{K}}$ is elementary submodel.*

Assumption 13.3 We work in this section entirely in the following context. $\boldsymbol{K}$ is the class of atomic models of a complete first order theory T. Note that with $\prec_{\boldsymbol{K}}$ as $\prec$, elementary submodel, this is an abstract elementary class. Moreover, $\boldsymbol{K}$ is $\aleph_0$-categorical and every member of $\boldsymbol{K}$ is $\aleph_0$-homogeneous. We write $\mathbb{M}$ for the monster model of T; in interesting cases $\mathbb{M}$ is not in $\boldsymbol{K}$.

Definition 13.4 *Let A be an atomic set; $S_{\text{at}}(A)$ is the collection of $p \in S(A)$ such that if $\boldsymbol{a} \in \mathbb{M}$ realizes p, $A\boldsymbol{a}$ is atomic.*

If A is countable and $p \in S_{\text{at}}(A)$ then p is realized in the unique countable $M \in \boldsymbol{K}$. But this may fail for uncountable A; indeed unless $\boldsymbol{K}$ is homogeneous, there will be some A and $p \in S_{\text{at}}(A)$ which is not realized in a model ([21]).

Definition 13.5 *$\boldsymbol{K}$ is λ-stable if for every M of cardinality λ, $|S_{\text{at}}(M)| = \lambda$.*

To say $\boldsymbol{K}$ is ω-stable in this sense is strictly weaker than requiring $|S_{\text{at}}(A)| = \aleph_0$ for arbitrary countable A ([21]).

Example 13.6 *Consider two structures $(\mathbb{Q}, <)$ and $(\mathbb{Q}, +, \cdot, <)$. If $\boldsymbol{K}_1$ is class of atomic models of the theory of dense linear order without endpoints, then $\boldsymbol{K}_1$ is not ω-stable; $\text{tp}(\sqrt{2}; \mathbb{Q}) \in S_{\text{at}}(\mathbb{Q})$. If $\boldsymbol{K}_2$ is class of atomic models of the theory of the ordered field of rationals, then $\boldsymbol{K}_2$ is ω-stable; $\text{tp}(\sqrt{2}; \mathbb{Q}) \notin S_{\text{at}}(\mathbb{Q})$.*

Definition 13.7 *We say B is* atomic over *A if for every $\mathbf{b} \in B$, there is a formula $\phi(\mathbf{x}, \boldsymbol{a})$ such that $\phi(\mathbf{x}, \boldsymbol{a}) \rightarrow \psi(\mathbf{x}, \boldsymbol{a}')$ for every $\psi(\mathbf{x}, \boldsymbol{a}')$ such that $\psi(\mathbf{b}, \boldsymbol{a}')$.*

This formulation assumes we have the diagram of A. Since A is atomic, we can further for any $\boldsymbol{a}' \in A$, if $\theta(\mathbf{w}, \mathbf{y})$ generates $\text{tp}(\boldsymbol{a}'\boldsymbol{a}/\emptyset)$ then $\theta(\mathbf{w}, \mathbf{y}) \wedge \phi(\mathbf{x}, \boldsymbol{a}) \rightarrow \psi(\mathbf{x}, \mathbf{y})$.

The following result is a standard combination of generating formulas.

Exercise 13.8 *If C is atomic over B and B is* atomic over *A then C is atomic over A.*

Definition 13.9 *M is primary over A if there is a sequence $M = A \cup \langle e_i : i < \lambda \rangle$ and $\text{tp}(e_j/AE_{<j})$ is isolated for each j.*

Definition 13.10 *$M \in \boldsymbol{K}$ is prime over A if every elementary map from A into $N \in \boldsymbol{K}$ extends to an elementary map from M into N.*

Now it is an easy induction to show:

Exercise 13.11 *If M is primary over A, then M is atomic over A and prime over A.*

Definition 13.12 *$M \in \boldsymbol{K}$ is prime over A if every elementary map from A into $N \in \boldsymbol{K}$ extends to an elementary map from M into N.*

14
The ω_1-order property

In this chapter $\boldsymbol{K}$ is atomic.

Definition 14.1 *The formula* $\phi(\mathbf{x}, \mathbf{y})$ *has the* κ-order property *for* $\boldsymbol{K}$ *if there exist* $\langle a_i, b_i : \kappa\rangle$ *such that*

$$\phi(\boldsymbol{a}_i, \mathbf{b}_j) \text{ if and only if } i < j.$$

Theorem 14.2 *If* $I(\boldsymbol{K}, \aleph_1) < 2^{\aleph_1}$*, then no formula* ϕ *has the* $\aleph_1$*-order property for* $\boldsymbol{K}$*.*

proof to follow - 5.1-5.4 of [28]

15
Independence in ω-stable Classes

We work in an atomic $\boldsymbol{K}$ which is ω-stable.

We want to define a notion of independence over X. In this section we will describe the basic properties of this relation. As in Chapter II of [1], we can use these as properties of an abstract dependence relation to demonstrate other technical conditions.

We begin with defining an appropriate rank function.

Definition 15.1 *Let $N \in \boldsymbol{K}$ and $\phi(\mathbf{x})$ a formula with parameters from N. We define $R_N(\phi) \geq \alpha$ by induction on α.*

1. *$R_N(\phi) \geq 0$ if ϕ is realized in N.*
2. *For a limit ordinal δ, $R_N(\phi) \geq \delta$, if $R_N(\phi) \geq \alpha$ for each $\alpha < \delta$.*
3. *$R_N(\phi) \geq \alpha + 1$ if*
    - *(a) There is an $\boldsymbol{a} \in M$ and a formula $\psi(\mathbf{x}, \mathbf{y})$ such that both $\phi(x) \wedge \psi(x, \boldsymbol{a})$ and $\phi(x) \wedge \neg\psi(x, \boldsymbol{a})$ have rank at least α;*
    - *(b) for each $\mathbf{c} \in M$ there is a formula $\chi(x, \mathbf{c})$ isolating a complete type over $\mathbf{c}$ and $\phi(x) \wedge \chi(x, \mathbf{c})$ has rank at least α.*

We write $R_N(\phi)$ is -1 if ϕ is not realized in N. As usual the rank of a formula is the least α such that $R_N(\phi) \not\geq \alpha + 1$, and $R_N(\phi) = \infty$ if it is greater than or equal every ordinal. And we let the rank of type be the minimum of the ranks of (finite conjunctions of) formulas in the type.

Note that the rank of a formula $\phi(\mathbf{x}, \mathbf{b})$ in M depends only on the formula $\phi(\mathbf{x}, \mathbf{y})$ and the type of $\mathbf{b}$ over the empty set in the sense of M. We will drop the subscript N in most cases. For every type p, there is a formula ϕ in p with $R(p) = R(\phi)$. One can easily prove by induction that if $p \subset q$, $R(p) \geq R(q)$ and that there is a countable ordinal α such that if $R(p) > \alpha$ then $R(p) = \infty$.

Theorem 15.2 *If $\boldsymbol{K}$ is ω-stable, then for every p, $R(p) < \infty$.*

Proof. Suppose not; i.e. there is a type $p \in S_{at}(M)$ with M atomic and $R(p) = \infty$. Then for any finite subset C of M and any $c \in M$ there are finite C' containing cC and $p' \in S_{\text{at}}(M)$ such that $p \restriction C = p' \restriction C$, but $p \restriction C'$ and $p' \restriction C'$ are contradictory and principal. (For this, note that if $\phi(x)$ generates $p \restriction C$, $R(\phi) = \infty \geq \omega_1 + 2$. Thus there is $\boldsymbol{a}$ and ψ witnessing 3a) of the definition of rank so both $\phi(x) \wedge \psi(x, \boldsymbol{a})$ and $\phi(x) \wedge \neg\psi(x, \boldsymbol{a})$ have rank at least $\omega_1 + 1$. Letting $C' = C\boldsymbol{a}c$ and applying 3b) we find a complete extension over C' with rank at least ω_1.

Thus, we can choose by induction finite sets C_s and formulas ϕ_s for $s \in 2^{<\omega}$ such that:

1. If $s \subset t$, $C_s \subset C_t$ and $\phi_t \rightarrow \phi_s$.
2. For each $\sigma \in 2^\omega$, $\bigcup_{s \subset \sigma} C_s = M$.
3. $\phi_{s0}(x)$ and $\phi_{s1}(x)$ are over C_s and each generates a complete type over C_s.
4. ϕ_{s0} and ϕ_{s1} are contradictory.

In this construction the fact that we choose C' above to include an arbitrary a allows us to do 2) and the ϕ_{s0} and ϕ_{s1} generate appropriate choices of $p \restriction C_s$, $p' \restriction C_s$. Now, each p_σ generated by $\langle \phi_s : s \subset \sigma \rangle$ is in $S_{\text{at}}(M)$ by conditions 2) and 3) so we contradict ω-stability. $\square_{15.2}$

Definition 15.3 *A complete type p over A* splits *over $B \subset A$ if there are $\mathbf{b}, \mathbf{c} \in A$ which realize the same type over B and a formula ϕ with $\phi(\mathbf{x}, \mathbf{b}) \in p$ and $\neg\phi(\mathbf{x}, \mathbf{c}) \in p$.*

We will want to work with extensions of sets that behave much like elementary extension.

