Lecture 11: Excellence implies Categoricity: A

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The following assertion is an exercise in Lecture 4.

Let ψ be a complete sentence in $L_{\omega_1,\omega}$ in a countable language L. Then there is a countable language L' extending L and a first order L'-theory T such that reduct is a 1-1 map from the atomic models of T onto the models of ψ. So in particular, *any complete sentence of $L_{\omega_1,\omega}$ can be replaced (for spectrum purposes) by considering the atomic models of a first order theory.*

This section is indirectly based on [**?**, **?**, **?**], where most of the results were originally proved. But our exposition owes a great deal to [**?**, **?**, **?**].

Recall that a model M is *atomic* if every finite sequence in M realizes a principal type over the *empty set*. Thus if T is $\aleph_0$-categorical every model of T is atomic.

Assumption 1 We work in this section entirely in the following context. $\boldsymbol{K}$ is the class of atomic models of a complete first order theory T. Note that with $\prec_{\boldsymbol{K}}$ as $\prec$, elementary submodel, this is an abstract elementary class. Moreover, $\boldsymbol{K}$ is $\aleph_0$-categorical and every member of $\boldsymbol{K}$ is $\aleph_0$-homogeneous. We write $\mathbb{M}$ for the monster model of T; in interesting cases $\mathbb{M}$ is not in $\boldsymbol{K}$.

Definition 2 *$S_{\mathrm{at}}(A)$ is the collection of $p \in S(A)$ such that if $\boldsymbol{a} \in \mathbb{M}$ realizes p, $A\boldsymbol{a}$ is atomic.*

Definition 3 *$\boldsymbol{K}$ is ω-stable if for every countable M, $|S_{\mathrm{at}}(M)| = \aleph_0$.*

This is strictly weaker than requiring $|S_{\mathrm{at}}(A)| = \aleph_0$ for arbitrary countable A.

Example 4 *Consider two structures $(\mathbb{Q}, <)$ and $(\mathbb{Q}, +, \cdot, <)$. If $\boldsymbol{K}_1$ is class of atomic models of the theory of dense linear order without endpoints, then $\boldsymbol{K}_1$ is not ω-stable;* $\mathrm{tp}(\sqrt{2}; \mathbb{Q}) \in S_{\mathrm{at}}(\mathbb{Q})$. *If $\boldsymbol{K}_1$ is class of atomic models of the theory of the ordered field of rationals, then $\boldsymbol{K}_2$ is ω-stable;* $\mathrm{tp}(\sqrt{2}; \mathbb{Q}) \notin S_{\mathrm{at}}(\mathbb{Q})$.

Definition 5 *M is primary over A if there is a sequence $M = A \cup \langle e_i : i < \lambda\rangle$ and $\mathrm{tp}(e_j/AE_{<j})$ is isolated for each j.*

Following Lessmann, we give another meaning to ‘excellent’:

Definition 6 *The atomic class $\boldsymbol{K}$ is* *-excellent *if*

1. *$\boldsymbol{K}$ is ω-stable.*

2. *$\boldsymbol{K}$ satisfies the amalgamation property*

3. *Let p be a complete type over a model $M \in \boldsymbol{K}$ such that $p \restriction C$ is realized in M for each finite $C \subset M$, then there is a model $N \in \boldsymbol{K}$ with N primary over $M\boldsymbol{a}$ such that p is realized by $\boldsymbol{a}$ in N.*

Our goals are:

1. Show in ZFC that *-excellent classes satisfy Morley's theorem.

2. Show assuming the weak continuum hypothesis that if an atomic class $\boldsymbol{K}$ is categorical up to $\aleph_\omega$, then it is $*$-excellent.

Independent cubes will appear as an intermediary in this argument.

Lessmann ([**?**] also assumes that $\boldsymbol{K}$ has arbitrarily large models but as we see below that is a actually a consequence of $*$-excellence as described here. That hypothesis yields ω-stability easily by Lecture 5, but ω-stability can be gotten more cheaply (or at least at a different price) as we will see below.