Definition 15.4 *Let $A \subset B \subseteq M \in \boldsymbol{K}$. We say A is Tarski-Vaught in B and write $A \leq_{\text{TV}} B$ if for every formula $\phi(\mathbf{x}, \mathbf{y})$ and any $\boldsymbol{a} \in A$, $\mathbf{b} \in B$, if $M \models \phi(\boldsymbol{a}, \mathbf{b})$ there is a $\mathbf{b}' \in A$ such that $M \models \phi(\boldsymbol{a}, \mathbf{b}')$.*

Exercise 15.5 *If $M \in \boldsymbol{K}$ and MB is atomic then $M \leq_{\text{TV}} MB$.*

Lemma 15.6 (Weak Extension) *For any $p \in S_{\text{at}}(A)$; if $A \leq_{\text{TV}} B$, B is atomic and p does not split over some finite subset C of A, there is an extension of p to $\hat{p} \in S_{\text{at}}(B)$ which does not split over C.*

Proof. Put $\phi(\mathbf{x}, \mathbf{b}) \in \hat{p}$ if and only if there is a $\mathbf{b}'$ in A which realizes the same type as $\mathbf{b}$ over C and $\phi(\mathbf{x}, \mathbf{b}') \in p$. It is easy to check that $\hat{p}$ is well-defined, consistent, and doesn't split over C, let alone A. Suppose for contradiction that $\hat{p} \notin S_{\mathrm{at}}(B)$. Then for some $\mathbf{e}$ realizing $\hat{p}$ and some $\mathbf{b} \in B$, $C\mathbf{be}$ is not an atomic set. Let $\mathbf{b}' \in A$ realize $\mathrm{tp}(\mathbf{b}/C)$; since $\mathbf{e}$ realizes $\hat{p} \restriction A = p \in S_{\mathrm{at}}(A)$, there is $\theta(\mathbf{x}, \mathbf{y}, \mathbf{z})$ that implies $\mathrm{tp}(\mathbf{cb}'\mathbf{e}/\emptyset)$. By the definition of $\hat{p}$, $\theta(\mathbf{cb}, \mathbf{x}) \in \hat{p}$). Thus, $\theta(\mathbf{cbe})$ holds and $C\mathbf{be}$ is an atomic set after all. $\square_{15.6}$

Lemma 15.7 *Let $\boldsymbol{K}$ be ω-stable. Suppose $p \in S_{\mathrm{at}}(M)$ for some countable $M \in \boldsymbol{K}$. Then there is a finite $C \subset M$ such that p does not split over M.*

Proof. Choose finite C such that $p' = p \restriction C$ satisfies $R(p') = R(p)$. Clearly, p does not split over C. $\square_{15.7}$

Theorem 15.8 (Extension) *If $p \in S_{\mathrm{at}}(M)$ and $M \prec N$, then there is an extension of p to $\hat{p} \in S_{\mathrm{at}}(N)$ which does not split over M.*

Proof. Choose any countable $M_0 \prec M$. By Lemma 15.7, there is a finite $C \in M_0$ such that $p_0 = p \restriction M_0$ does not split over C. By Lemma 15.6, p_0 has a unique extension to $S_{\mathrm{at}}(N)$ which does not split over B and so not over M. $\square_{15.8}$

Remark 15.9 *Note that there is no difficulty in applying this theorem when p is not the type of a finite sequence but the type of an infinite set A, provided AM is atomic.*

Definition 15.10 *Let ABC be atomic. We write $A \underset{C}{\perp} B$ and say A is free or independent from B over C if for any finite sequence $\boldsymbol{a}$ from A, $\mathrm{tp}(\boldsymbol{a}/B)$ does not split over some finite subset of C.*

This notion satisfies many of the same properties as non-forking on an ω-stable first order theory but with certain restrictions on the domains of types. In some ways, the current setting is actually simpler that the first order setting. Every model is 'ω-saturated' in the sense that if A is a finite subset of $M \in \boldsymbol{K}$ and $p \in S_{\mathrm{at}}(A)$ then p is realized in M. In particular note that monotonicity and transitivity are immediate.

Fact 15.11

Monotonicity 1. *$A \underset{C}{\perp} B$ implies $A \underset{C'}{\perp} B$ if $C \subseteq C' \subseteq B$*

2. *$A \underset{C}{\perp} B$ implies $A \underset{C}{\perp} B'$ if $C \subseteq B' \subseteq B$*

Transitivity *If $B \leq_{\mathrm{TV}} C \leq_{\mathrm{TV}} D$, $A \underset{C}{\perp} D$ and $A \underset{B}{\perp} C$ implies $A \underset{B}{\perp} D$.*

Definition 15.12 *1. $p \in S_{\rm at}(A)$ is* stationary *if for every (some) $M \in \boldsymbol{K}$ with $A \subset M$,there is a unique extension $\hat{p}$ of p to $S_{\rm at}(M)$ that does not split over A.*

2. If $p \in S_{\rm at}(M)$ does not split over A and $p \restriction A$ is stationary then we say p is based *on A.*

Lemma 15.13 *1. If $p \in S(M)$, then*

(a) p is stationary

(b) there is finite $C \subset M$ such that $P \restriction C$ is stationary.

2. If $\boldsymbol{K}$ is ω-stable, $\boldsymbol{K}$ is stable in every cardinality.

1a) is trivial; for 1b) choose a formula over a finite subset with the same rank. 2) is now obvious.

Note this does not imply for arbitrary atomic classes that $\hat{p}$ is realized in a model.

We need a strengthening of the extension property.

Lemma 15.14 *Suppose A is finite. If $p \in S_{\rm at}(A)$ is stationary and then for any B containing A, there is a unique non-splitting extension of p to $\hat{p} \in S_{\rm at}(B)$.*

Proof. Fix $M \in \boldsymbol{K}$ containing A and q is the unique non-splitting extension of p to $S_{\rm at}(M)$. Put $\phi(\mathbf{x}, \mathbf{c}) \in \hat{p}$ if and only if there is a $\mathbf{b}'$ in A which realizes the same type as $\mathbf{b}$ over A and $\phi(\mathbf{x}, \mathbf{b}') \in q$. Since M is ω-saturated, it is easy to check that $\hat{p}$ is well-defined, consistent, and doesn't split over A. Suppose for contradiction that $\hat{p} \not\in S_{\rm at}(B)$. Then for some $\mathbf{e}$ realizing $\hat{p}$ and some $\mathbf{b} \in B$, $A\mathbf{be}$ is not an atomic set. Let $\mathbf{b}' \in M$ realize ${\rm tp}(\mathbf{b}/A)$. By definition, for any $\theta(\mathbf{x}, \mathbf{y}, \mathbf{z})$, $\theta(\mathbf{x}, \mathbf{b}, \boldsymbol{a}) \in \hat{p}$ if and only if $\theta(\mathbf{x}, \mathbf{b}', \boldsymbol{a}) \in q$. But $q \restriction \boldsymbol{a}\mathbf{b}'$ is principal so $\hat{p} \restriction \mathbf{b}, \boldsymbol{a}$ is principal as required. $\square_{15.14}$

We have justified the following notation.

Notation 15.15 *If $p \in S_{\rm at}(A)$ is based on $\boldsymbol{a}$, for any $B \supseteq \boldsymbol{a}$ we denote by $p|B$ the unique nonsplitting extension of $p \restriction \boldsymbol{a}$ to $S_{\rm at}(B)$.*

Probably the most difficult result to establish is symmetry; we need an auxiliary notion.

Definition 15.16 *The formula $\phi(\mathbf{x}, \mathbf{y})$ has the κ* order property *for $\boldsymbol{K}$ if there exist $\langle a_i, b_i : \kappa\rangle$ such that*

$$\phi(\boldsymbol{a}_i, \mathbf{b}_j) \text{ if and only if } i < j.$$

Remark 15.17 *Depending on what we know about the size of models we can derive κ-order property from the failure of symmetry for various κ.*

In Chapter 14, we show that if $\boldsymbol{K}$ has few models in $\aleph_1$, then $\boldsymbol{K}$ does not have the $\aleph_1$-order property. If $\boldsymbol{K}$ has arbitrarily large models and is ω-stable we get a somewhat weaker result.

We write A_i for $\{\boldsymbol{a}_j : j < i\}$.

Lemma 15.18 *If an atomic class $\boldsymbol{K}$ with vocabulary τ has the $\beth_{\omega_1}$-order property then it is not ω-stable.*

Proof. Let $Y = \langle a_i, b_i : i < \beth_{\omega_1} \rangle$ and $\phi(\mathbf{x}, \mathbf{y})$ witness the order property. Note that Y is contained in $N \in \boldsymbol{K}$, an atomic model. By Theorem 12.1, we can find a vocabulary τ^* extending τ and a sequence Z of order indiscernibles such that $\Phi = \mathbf{D}(Z) \subseteq \mathbf{D}(Y)$. By compactness, we can assume that Z has the order type of the reals. By Theorem 12.1, $EM_\tau(Z, \Phi) \in \boldsymbol{K}$. Now if W is a dense countable subset of Z, $S_{\rm at}(W)$ is uncountable and we finish. $\square_{15.18}$

To show symmetry we want to deduce some version of the order property from it's failure. We need to build long sequences of non-splitting extensions; we start with ω_1.

Definition 15.19 *The atomic set $A \subset \mathbb{M}$ is* good *if the isolated complete types over A are dense in $S(A)$.*

Note that if A and $S_{\rm at}(A)$ are countable, then A is good, since the isolated types are dense in any countable Stone space.

Is that line a cheat?

Moreover, if $M\boldsymbol{a}$ is atomic, $|S^n_{\rm at}(M\boldsymbol{a})| \leq |S^{n+m}_{\rm at}(M)|$ where $\lg(\boldsymbol{a}) = m$ and the superscript denotes the arity of the type. (Mapping $\operatorname{tp}(\mathbf{d}/M\boldsymbol{a})$ to $\operatorname{tp}(\boldsymbol{a}\mathbf{d})$ is an injection.) So we have:

Lemma 15.20 *If M is countable then for any $p \in S_{\rm at}(M)$, and $\boldsymbol{a}$ realizing p, $M\boldsymbol{a}$ is good.*

Lemma 15.21 *Suppose M is countable and let $A \subseteq M$ and $p \in S_{\rm at}(A)$. If p extends to a type p_0 in $S_{\rm at}(M)$ then there exist $\langle a_i : i < \omega_1 \rangle$ of realizations of p in a model $M \in \boldsymbol{K}$.*

Proof. Define $\langle a_i : i < \omega_1 \rangle$ and $\langle M_i : i < \omega_1 \rangle$ by induction. $M_0 = M$ and a_0 is realization of p_0. Take limits at successors. Let a_i realize a non-splitting extension p_i of p_0 to M_i; p_i is not realized in M' because p_i does not split over M. Let M_{i+1} be primary over $M_i a_i$ (M_{i+1} exists since $M_i a_i$ is countable and good).

Remark 15.22 *Note that we can carry this induction on countable sets because of Lemma 15.20. Extending this induction beyond ω_1 requires the use of n-dimensional cubes in $\aleph_0$.*

Lemma 15.23 *Suppose* tp(ab/A) *is based on some* $C \subseteq A$. *If* $a \underset{A}{\downarrow} b$ *and* $b \underset{A}{\not\downarrow} a$ *then some formula has the* $\aleph_1$*-order property for* $\boldsymbol{K}$.

Proof. Suppose for contradiction that $b \underset{A}{\not\downarrow} a$. Then there is a formula $\phi(x, \mathbf{z}, y)$ and $\mathbf{m} \in A$ such that if $b' \underset{A}{\downarrow} a$ and $b' \equiv_A b$, then $\neg\phi(a, \mathbf{m}, b')$ but $\phi(a, \mathbf{m}, b)$. Now, let $a = a_0, b = b_0$; invoke Lemma 15.21 to choose for $i < \omega_1$, b_{i+1} to realize the nonsplitting extension of tp(b/M) to A_iB_i (guaranteed by Lemma 15.14 and then a_{i+1} to realize the nonsplitting extension of tp(a/M) to A_iB_{i+1}.

Case $i < j$: Since $b_j \underset{C}{\downarrow} a_0$, by the choice of ϕ, we have $\neg\phi(a_0, \mathbf{m}, b_j)$ and since $a_0\mathbf{m} \equiv_C a_i\mathbf{m}$, and $b_j \underset{C}{\downarrow} A_{i+1}$, $\neg\phi(a_i, \mathbf{m}, b_j)$.

Case $i \geq j$: We have $\phi(a_0, \mathbf{m}, b_0)$ and $a_0 \equiv_{C\mathbf{m}b_0} a_i$ so $\phi(a_i, \mathbf{m}, b_0)$ holds. Further tp(a_i/AB_{j+1}) does not split over A and $b_j \equiv_M b_0$ so $\phi(a_i, \mathbf{m}, b_j)$

Thus, $\phi(x, \mathbf{m}, y)$ has the ω_1-order property. $\square_{15.23}$

We need to write in the exact hypothesis when to order property situation is resolved.

Theorem 15.24 *If* $a \underset{A}{\downarrow} b$ *then* $b \underset{A}{\downarrow} a$.

The 'ω-saturation' eliminates for types over models the distinctions between orthogonality and weak orthogonality that are important in the first order case.

Definition 15.25 *Let* $p, q \in S_{\mathrm{at}}(A)$ *be stationary.*

1. *We say* p *is* weakly orthogonal *to* q *and write* $p \perp^w q$ *if any realizations of* p *and* q *respectively are independent over* M.
2. *We say* p *is* orthogonal *to* q *and write* $p \perp q$ *if for any* $B \supseteq A$ *with* $B \subseteq M \in \boldsymbol{K}$, $p|B \perp^w q|B$.

Note that $p|B \perp^w q|B$ means $p|B \models p|B\boldsymbol{a}$, if $\boldsymbol{a} \models q|B$.

Since the nonsplitting independence relation satisfies transitivity, symmetry and monotonicity, we have by the same abstract argument as Corollary II.2.10 of [1].

Lemma 15.26 *Suppose* tp(a/A) *and* tp(b/A) *are based in* A. *If* $ab \underset{A}{\downarrow} B$ *then* $a \underset{A}{\downarrow} b$ *iff* $a \underset{B}{\downarrow} b$.

Exercise 15.27 *Prove Lemma 15.26.*

Using Lemma 15.26 again, we see that in this context weak orthogonality over models implies orthogonality. This is analogous to the same proposition for ω-saturated models of countable first order ω-stable theories.

Lemma 15.28 *If* $M \in \boldsymbol{K}$ *and* $a \underset{M}{\perp} b$*, then for any* $N \supset M$*, if* $a' \models \mathrm{tp}(a/M)|N)$ *and* $b' \models \mathrm{tp}(a/M)|N)$ *then* $a' \underset{N}{\perp} b'$*.*

Proof.

Suppose $a' \underset{N}{\not\perp} b'$. Choose finite $\mathbf{e} \in N$ so that $\mathrm{tp}(a'b'/N)$ is based on $\mathbf{e}$ and $p|M\mathbf{e} \not\models q|M\mathbf{e}$; choose finite $\mathbf{d} \in M$ so that $\mathrm{tp}(ab\mathbf{e}/M)$ is based on $\mathbf{d}$. Choose $\boldsymbol{a}^*\mathbf{b}^*\mathbf{e}^* \in M$ with $\boldsymbol{a}^*\mathbf{b}^*\mathbf{e}^* \equiv_{\mathbf{d}} \boldsymbol{a}'\mathbf{b}'\mathbf{e}$. Now $\boldsymbol{a} \equiv_{\mathbf{d}} \boldsymbol{a}^*$ and $\mathbf{b} \equiv_{\mathbf{d}} \mathbf{b}^*$. By Lemma 15.26, $\boldsymbol{a}^* \underset{\mathbf{ed}}{\not\perp} \mathbf{b}^*$, i.e. $p|\mathbf{ed} \not\models q|\mathbf{ed}$. But this contradicts the choice of $\mathbf{e}$ and $\boldsymbol{a}^*\mathbf{b}^*\mathbf{e}^*$. $\square_{15.28}$

16
Consequences of Excellence

Definition 16.1 *The atomic aec* $\boldsymbol{K}$ *is* excellent *if*

1. $\boldsymbol{K}$ *is* ω*-stable;*
2. *Nonsplitting is symmetric.*
3. *For every* n*,* $\boldsymbol{K}$ *satisfies the* $(\aleph_0, n)$*-existence property.*

We need a little notation to define the $(\aleph_0, n)$-existence property.

Notation 16.2 *An independent* (λ, n)*-system is a family of models* $\langle M_s : s \subset n \rangle$ *such that:*

1. *Each* $M_s \in \boldsymbol{K}$ *has cardinality* λ*.*
2. *If* $s \subseteq t$*,* $M_s \prec_{\boldsymbol{K}} M_t$*.*
3. *For each* s*,* $A_s = \bigcup_{t \subset s} M_t$ *is atomic.*
4. *For each* s*,* $M_s \underset{A_s}{\downarrow} B_s$ *where* $B_s = \bigcup_{t \not\subset s} M_t$*.*

We follow [20] here; Shelah [31] uses (λ, n)-existence for a slightly different property.

Definition 16.3 $\boldsymbol{K}$ *satisfies* (λ, n)*-existence if there is a primary model over* $\bigcup_{t \subset n} M_t$ *for every independent* (λ, n)*-system.*

The main result is to show that if the $(\aleph_0, n)$-existence property holds for all n then the (λ, n)-existence property holds for all λ. But first we deduce the actual property that we use later.

We need one technical lemma first. Compare below.