Note that types over sets make syntactic sense as in first order logic, but we have to be careful about whether they are realized. By 2) of Definition 6, $p \in S_{\mathrm{at}}(M)$ if and only if $p \restriction C$ is realized in M for each finite $C \subset M$.

Exercise 7 *If $M \prec N \in \boldsymbol{K}$, where $\boldsymbol{K}$ is *-excellent and $p \in S_{\mathrm{at}}(M)$ then p extends to $q \in S_{\mathrm{at}}(N)$.*

Lemma 8 *If $\boldsymbol{K}$ is *-excellent then Galois types are the same as syntactic types in S_{at}.*

Proof. Equality of Galois types is always finer than equality of syntactic types. But if a, b realize the same $p \in S_{\mathrm{at}}(M)$, by 2) of Definition 6, we can map Ma into any model containing Mb and take a to b so the Galois types are the same. $\square_8$

Definition 9 *A complete type p over A* splits *over $B \subset A$ if there are $\mathbf{b}, \mathbf{c} \in A$ which realize the same type over B and a formula ϕ with $\phi(\mathbf{x}, \mathbf{b}) \in p$ and $\neg\phi(\mathbf{x}, \mathbf{c}) \in p$.*

We will want to work with extensions of sets that behave much like elementary extension.

Definition 10 *Let $A \subset B \subseteq M \in \boldsymbol{K}$. We say A is Tarski-Vaught in B and write $A \leq_{\mathrm{TV}} B$ if for every formula $\phi(\mathbf{x}, \mathbf{y})$ and any $\boldsymbol{a} \in A$, $\mathbf{b} \in B$, if $M \models \phi(\boldsymbol{a}, \mathbf{b})$ there is a $\mathbf{b}' \in A$ such that $M \models \phi(\boldsymbol{a}, \mathbf{b}')$.*

Exercise 11 *If $M \in \boldsymbol{K}$ and MB is atomic then $M \leq_{\mathrm{TV}} MB$.*

Lemma 12 (Weak Extension) *For any $p \in S_{\mathrm{at}}(A)$; if $A \leq_{\mathrm{TV}} B$, B is atomic and p does not split over some finite subset C of A, there is an extension of p to $\hat{p} \in S_{\mathrm{at}}(B)$ which does not split over C.*

Proof. Put $\phi(\mathbf{x}, \mathbf{b}) \in \hat{p}$ if and only if there is a $\mathbf{b}'$ in A which realizes the same type as $\mathbf{b}$ over C and $\phi(\mathbf{x}, \mathbf{b}') \in p$. It is easy to check that $\hat{p}$ is well-defined, consistent, and doesn't split over C, let alone A. Suppose for contradiction that $\hat{p} \notin S_{\mathrm{at}}(B)$. Then for some $\mathbf{e}$ realizing $\hat{p}$ and some $\mathbf{b} \in B$, $C\mathbf{be}$ is not an atomic set. Let $\mathbf{b}' \in A$ realize $\mathrm{tp}(\mathbf{b}/C)$; since $\mathbf{e}$ realizes $\hat{p} \restriction A = p \in S_{\mathrm{at}}(A)$, there is $\theta(\mathbf{x}, \mathbf{y}, \mathbf{z})$ that implies $\mathrm{tp}(\mathbf{cb}'\mathbf{e}/\emptyset)$. By the definition of $\hat{p}$, $\theta(\mathbf{cb}, \mathbf{x}) \in \hat{p}$). Thus, $\theta(\mathbf{cbe})$ holds and $C\mathbf{be}$ is an atomic set after all. $\square_{12}$

Lemma 13 *Let $\boldsymbol{K}$ be $*$-excellent. Suppose $p \in S_{\rm at}(M)$ for some countable $M \in \boldsymbol{K}$. Then there is a finite $C \subset M$ such that p does not split over M.*