Lemma 16.4 *Suppose $B \leq_{\mathrm{TV}} C$. If for some $\mathbf{b} \in B$, $\phi(\mathbf{x}, \mathbf{b})$ isolates $\mathrm{tp}(\mathbf{d}/B)$ and $\mathbf{d}C$ is atomic then $\phi(\mathbf{x}, \mathbf{b})$ isolates $\mathrm{tp}(\mathbf{d}/C)$.*

Proof. Fix any $\psi(\mathbf{x}, \mathbf{c})$ (with $\mathbf{c} \in C$) that is satisfied by $\mathbf{d}$. Since $B \leq_{\mathrm{TV}} C$, there is a $\mathbf{c}' \in B$ with $\mathbf{c} \equiv_{\mathbf{b}} \mathbf{c}'$. Now $\phi(\mathbf{x}, \mathbf{b})$ implies either $\psi(\mathbf{x}, \mathbf{c}')$ or $\neg\psi(\mathbf{x}, \mathbf{c}')$ and the same implication holds for $\psi(\mathbf{x}, \mathbf{c}')$. Since we have $\phi(\mathbf{d}, \mathbf{b}) \wedge \psi(\mathbf{d}, \mathbf{c})$, it must be that $(\forall \mathbf{x})\phi(\mathbf{x}, \mathbf{b}) \rightarrow \psi(\mathbf{x}, \mathbf{c})$. $\square_{16.4}$

Lemma 16.5 *If for all $\mu < \lambda$, the $(\mu, 2)$-existence property holds then for any model M of cardinality λ and any $\boldsymbol{a}$ such that $M\boldsymbol{a}$ is atomic, there is a primary model N over $M\boldsymbol{a}$.*

Proof. Write M as an increasing continuous chain of M_i with $|M_i| = |i| + \aleph_0$. Since M_0 is countable, there is a primary model N_0 over $M_0\boldsymbol{a}$. By the extension axiom, we may assume $N_0 \underset{M_0}{\downarrow} M$. Suppose we have constructed an increasing continuous elementary chain N_i for $i < j$ with $N_i \underset{M_i}{\downarrow} M$. If j is a limit take $\bigcup_{i<j} N_i$ as N_j and note that by finite character $N_j \underset{M_j}{\downarrow} M$. If $j = i + 1$, note that by induction (M_i, N_i, M_j) is an $(|i| + \aleph_0, 2)$ system. Choose N_j primary over $N_i \cup M_j$ by the $(|i| + \aleph_0, 2)$-existence property. This completes the construction; it only remains to note that $N = \bigcup_{i<\lambda} N_i$ is primary over $M\boldsymbol{a}$. But this follows by induction using Lemma 16.4. $\square_{16.5}$

Exercise 16.6 *Verify that N_δ is primary over $M_\delta\boldsymbol{a}$ for limit δ.*

> Not quite sure where this goes. Dominance; note hypothesis is really that every tuple in N has principal type over $M\boldsymbol{a}$. This may be interesting before symmetry; note that the conclusion (without symmetry) is backwards from natural intuition.

Lemma 16.7 *Suppose N is prime over $M\boldsymbol{a}$ and $\mathbf{d} \in N$. Then $\mathrm{tp}(\boldsymbol{a}/M\mathbf{d})$ splits over M.*

Proof. For some $\mathbf{c} \in M$, there is a formula $\phi(\mathbf{c}, \boldsymbol{a}, \mathbf{x})$, satisfied by $\mathbf{d}$, which implies $\mathrm{tp}(\mathbf{d}/M\boldsymbol{a})$. In particular; $\phi(\mathbf{c}, \boldsymbol{a}, \mathbf{x}) \rightarrow \mathbf{x} \neq \mathbf{m}$ for any $\mathbf{m} \in M$. Now let C' be any finite subset of M, which contains $\mathbf{c}$. We show $\mathrm{tp}(\boldsymbol{a}/M\mathbf{d})$ splits over C. Namely, choose $\mathbf{d}' \in M$ with $\mathbf{d} \equiv_C \mathbf{d}'$. Then, we must have $\neg\phi(\mathbf{c}, \boldsymbol{a}, \mathbf{d}')$ as required. $\square_{16.7}$

17
Quasiminimal Sets in Excellent Classes

In this chapter we introduce a notion of *-excellence which is intermediate in strength between Shelah's and Zilber's notion. We will construct quasiminimal formulas in a *-excellent class and begin the definition of an independence notion analogous to nonforking in first order logic.

This chapter is indirectly based on [28, 30, 31], where most of the results were originally proved. But our exposition owes a great deal to [20, 19, 17, 10].

Recall that a model M is *atomic* if every finite sequence in M realizes a principal type over the *empty set*. Thus if T is $\aleph_0$-categorical every model of T is atomic.

Following Lessmann, we give another meaning to 'excellent':

Definition 17.1 *The atomic class* $\boldsymbol{K}$ *is* *-excellent *if*

1. $\boldsymbol{K}$ *is* ω*-stable.*
2. $\boldsymbol{K}$ *satisfies the amalgamation property*
3. *Let* p *be a complete type over a model* $M \in \boldsymbol{K}$ *such that* $p \restriction C$ *is realized in* M *for each finite* $C \subset M$*, then there is a model* $N \in \boldsymbol{K}$ *with* N *primary over* $M\boldsymbol{a}$ *such that* p *is realized by* $\boldsymbol{a}$ *in* N*.*

Our goals are:

1. Show in ZFC that *-excellent classes satisfy Morley's theorem.
2. Show assuming the weak continuum hypothesis that if an atomic class $\boldsymbol{K}$ is categorical up to $\aleph_\omega$, then it is $*$-excellent.

We have verified in Chapter **??** that excellence implies Conditions 2 (STILL TO DO) and 3. We now show that this implies that Galois-types are syntactic types in this context and that $\boldsymbol{K}$ has arbitrarily large models.

delete?? Lessmann ([19] also assumes that $\boldsymbol{K}$ has arbitrarily large models but as we see below that is a actually a consequence of $*$-excellence as described here. That hypothesis yields ω-stability easily by Lecture 5, but ω-stability can be gotten more cheaply (or at least at a different price) as we will see below.

Note that types over sets make syntactic sense as in first order logic, but we have to be careful about whether they are realized. By 3) of Definition 17.1, $p \in S_{\mathrm{at}}(M)$ if and only if $p \restriction C$ is realized in M for each finite $C \subset M$.

think about next two.

Exercise 17.2 *If $M \prec N \in \boldsymbol{K}$, where $\boldsymbol{K}$ is *-excellent and $p \in S_{\mathrm{at}}(M)$ then p extends to $q \in S_{\mathrm{at}}(N)$.*

Lemma 17.3 *If $\boldsymbol{K}$ is *-excellent then Galois types over a model M are the same as syntactic types in $S_{\mathrm{at}}(M)$.*

Proof. Equality of Galois types is always finer than equality of syntactic types. But if a, b realize the same $p \in S_{\mathrm{at}}(M)$, by 2) of Definition 17.1, we can map Ma into any model containing Mb and take a to b so the Galois types are the same. $\square_{17.3}$

Note however, we have more resources here than in a general AEC. The types in $S_{\mathrm{at}}(A)$ for A atomic play an important role; in general there is no notion of types over all subsets of models of $\boldsymbol{K}$.

Lemma 17.4 *Let $A \subseteq M$ and $p \in S_{\mathrm{at}}(A)$. TFAE:*

1. *There is an N with $M \prec N$ and $c \in N - M$ realizing p; i.e. p extends to a type in $S_{\mathrm{at}}(M)$.*
2. *For all M' with $M \prec M'$ there is an N', $M' \prec N'$ and some $d \in N' - M'$ realizing p.*

Proof: 2) implies 1) is immediate. For the converse, assume 1) holds. Without loss of generality, by amalgamation, M' contains N. Let $q = \mathrm{tp}(\mathbf{c}/M)$. By Theorem 15.8, there is a nonsplitting extension $\hat{q}$ of q to $S_{\mathrm{at}}(M')$. By Assumption 17.1 3) $\hat{q}$ is realized in $N' \in \boldsymbol{K}$. Moreover, it is not realized in M' because $\hat{q}$ does not split over M. $\square$17.4

For countable M', we will see below how to get M' via the omitting types theorem. But the existence of N' for uncountable cardinalities requires the use of n-dimensional cubes in $\aleph_0$.

Definition 17.5 *The type p over $A \subseteq M \in \boldsymbol{K}$ is* big *if for any $M' \supseteq A$ there exists an N' with $M' \prec_{\boldsymbol{K}} N'$ and with a realization of p in $N' - M'$.*

By iteratively applying Lemma 17.4, we can show:

Corollary 17.6 *Let $A \subseteq M$ and $p \in S_{\mathrm{at}}(A)$. If there is an N with $M \prec N$ and $c \in N - M$ realizing p then*

1. *p is big and*
2. *$\boldsymbol{K}$ has arbitrarily large models.*

Thus every nonalgebraic type over a model and every type with uncountably many realizations (check the hypothesis via Lowenheim-Skolem) is big. But if we consider $\boldsymbol{K}$ to contain only one model: two copies of (Z, S), we see a type over a finite set can have infinitely many realizations without being big.

With this technology we can prove a nice result. Let's update our notion of saturation for this context.

Definition 17.7 1. *The model $M \in \boldsymbol{K}$ is λ-full over $A \subset M$ if for every $C \subset M$ with $|C| < \lambda$, if p is based on A then $p|AC$ is realized in M.*

2. *M is λ full if for every $A \subset M$ with $|A| < \lambda$ and every $p \in S_{\mathrm{at}}(A)$ that is based in A, is realized in M.*

We use clause 1) when constructing for example countable models M which are full over a countable set A.

> We would like to have the hypothesis be ω-stable atomic class with amalgamation. But we don't know in general that 'every p based on A can be extended to $\hat{p} \in S_{\mathrm{at}}(M_A)$ for M_A with $A \subset M_A \subset M$'.

Lemma 17.8 *Let $\boldsymbol{K}$ be a *-excellent class.*

Suppose $|M| = \lambda > \aleph_0$. Then M is Galois-saturated if and only if M is λ-full.

Proof. Since ω-stability implies that if $p \in S_{\mathrm{at}}(N)$ then p is based on N, a full model is Galois-saturated. But since every p based on A can be extended to $\hat{p} \in S_{\mathrm{at}}(M_A)$ for M_A with $A \subset M_A \subset M$ and $|A| = |M_A|$, every Galois-saturated model is full. $\square_{17.8}$

Lemma 17.9 *Let $\boldsymbol{K}$ be an ω-stable atomic class. If $M_i : i < \delta$ for $\delta \leq \lambda$ are λ-full with λ uncountable then $M = \bigcup_{i<\delta} M_i$ is also λ-full.*

Proof. This is immediate if $\delta < \mathrm{cf}(\lambda)$. Let $A \subset M$ with $|A| < \lambda$ and suppose $p \in S_{\mathrm{at}}(A)$ is based on $\mathbf{d} \in A$. Let $r \in S_{\mathrm{at}}(M)$ be the unique nonsplitting extension of p to $S_{\mathrm{at}}(M)$. Without loss of generality $\mathbf{d} \in M_0$. Let $\boldsymbol{a}_i$ for $i < \delta$ enumerate all finite tuples from A. For each $\boldsymbol{a}_i$, choose a finite subset X_i of M_0 such that $\mathrm{tp}(\boldsymbol{a}_i/M_0)$ is based on X_i, Using the λ-fullness of M_0, construct $E \subset M_0$ such that $E = \langle e_i : i < \delta + 1\rangle$ and N_i for $i < \lambda$ such that: $\mathbf{d} \in N_0$, for $i < \delta$, $X_i \in N_i$ $|N_i| = |E_i| + \mathrm{LS}(\boldsymbol{K})$, $M_0 E_i \subset N_i$, $\mathrm{tp}(e_i/N_i) = r|N_i$. Let note $|N_\delta| < \lambda$. Then $A \underset{N}{\downarrow} M_0$. Let $e = e_\delta$. By symmetry $e \underset{N}{\downarrow} A$ and by the choice of N, $e \underset{\mathbf{d}}{\downarrow} N$. By transitivity e realizes $r|A = p$ and we finish.

Theorem 17.10 *Let $\boldsymbol{K}$ be a *-excellent class. Then $\boldsymbol{K}$ has λ-full models of cardinality λ for all λ.*

Proof. Note that by the proof of Lemma 11.15 (Note only stability was used to get the saturated model.) we have a Galois-saturated model of cardinality λ for every uncountable regular λ. Now for singular λ take a sequence of λ_i saturated models for λ_i tending to λ and apply Lemma 17.9 to get a model M which is λ_i-saturated for each λ_i. But then it is λ-saturated since the λ_i sup to λ. $\square_{17.10}$

Definition 17.11 *The type $p \in S_{\mathrm{at}}(A)$ is* quasiminimal *if p is big and for any M containing A, p has a unique extension to a type over M which is not realized in M.*

Note that whether $q(x, \boldsymbol{a})$ is big or quasiminimal is a property of $\mathrm{tp}(\boldsymbol{a}/\emptyset)$. Since every model is ω-saturated the minimal vrs strongly minimal difficulty does not arise.

Now almost as one constructs a minimal set in the first order context, we find a quasiminimal type; for details see [19]

Lemma 17.12 *Let $\boldsymbol{K}$ be excellent. For any $M \in \boldsymbol{K}$, there is a $\mathbf{c} \in M$ and a formula $\phi(x, \mathbf{c})$ which is quasiminimal.*

Proof. It suffices to show the countable model has a quasiminimal formula $\phi(x, \mathbf{c})$ (since quasiminimality of depends on the type of $\mathbf{c}$ over the empty set). As in the first order case, construct a tree of formulas which are contradictory at each stage and are big. But as in the proof of Lemma 15.7 make sure the parameters in each infinite path exhaust M. Then, if we can construct the tree ω-stability is contradicted as in Lemma 15.7. So there is a quasiminimal formula. $\square_{17.12}$

Definition 17.13 *Let $\mathbf{c} \in M \in \boldsymbol{K}$ and suppose $\phi(x, \mathbf{c})$ generates a quasiminimal type over M. For any elementary extension N of M define* cl *on the set of realizations of $\phi(x, \mathbf{c})$ in N by $a \in \mathrm{cl}(A)$ if* $\mathrm{tp}(a/A\mathbf{c})$ *is not big.*

Equivalently, we could say $a \in \mathrm{cl}(A)$ if every realization of $\mathrm{tp}(a/A\mathbf{c})$ is contained in each $M' \in \boldsymbol{K}$ which contains $A\mathbf{c}$.

Lemma 17.14 *Let* $\mathbf{c} \in M \in \boldsymbol{K}$ *and suppose* $\phi(x, \mathbf{c})$ *generates a quasiminimal type over* M. *If the elementary extension* N *of* M *is full with* $|N| > |M|$, *then* cl *defines a pregeometry on the realizations of* $\phi(x, \mathbf{c})$ *in* N.

Proof. Clearly for any a and A, $a \in A$ implies $a \in \mathrm{cl}(A)$. To see that cl has finite character note that if $\mathrm{tp}(a/A\mathbf{c})$ is not big, then it differs from the unique big type over $A\mathbf{c}$ and this is witnessed by a formula so a is in the closure of the parameters of that formula.

For idempotence, suppose $a \in \mathrm{cl}(B)$ and $B \subseteq \mathrm{cl}(A)$. Use the comment after Definition 17.13 Every $M \in \boldsymbol{K}$ which contains A contains B and every $M \in \boldsymbol{K}$ which contains B contains a; the result follows.

It is only to verify exchange that we need the fullness of N. Suppose $a, b \models \phi(x, \mathbf{c})$, each realizes a big type over $A \subseteq \phi(N)$ and $r = \mathrm{tp}(b/Aa\mathbf{c})$ is big. Since $r = \mathrm{tp}(a/A\mathbf{c})$ is big and N is full we can choose λ realizations a_i of r in N. Let $M' \prec N$ contain the a_i and let b' realize the unique big type over M' containing $\phi(x, \mathbf{c})$. Since $\mathrm{tp}(b/Aa\mathbf{c})$ is big, the uniqueness yields all pairs (a_i, b') realize the same type $p(x, y) \in S(A\mathbf{c})$ as (a, b). But then the a_i are uncountably many realization of $\mathrm{tp}(a/Ab\mathbf{c})$ so this type is big as well; this yields exchange by contraposition. $\square_{17.14}$

So the dimension of the quasiminimal set is well-defined. To conclude categoricity, we must show that dimension determines the isomorphism type of the model; this is the topic of the next chapter.

18

Two Cardinal Models and Categoricity Transfer

We work in a *-excellent atomic class $\boldsymbol{K}$. That is, $\boldsymbol{K}$ is the class of atomic models of a first order theory T, which was obtained from a complete sentence in $L_{\omega_1,\omega}$ by adding predicates for all formulas in a countable fragment L^* of $L_{\omega_1,\omega}$. The vocabulary for $\boldsymbol{K}$ is τ.

Definition 18.1 *The type $p(\mathbf{x}) \in S_{at}(M)$ is definable over the finite set $\mathbf{c}$ if for each formula $\phi(\mathbf{x})$ there is a formula $(d_p x)\phi(x, \mathbf{y})[\mathbf{y}, \mathbf{c}]$ with free variable $\mathbf{y}$ such that $(d_p x)\phi(x, \mathbf{y})[\mathbf{m}, \mathbf{c}]$ holds for exactly those $\mathbf{m} \in M$ such that $\phi(\mathbf{x}, \mathbf{m}) \in p$. This is a* defining schema *for p.*

The following lemma is taken without proof (or even mention) in the proof of Lemma 4.2 of [28]. In the proof we expand the language but in a way that does no harm.

Lemma 18.2 *There is an atomic class $\boldsymbol{K}_1$ in a vocabulary τ_1, whose models are in* 1-1 *correspondence with those of $\boldsymbol{K}$ such that: for each τ_1-formula $\phi(\mathbf{x}, \mathbf{y})$ and countable ordinal α, there is a τ_1-formula $P_{\phi,\alpha}(\mathbf{y})$ such that in any model M in $\boldsymbol{K}_1$, $P_{\phi,\alpha}(\mathbf{m})$ holds if and only if $R_M(\phi(\mathbf{x}, \mathbf{m})) \geq \alpha$.*

Proof. Define a sequence of classes and vocabularies $\tau^i, \boldsymbol{K}^i$ by adjoining predicates in τ^{i+1} which define rank for τ^i-formulas. Note that reduct is a 1-1 map from τ^ω structures to τ-structures. Then $\tau^\omega, \boldsymbol{K}^\omega$ are the required $\tau_1, \boldsymbol{K}_1$. $\square_{18.2}$

Henceforth, we assume $\boldsymbol{K}$ satisfies the conclusion of Lemma 18.2.