Proof. Suppose $p \in S_{\rm at}(M)$ splits over every finite subset of M. Then for any $a \in M$ there are finite C' containing a and $p' \in S_{\rm at}(M)$ such that $p \restriction C'$ and $p' \restriction C'$ are contradictory and principal. Thus, we can choose by induction finite sets C_s and formulas ϕ_s for $s \in 2^{<\omega}$ such that

1. If $s \subset t$, $C_s \subset C_t$ and $\phi_t \to \phi_s$.
2. For each $\sigma \in 2^\omega$, $\bigcup_{s\subset\sigma} C_s = M$.
3. $\phi_{s0}(x)$ and $\phi_{s1}(x)$ are over C_s and each generates a complete type over C_s.
4. ϕ_{s0} and ϕ_{s1} are contradictory.

In this construction the fact that we choose C' above to include an arbitrary a allows us to do 2) and the ϕ_{s0} and ϕ_{s1} generate appropriate choices of $p \restriction C_s$, $p' \restriction C_s$. Now, each p_σ generated by $\langle \phi_s : s \subset \sigma\rangle$ is in $S_{\rm at}(M)$ and we contradict ω-stability. $\square_{13}$

Theorem 14 (Extension) *If $p \in S_{\rm at}(M)$ and $M \prec N$, then there is an extension of p to $\hat{p} \in S_{\rm at}(N)$ which does not split over M.*

Proof. Choose any countable $M_0 \prec M$. By Lemma 13, there is a finite $C \in M_0$ such that $p_0 = p \restriction M_0$ does not split over C. By Lemma 12, p_0 has a unique extension to $S_{\rm at}(N)$ which does not split over B and so not over M. $\square_{14}$

Lemma 15 *Let $A \subseteq M$ and $p \in S_{\rm at}(A)$. TFAE:*

1. *There is an N with $M \prec N$ and $c \in N - M$ realizing p; i.e. p extends to a type in $S_{\rm at}(M)$.*
2. *For all M' with $M \prec M'$ there is an N', $M' \prec N'$ and some $d \in N' - M'$ realizing p.*

Proof: 2) implies 1) is immediate. For the converse, assume 1) holds. Without loss of generality, by amalgamation, M' contains N. Let $q = \mathrm{tp}(\mathbf{c}/M)$. By Theorem 14, there is a nonsplitting extension $\hat{q}$ of q to $S_{\rm at}(M')$. By Assumption 6 2) $\hat{q}$ is realized in $N' \in \boldsymbol{K}$. Moreover, it is not realized in M' because $\hat{q}$ does not split over M. $\square 15$

For countable M', we will see below how to get M' via the omitting types theorem. But the existence of N' for uncountable cardinalities requires the use of n-dimensional cubes in $\aleph_0$.

Definition 16 *The type p over $A \subseteq M \in$ bK is* big *if for any $M' \supseteq A$ there exists an N' with a realization of p in $N' - M'$. A formula $\phi(x, \mathbf{c})$ is big if there is a $\phi'(x, \mathbf{c})$, that ϕ, and generates a complete big type over $\mathbf{c}$.*

By iteratively applying Lemma 15, we can show:

Corollary 17 *Let $A \subseteq M$ and $p \in S_{\rm at}(A)$. If there is an N with $M \prec N$ and $c \in N - M$ realizing p then*

1. *p is big and*

2. $\boldsymbol{K}$ *has arbitrarily large models.*

Thus every nonalgebraic type over a model and every type with uncountably many realizations (check the hypothesis via Lowenheim-Skolem) is big. But if we consider $\boldsymbol{K}$ to contain only one models: two copies of (Z, S), we see a type over a finite set can have infinitely many realizations without being big.

Definition 18 *The type* $p \in S_{\mathrm{at}}(A)$ *is* quasiminimal *if* p *is big and for any* M *containing* A, p *has a unique extension to a type over* M *which is not realized in* M.

Note that whether $q(x, \boldsymbol{a})$ is big or quasiminimal is a property of $\mathrm{tp}(\boldsymbol{a}/\emptyset)$. Since every model is ω-saturated the minimal vrs strongly minimal difficulty does not arise.