Lemma 18.3 *Let $\boldsymbol{K}$ be ω-stable. Every type over a model is definable.*

Proof. Now let N be an atomic model of T and let $p \in S_{at}(N)$; choose $\phi(\mathbf{x}, \mathbf{c})$ so that $R(p) = R(\phi(x, \mathbf{c})) = \alpha$. Now for any $\psi(\mathbf{x}, \mathbf{d})$, $\psi(\mathbf{x}, \mathbf{d}) \in p$ if and only if $R(\phi(x, \mathbf{c}) \wedge \psi(\mathbf{x}, \mathbf{d})) = \alpha$. And, the collection of such $\mathbf{m}$ is defined by $P_{\phi(x,\mathbf{c}) \wedge \psi(\mathbf{x},\mathbf{y}),\alpha}(\mathbf{y}, \mathbf{c})$. $\square_{18.3}$

Note that if p doesn't split over C with $C \subset M \prec N$ and $\hat{p} \in S_{\mathrm{at}}(M)$ is a nonsplitting extension of p, $\hat{p}$ is defined by the same schema as p.

We abuse standard notation from e.g. [18] in our context. Note that we have restrict our attention to **big** formulas. This will give us two cardinal transfer theorems that read exactly as those for first order but actually have different content because the first order versions refer arbitrary infinite definable sets.

Definition 18.4 *1. A triple (M, N, ϕ) where $M \prec N \in \boldsymbol{K}$ with $M \neq N$, ϕ is defined over M, ϕ big, and $\phi(M) = \phi(N)$ is called a* Vaughtian triple*.*

2. We say $\boldsymbol{K}$ admits (κ, λ), witnessed by ϕ, if there is a model $N \in \boldsymbol{K}$ with $|N| = \kappa$ and $|\phi(N)| = \lambda$ and ϕ is big.

Of course, it is easy in this context to have definable sets which are countable in all models. But we'll show that this is really the only sense in which excellent classes differ from stable theories as far as two cardinal theorems are concerned.

The overall structure of the proof of the next result is based on Proposition 2.21 of [20]; but in the crucial type omitting step we expand the argument of Theorem IX.5.13 in [1] rather than introducing nonorthogonality arguments at this stage.

Lemma 18.5 *Suppose $\boldsymbol{K}$ is *-excellent.*

1. If $\boldsymbol{K}$ admits (κ, λ) for some $\kappa > \lambda$ then $\boldsymbol{K}$ has a Vaughtian triple.

2. If $\boldsymbol{K}$ has a Vaughtian triple, for any (κ', λ') with $\kappa' > \lambda'$, $\boldsymbol{K}$ admits (κ', λ').

Proof. Suppose $N \in \boldsymbol{K}$ with $|N| = \kappa$ and $|\phi(N)| = \lambda$. For notational simplicity we add the parameters of ϕ to the language. By Löwenheim-Skolem, we can embed $\phi(N)$ in a proper elementary submodel M and get a Vaughtian triple. We may assume that M and N are countable. To see this, build within the given M, N countable increasing sequences of countable models M_i, N_i, fixing one element $b \in N - M$ to be in N_0 and choosing $M_i \prec M$, $N_i \prec N$, $M_i \prec N_i$ and $\phi(N_i) \subset \phi(M_{i+1})$. Then M_ω, N_ω are as required.

Now for any κ', we will construct a (κ', ω) model. Say $b \in N - M$ and let $q = \mathrm{tp}(b/M)$. Now construct N_i for $i < \kappa'$ so that N_{i+1} is primary over the $N_i b_i$ where b_i realizes the non-splitting extension of q to $S_{\mathrm{at}}(N_i)$. Fix finite C contained in M so that q does not split over C. We prove by induction that each $\phi(N_i) = \phi(M)$. Suppose this holds for i, but there is an $e \in \phi(N_{i+1}) - \phi(M)$. Fix $\mathbf{m} \in M_i$ and $\theta(x, \mathbf{z}, y)$ such that $\theta(b_i, \mathbf{m}, y)$ isolates $\mathrm{tp}(e/M_i)$. We will obtain a contradiction.

For every $\mathbf{n} \in N$, if $(\exists y)(\theta(b, \mathbf{n}, y) \wedge \phi(y)$ then for some $d \in M$, $\theta(b, \mathbf{n}, d) \wedge \phi(d)$ holds. Thus,

$$(\forall \mathbf{z})[(d_q x)(\exists y)\theta(x, \mathbf{z}, y) \wedge \phi(y))[\mathbf{z}, \mathbf{c}] \rightarrow (\exists y)\phi(y) \wedge (d_q x)\theta(x, \mathbf{z}, y)[\mathbf{z}, y, \mathbf{c}]].$$

We have $\theta(b_i, \mathbf{m}, e)$, so $M_i \models (d_q x)((\exists y)\theta(x, \mathbf{z}, y) \wedge \phi(y)))[\mathbf{m}, \mathbf{c}]$. Thus by the displayed formula $M_i \models (\exists y)\phi(y) \wedge (d_q x)(\theta(x, \mathbf{z}, y))[\mathbf{m}, y, \mathbf{c}]$. That is, for some $d \in M$, $M_i \models (d_q x)(\theta(x, \mathbf{z}, y))[\mathbf{m}, d, \mathbf{c}]$. Since $\mathrm{tp}(b_i/M_i$ is defined by d_q, we have $\theta(\mathbf{m}, d, \mathbf{c})$. But this contradicts the fact that $\theta(b_i, \mathbf{m}, y)$ isolates $\mathrm{tp}(e/M_i)$.

Thus, we have constructed a model M_μ of M_μ power μ where ϕ is satisfied only countably many times. To construct a (κ', λ') model, iteratively realize the non-splitting extension of ϕ, λ' times.

$\square_{18.5}$

We need one further corollary of Therorem 16.5.

Lemma 18.6 *If $p \in S(M_0)$ is quasiminimal and X is an independent set of realizations of p, there is a primary model over MX.*

Proof. Let $X = \{x_i : i < \lambda\}$. By Theorem 16.5 define M_{i+1} to be primary over $M_i x_i$, taking unions at limits. $\square_{18.6}$

Exercise 18.7 *Use the independence of X to verify that for limit δ, M_δ is in fact primary over X_δ.*

Now we conclude that categoricity transfers.

Theorem 18.8 *Suppose $\boldsymbol{K}$ is *-excellent. The following are equivalent.*

1. *$\boldsymbol{K}$ is categorical in some uncountable cardinality.*
2. *$\boldsymbol{K}$ has no two cardinal models.*
3. *$\boldsymbol{K}$ is categorical in every uncountable cardinal.*

Proof. We first show 1) implies 2). Suppose for contradiction that there is a two-cardinal model (M, N, ϕ) even though $\boldsymbol{K}$ is κ-categorical for some uncountable κ.By Theorem 18.5 $\boldsymbol{K}$ has $(\kappa, \aleph_0)$-model. But by Theorem 17.10, if it is categorical there is a full model in the categoricity cardinal and every big definable subset of a full model has the same cardinality as the model.

3) implies 1) is obvious; it remains to show 2) implies 3). Let M_0 be the unique countable model. By Lemma 17.12, there is a quasiminimal formula $\phi(x, \mathbf{c})$ with parameters from M_0. For any λ, by Theorem 17.10, there is a full model N of $\boldsymbol{K}$ extending M_0 with cardinality λ. By Lemma 17.14, cl is a pregeometry on $\phi(N)$. Note that $\phi(M)$ is closed since by definition any element a of $\mathrm{cl}(\phi(M))$ both satisfies ϕ and is in every model containing $\phi(M)$,

including M. Thus we can choose a basis X for $\phi(M)$. By Theorem 18.6, there is a prime model $M_{|X|}$ over MX. But $X \subset \phi(M_{|X|}) \subset \phi(M)$ so $\phi(M_{|X|}) = \phi(M)$; whence since we assume there are no two cardinal models, $M_{|X|} = M$ and M is prime and minimal over MX.

Now we show categoricity in any uncountable cardinality. If M, N are models of power λ, they are each prime and minimal over X, a basis for $\phi(M)$ and Y, a basis for $\phi(N)$, respectively. Now any bijection between X and Y is elementary by the moreover clause in Lemma 17.14. It extends to a map from M into N by primeness and it must be onto; otherwise there is a two cardinal model. $\square_{18.8}$

19
Demystifying Non-excellence

In this chapter we expound the Hart-Shelah example of a sentence ψ in $L_{\omega_1,\omega}$ which is categorical in $\aleph_0, \aleph_1$ but not in $2^{\aleph_1}$. More generally for each k the construction provides a sentence which is categorical up to $\aleph_k$ but not categorical everywhere. We outline the general construction but specialize to $k = 2$ for many specific arguments.

This example is a descendent of the example in [3] of an $\aleph_1$-categorical theory which is not almost strongly minimal. That is, the universe is not in the algebraic closure of a strongly minimal set. Here is a simple way to describe such a model. Let G be a strongly minimal group and let π map X onto G. Add to the language a binary function t for the fixed-point free action of G on $\pi^{-1}(g)$ for each $g \in G$. This guarantees that each fiber has the same cardinality as G and π guarantees the number of fibers is the same as $|G|$. Since there is no interaction among the fibers, categoricity in all uncountable powers is easy to check.