Now almost as one constructs a minimal set in the first order context, we find a quasiminimal type; for details see [**?**]

Lemma 19 *Let* $\boldsymbol{K}$ *be excellent. For any* $M \in \boldsymbol{K}$, *there is a* $\mathbf{c} \in M$ *and a formula* $\phi(x, \mathbf{c})$ *which is quasiminimal.*

Proof. It suffices to show the countable model has a quasiminimal formula $\phi(x, \mathbf{c})$ (since quasiminimality of depends on the type of $\mathbf{c}$ over the empty set). As in the first order case, construct a tree of formulas which are contradictory at each stage and are big. But as in the proof of Lemma 13 make sure the parameters in each infinite path exhaust M. Then, if we can construct the tree ω-stability is contradicted as in Lemma 13. So there is a big formula. $\square_{19}$

Definition 20 *The model* $N \in \boldsymbol{K}$ *is* λ*-full (or Galois-saturated) if for any* $N \prec M$ *with* $|N| < \lambda$, *any* $p \in S_{\mathrm{at}}(N)$ *is realized in* M.

Since $\boldsymbol{K}$ is stable in all cardinalities by Lecture 12, there are Galois-saturated models in all regular cardinalities as in Lecture 5???. Singular to be done later. splitting

Definition 21 *Let* $\mathbf{c} \in M \in \boldsymbol{K}$ *and suppose* $\phi(x, \mathbf{c})$ *generates a quasiminimal type over* M. *For any elementary extension* N *of* M *define* cl *on the set of realizations of* $\phi(x, \mathbf{c})$ *in* N *by* $a \in \mathrm{cl}(A)$ *if* $\mathrm{tp}(a/A\mathbf{c})$ *is not big.*

Equivalently, we could say $a \in \mathrm{cl}(A)$ if every realization of $\mathrm{tp}(a/A\mathbf{c})$ is contain in each $M' \in \boldsymbol{K}$ which contains $A\mathbf{c}$.

Lemma 22 *Let* $\mathbf{c} \in M \in \boldsymbol{K}$ *and suppose* $\phi(x, \mathbf{c})$ *generates a quasiminimal type over* M. *If the elementary extension* N *of* M *is full with* $|N| > |M|$, *then* cl *defines a pregeometry on the realizations of* $\phi(x, \mathbf{c})$ *in* N.

Moreover, if $X \subset \phi(N, \mathbf{c})$ *and* $Y \subset \phi(M, \mathbf{c})$ *are equicardinal independent subsets of* $\phi(N, \mathbf{c})$, $\phi(M, \mathbf{c})$ *respectively, there is an elementary map from* X *to* Y.

Proof. Clearly for any a and A, $a \in A$ implies $a \in \mathrm{cl}(A)$. To see that cl has finite character note that if $\mathrm{tp}(a/A\mathbf{c})$ is not big, then it differs from the unique big type over $A\mathbf{c}$ and this is witnessed by a formula so a is in the closure of the parameters of that formula.

For idempotence, suppose $a \in \mathrm{cl}(B)$ and $B \subseteq \mathrm{cl}(A)$. Use the comment after Definition 21 Every $M \in \boldsymbol{K}$ which contains A contains B and every $M \in \boldsymbol{K}$ which contains B contains a; the result follows.

It is only to verify exchange that we need the fullness of N. Suppose $a, b \models \phi(x, \mathbf{c})$, each realizes a big type over $A \subseteq \phi(N)$ and $r = \mathrm{tp}(b/Aa\mathbf{c})$ is big. Since $r = \mathrm{tp}(a/A\mathbf{c})$ is big and N is full we can choose λ realizations a_i of r in N. Let $M' \prec N$ contain the a_i and let b' realize the unique big type over M' containing $\phi(x, \mathbf{c})$. Since $\mathrm{tp}(b/Aa\mathbf{c})$ is big, the uniqueness yields all pairs (a_i, b') realize the same type $p(x, y) \in S(A\mathbf{c})$ as (a, b). But then the a_i are uncountably many realization of $\mathrm{tp}(a/Ab\mathbf{c})$ so this type is big as well; this yields exchange by contraposition.