19.1 The basic structure

Notation 19.1 *The formal language for this example contains unary predicates $W, Z_2, I, K, G^a, G^*, H^a, H^*$; binary functions e_G taking $G \times K$ to Z_2 and e_H taking $H \times K$ to Z_2; function π_G mapping G^* to $W \times K$, function π_H mapping H^* to K, a 5-ary relation t on $W \times K \times G^a \times G^* \times G^*$), a 4-ary relation symbol h, $K \times G^a \times H^* \times H^*$), and ternary relations q_ℓ on* *for $\ell < \omega$ on $G^* \times G^* \times H^*$. Certain other projection functions are in the language but not expressly described.*

We will construct a structure $M(I)$ from any infinite set I. The structure will be a disjoint union of sets I, K, Z_2, G^a, H^a, G^* and H^*. All except G^* and H^* will be completely determined by I. For standard structures G^* and H^* will also be completely determined. Let $K = [I]^k$ be the set of k element subsets of I and let G be the direct sum of K copies of Z_2 so G and K have the same cardinality. We are going to put a structure on $K \times G$. (A priori, we might as well have indexed by I! The connection with K will only appear with the introduction of the Q_ℓ at the end of the construction.)

The basic object in the construction is a family of copies of the group G, indexed by a set $W \times K$. W will be a copy of ω, which is required to provide some coding in the non-categoricity proof; we point this out at the appropriate time. We add constants c_ℓ for the elements of W and the sentence ψ saying they exhaust W. The basic object should be categorical in all powers. This is achieved in two steps. We include K, G^a (naming G) and Z_2 as sorts of the structure with the evaluation function: for $\gamma \in G$ and $k \in K$, $e_G(\gamma, k) = \gamma(k) \in Z_2$. So in $L_{\omega_1,\omega}$ we can say that the predicate G^a denotes exactly the set of elements with finite support of ${}^K Z_2$. Now, we introduce a set G^*. We say a model is *G-standard* if G^* is $W \times K \times G$ but we don't have all the projection functions which would allow us to say that. Rather we have only the projection function π_G from G^* onto $W \times K$. We write $v \in G_{\ell,u}$ when $\pi_G(v) = (\ell, u)$ and say v is in the (G)-stalk over ℓ, u. Thus, $W \times K$ indexes a partition of G^*. The relation $t(\ell, u, v, w, x)$ is the graph of the action (by addition) of G^a on each member $G_{\ell,u}$ of the partition indexed by $W \times K$. That is $t(\ell, u, v, w, x)$ holds where $w = (\ell, u, w'), x = (\ell, u, x')$, $v \in G$ and $v + w' = x'$. So each class is an (affine) copy of G.

This much of the structure is clearly categorical (and homogeneous). So we must work harder.

We consider a new group H, the direct sum of ω copies of Z_2 (again explicitly represented as a subset H^a of ${}^W Z_2$, with a sort for W and the evaluation function, e_H). Again we add a set H^* and a function π_H taking H^* to K. When H^* is $K \times H$, we say the model is *H-standard.* As before $\pi_H(x) = u$ holds if x has the form (u, x'). Finally $h(u, v, w, x)$ is the graph of the action (by addition) of H^a on each member H_u of the partition of H^* indexed by K.

This structure is still categorical. To see this, suppose two such models have been built on I and I' of the same cardinality. Take any bijection between I and I'. To extend the map to G^* and H^* fix one element in each partition class in each model. The natural correspondence (linking those selected in corresponding classes) extends to an isomorphism.

The key step is the definition of a family of relations Q_ℓ that witness the failure of k-dimensional amalgamation. For concreteness we fix $k = 2$. The structure is imposed by a family of 3-ary relations Q_ℓ on $G^* \times G^* \times H^*$, which have a local character. Q_ℓ only holds of elements with first coordinate ℓ. Moreover, $Q_\ell((\ell, u_1, x_1), (\ell, u_2, x_2), (u_3, x_3))$ implies that u_1, u_2, u_3 are the three two element subsets of a 3 element subset of I. We call u_1, u_2, u_3 a *compatible triple*. (Suitable projections are in the language to express this.) Finally, $Q_\ell((\ell, u_1, x_1), (\ell, u_2, x_2), (u_3, x_3))$ holds just if

$$x_1(u_3) + x_2(u_3) = x_3(\ell).$$

This completes the description of the example.

19.2 $k = 2$, $\aleph_1$-categoricity

We now show that with $k = 2$ the structure is $\aleph_1$-categorical. We choose global sections for the maps π_G and π_H.

Definition 19.2 *Fix a model M. A* solution *for M is a selector f that chooses one element of the fiber in G^* above each element of $W \times K$ and one element of the fiber in H^* above each element of K. Formally, f is a pair of functions (f_0, f_1), where $f_0 : W(M) \times K(M) \to G^*(M)$ and $f_1 : K(M) \to H^*(M)$ such that $\pi_G f_0$ and $\pi_H f_1$ are the identity and for each ℓ and compatible triple u_1, u_2, u_3,:*

$$Q_\ell([(\ell, u_1), f_0(\ell, u_1)], [(\ell, u_2), f_0(\ell, u_2)], (u_3, f_1(u_3))).$$

Lemma 19.3 *If M and N have the same cardinality and have solutions f_M and f_N then $M \cong N$.*

Proof. Without loss of generality, $M = M(I)$, $N = M(I')$. Let α be an arbitrary bijection between I and I'. Extend naturally to a map from $K(M)$ to $K(N)$ and from $G^a(M), H^a(M)$ to $G^a(N), H^a(N)$; let $\alpha(f_M(\ell, u))$ be $f_N(\ell, \alpha(u))$. If $x \in G^*(M)$, and $M \models \pi_G(x) = (\ell, u)$, there is a unique $a \in G^a(M)$ such that

$$M \models t(c_\ell, u, a, f_M(\ell, u), x).$$

Let $\alpha(x)$ be the unique $y \in N$ such that

$$N \models t(c_\ell, \alpha(u), \alpha(a), f_N(\ell, u), y).$$

Do a similar construction for H^* and observe that each Q_ℓ is preserved. $\square_{19.3}$

Slightly more generally, we describe selectors over subsets of $I(M)$ rather than all of $I(M)$.

Definition 19.4 *There is a solution for the subset A of $I(M)$ if for each 2-set u from $[A]^2$ and each ℓ there are $f_0(\ell, u) \in G_{\ell,u}$ and $f_1(u) \in H_u$ such that if u_1, u_2, u_3 are a compatible triple from A^2, for every ℓ,*

$$Q_\ell([(\ell, u_1), f_0(\ell, u_1)], [(\ell, u_2), f_0(\ell, u_2)], (u_3, f_1(u_3))).$$

Fact 19.5 *If $|M| \leq \aleph_1$, M has a solution.*

Proof. Note first that it suffices to show that if there is a solution $f = (f_0, f_1)$ over a countable set $A \subset I(M)$ then for any $b \in I(M)$ it can be extended to a solution $f' = (f'_0, f'_1)$ over $A \cup \{b\}$. To show the extension property, enumerate A as a_i for $i < \omega$. For each i, choose arbitrarily $f'_1(a_i, b) \in H_{a_i,b}$. Then prove by induction on n that f_0 can be further extended to a solution f'_0 over all tuples $(\ell, (a_i, b))$ for $\ell, i \leq n$. At the induction step, we must define $f'_0(\ell, (a_{n+1}, b))$ compatibly with our previous choices of the $f'_0(\ell, (a_{n+1}, a_i))$ and $f'_1((a_i, b))$ for $i \leq n$. This ability is guaranteed by the following fact which holds in the original structure as the conditions fix only a finite number of values of the element to be chosen. $\square_{19.5}$

Fact 19.6 *If $\ell \in W$, $a_1, \ldots a_{n+1}, b$ are in I, $x_j \in G_{c_\ell,(a_n,a_j)}$ for $1 \leq j \leq n$, and for each $1 \leq j \leq n$, $y_j \in H_{b,a_j}$ then*

$$(\exists x) \bigwedge_{1 \leq j \leq n} \pi_G(x) = (\ell, (a_j, b)) \wedge Q_\ell(x, x_j, y_j).$$

But we are stymied at $\aleph_2$ since adding any new point to a set of size $\aleph_1$ gives us $\aleph_1$ conditions to meet.

19.3 $k = 3$, $\aleph_2$-categoricity

The argument above seems to depend only on $\aleph_0$-homogeneity and doesn't illustrate how the categoricty is carried to higher cardinalities by increasing k. But the idea becomes clearer if we take k equal to 3.

Here is a soft argument that moves from a configuration of countable models to categoricity in higher cardinals. We have already seen that if for $A \subseteq B$, both with cardinality $\aleph_n$, A-solutions extend to B-solutions then there is a solution for sets of cardinality $\aleph_{n+1}$. So to get a solution of cardinality $\aleph_2$ we need to have extension at $\aleph_1$. Again, it suffices to extend a set of cardinality $\aleph_1$ by one point. Writing the set as union of countable sets, it suffices to be able to amalgamate the solutions of two one point extensions of a countable set.

We show that 3-dimensional amalgamation holds in $\aleph_0$, when $k = 3$.

Lemma 19.7 *Let $k = 3$. Let A be a countable subset of I, $b_0, b_1 \in I - A$. If $f = (f_0, f_1)$ and $g = (g_0, g_1)$ are solutions over $A_0 = A \cup \{b_0\}$, $A_1 = A \cup \{b_1\}$*

respectively, which agree on A, there is a solution h over $A_{01} = A \cup \{b_0, b_1\}$ *which extends both g and f.*

Proof. With $k = 3$, the object of interest is a compatible four tuple u_0, u_1, u_2, u_3, the four 3-element subsets of b_0, b_1, b_2, b_3. For each ℓ there are two of the four arguments of Q_ℓ that have not been defined by g or f. Show the analog to Fact 2. $\square_{19.7}$

It is instructive to try to show amalgamation for countable sets with $k = 2$; the assymetry between the selectors for H^* and G^* and the diagonalization combine to frustrate such an attempt.

For larger n, one shows n-dimensional amalgamation on $\aleph_0$ descends step by step to 2-dimensional amalgamation on $\aleph_{n-2}$, and thus extension on $\aleph_{n-1}$ and categoricity in $\aleph_n$.

19.4 Failure of Categoricity in $2^{\aleph_1}$

In general ϕ is not $2^{\aleph_{k-1}}$-categorical; so for the $k = 2$ case, we will have ϕ is $\aleph_0$ and $\aleph_1$ categorical but not $2^{\aleph_1}$-categorical.

Fix the least λ with $\lambda^{\aleph_1} < 2^\lambda$. Clearly, λ is between $\aleph_1$ and $2^{\aleph_1}$. We will show ϕ has 2^λ models of power λ. It is easy to see that if K is not categorical in some κ; it not categorical in any larger κ'. There is a model of power κ' which has solution, so κ'-categoricity implies the unique model has a solution; but then so do all its submodels, which include all models of power κ. Thus, κ'-categoricity implies κ-categoricity and in particular if we show the failure of λ-categoricity we have the failure in $2^{\aleph_1}$.

Let H_1 denote $\otimes_\omega Z_2$.

Notation 19.8 *For any map h from K to* H_1/H *we define a model* $M^h(I)$. *The structure is* identical *to the standard structures defined above with one exception: the fibers* $\pi_H^{-1}(x)$ *are not copies of H but are the cosets* $h(u)$. *That is, we have a structure which is G-standard but not H-standard.*

As structures under the 'action language', $h(u)$ and H are isomorphic. But in the concrete representation the elements of H have finite support while the elements of $h(u)$ have infinite support. Our goal is to recognize this and more precisely to recognize $h(u)$. The $L_{\omega_1,\omega}$ sentence ψ forces G^a and H^a to be elements of finite support, but we can't force this for the fibers in G^* (H^*) because the projection (in the standard model) from G^* to G (H^* to H) is *not* in the language.

To show the failure of categoricity we will constructed a specific family of models, the M_A. Each model is G-standard but some stalks in $H^*(M_A)$ do not have finite support. With d as an oracle we will recover h from M_h.

Definition 19.9 *1. Fix a selector for G^* by defining a function d: $d((\ell, u)) = (\ell, u, e)$ where e is the identically 0 element of G; we call d the* identity selector.

2. Let $u_3 = \{y, z\}$, $u_1 = \{x, y\}$, $u_2 = \{x, z\}$ be in K. Define the map $\eta^d_M(u_3)$ from K into H_1/H by $\eta^d_M(u_3)\{\ell\} = 0$ if

$$M \models Q_\ell([\ell, u_1, d(\ell, u_1)], [\ell, u_2, d(\ell, u_2)], [u_3, c]).$$

where c is an arbitrary element of H_{u_3}. The choice of c will vary $\eta^d_M(u_3)$ within a coset of H_1/H.

3. We may write $\eta_M(u_3)$ when d is the identity selector.

So if we have the identity selector d, we can compute the coset $\eta_M(u)$ at every u and the value of $\eta_M(u)$ does not depend on which third point from I we choose to construct a compatible triple.

But this procedure names $|M|$ points; we fixed d. We will see how to proceed naming only $\aleph_1$-constants. Since $\lambda^{\aleph_1} < 2^\lambda$ this will cause no harm; we will guarantee that in the expanded language there are 2^λ models of cardinality λ.

We choose the set I in a very particular way and then the function h. Let I be $X \cup \aleph_1 \times \aleph_1 \cup A$ where $|X| = \aleph_0$ and A is a set of λ functions $a : \aleph_1 \to H/H_1$.

We define models $(M_h(I), d)$ but I depends on the choice of A and we will define the h uniformly for all A so we write: (M_A, d). The crucial point is the choice of the H-stalks over those members of K that have the form $(a, \langle\alpha, \beta\rangle)$. For $a \in A$ and $\langle\alpha, \beta\rangle \in \aleph_1 \times \aleph_1$, the map h is defined by $h(a, \langle\alpha, \beta\rangle) = a(\alpha)$. That is, the fiber $H_{(a,\langle\alpha,\beta\rangle)}$ is the coset $a(\alpha)$. To avoid interference, we set $h(u) = H$ for all other u. Again, define d by $d((\ell, u)) = (\ell, u, e)$ where e is the identically 0 element of G.

Computing $\eta^d_{M_A}$, yields $\eta^d_{M_A}(a, \langle\alpha, \beta\rangle)$ is $a(\alpha)$ since the coset $H_{(a,\langle\alpha,\beta\rangle)}$ is $a(\alpha)$. Now suppose d is replaced by a d' which agrees with d on $X \times (\aleph_1 \times \aleph_1)$ but may disagree elsewhere. (I.e. expand the language by naming $\aleph_1$ constants.)

The invariant $\eta^{d'}_{M_A}(\langle\alpha, \beta\rangle, a))$ is a function $\hat{h} \in H_1 = 2^\omega$. We complete $(\langle\alpha, \beta\rangle, a)$ to a triple by choosing an element n from X; $\hat{h}$ is computed in the model by $\hat{h}(\ell) = 0$ if and only if

$$M \models Q_\ell([\ell, (n, \langle\alpha, \beta\rangle), d'(\ell, (n, \langle\alpha, \beta\rangle))], [\ell, (n, a), d'(\ell, (n, a))], [(a, \langle\alpha, \beta\rangle), c])$$

where c is an arbitrary element of the H-stalk over $(\langle\alpha, \beta\rangle, a)$. Since d and d' are equal on stalks above elements of the form $(n, \langle\alpha, \beta\rangle)$, the value of $\hat{h}$ on the $(a, \langle\alpha, \beta\rangle)$-stalks is determined by $d'(\ell, (n, a))\{(\langle\alpha, \beta\rangle, a)\}$. The relevant values are at the $(\langle\alpha, \beta\rangle, a)$ where $d'(\ell, (n, a))\{(\langle\alpha, \beta\rangle, a)\} \neq 0$. There are of course only countably many of these for any (n, a) (finitely many for any $(\ell, (n, a))$. As long as there is no ℓ with $d'(\ell, (n, a))\{a, \langle\alpha, \beta\rangle\} \neq 0$, $\eta^z_{M_A}(\langle\alpha, \beta\rangle, a)$ is the same function whether calculated with z as d or d'. Since this happens for cocountably many β, we finish. That is $\hat{h} \in H_1/H$ is in the range of a just if for $\aleph_1$ elements i of I, $\eta^{d'}_{M_A}(i, a) = \hat{h}$. A is a set of λ functions from $\aleph_1$ into H_1/H. The range of an element of a is a subset of $2^{\aleph_0}$. So there are $2^{2^{\aleph_0}}$ possible sets of ranges and $2^{2^{\aleph_0}} \geq 2^\lambda$ so there are 2^λ nonisomorphic models M_A with cardinality λ.

This completes the argument for $k = 2$; we leave the nontrivial extension [11] of the coding to obtain the result for larger k to the reader.

19.5 Failure of tameness

We continue the notation of the non-categoricity argument in Section 19.4 and note that there is a syntactic type p over a model of cardinality λ which splits into several Galois types. Since over countable models Galois types and syntactic types are the same this implies that p is not determined by its restrictions to countable submodels. Thus mod(ϕ) is not $(\aleph_1, \aleph_0)$-tame. Let $A_0 \subset A_1$ be sets of functions of cardinality λ. By $\aleph_1$-categoricity M is saturated. Let $M = M_{A_0}$. By $\aleph_1$-categoricity M is saturated so it suffices to find a syntactic type over M which splits into several Galois types. Suppose $a, a' \in A_1 - A_0$ and suppose the range of a and the range of a' are different. In particular, assume there is $\hat{h} \in H_1/H$ which is in the range of a but not the range of a'. Then there can be no map fixing a model containing $X \cup \aleph_1 \times \aleph_1$, in particular M, which maps a to a'. Because for any d' which is the identity selector when restricted to M, for $\aleph_1$ elements i of I, $\eta^{d'}_{M_A}(i, a) = \hat{h}$ but $\eta^{d'}_{M_A}(i, a') = \hat{h}$ for only countably many i in I. But a and a' realize the same syntactic type over M, since we can map one the other by while fixing any finite subset of M.

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Index