For the moreover, use the fact that quasiminimal sets have a unique big type and induct. $\square_{22}$

So the dimension of the quasiminimal set is well-defined. To conclude categoricity, we must show that dimension determines the isomorphism type of the model.

We abuse standard notation from e.g. [**?**] in our context. Note that we have restrict our attention to **big** formulas. This will give us two cardinal transfer theorems that read exactly as those for first order but actually have different contact because the first order versions refer arbitrary infinite definable sets.

Definition 23 *1. A triple (M, N, ϕ) where $M \prec N \in \boldsymbol{K}$ with $M \neq N$, ϕ is defined over M, ϕ big, and $\phi(M) = \phi(N)$ is called a* Vaughtian triple.

2. We say $\boldsymbol{K}$ admits (κ, λ), witnessed by ϕ, if there is a model $N \in \boldsymbol{K}$ with $|N| = \kappa$ and $|\phi(N)| = \lambda$ and ϕ is big.

Of course, it is easy in this context to have definable sets which are countable in all models. But we'll show that this is really the only sense in which excellent classes differ from stable theories as far as two cardinal theorems are concerned.

In the first order case, one shows categoricity implies there are no two cardinal models (definable infinite subsets of smaller cardinality than the universe). We can't prove there are no two-cardinal formulas in our situation but from categoricity we show big formulas are not two-cardinal.

The overall structure of the proof of the next result is based on Proposition 2.21 of [**?**]; but in the crucial type omitting step we expand the argument of Theorem IX.5.13 in [**?**] rather than introducing nonorthogality arguments at this stage. We need one piece of notation.

Notation 24 *Suppose $p(\mathbf{x}) \in S(M)$ does not split over the finite set C, enumerated as $\mathbf{c}$, contained in M. For each formula $\phi(\mathbf{x})$ we write $(d_p x)\phi(x, \mathbf{y})[\mathbf{y}, \mathbf{c}]$ for the formula with free variable $\mathbf{y}$ which generates the principal type over $\mathbf{c}$ realized by exactly those $\mathbf{m} \in M$ such that $\phi(\mathbf{x}, \mathbf{m}) \in p$. This is a* defining schema *for p.*

Note that if p doesn't split over C with $C \subset M \prec N$ and $\hat{p} \in S_{\mathrm{at}}(M)$ is a nonsplitting extension of p, $\hat{p}$ is defined by the same schema as p.

Lemma 25 *Suppose $\boldsymbol{K}$ is *-excellent.*

1. *If $\boldsymbol{K}$ admits (κ, λ) for some $\kappa > \lambda$ then $\boldsymbol{K}$ has a Vaughtian triple.*
2. *If $\boldsymbol{K}$ has a Vaughtian triple, for any (κ', λ') with $\kappa' > \lambda'$, $\boldsymbol{K}$ admits (κ', λ').*

Proof. Suppose $N \in \boldsymbol{K}$ with $|N| = \kappa$ and $|\phi(N)| = \lambda$. For notational simplicity we add the parameters of ϕ to the language. By Löwenheim-Skolem, we can embed $\phi(N)$ in a proper elementary submodel M and get a Vaughtian triple. We may assume that M and N are countable. To see this, build within the given M, N

countable increasing sequences of countable models M_i, N_i, fixing one element $b \in N - M$ to be in N_0 and choosing $M_i \prec M$, $N_i \prec N$, $M_i \prec N_i$ and $\phi(N_i) \subset \phi(M_{i+1})$. Then M_ω, N_ω are as required.

Now for any κ', we will construct a (κ', ω) model. Say $b \in N - M$ and let $q = \mathrm{tp}(b/M)$. Now construct N_i for $i < \kappa'$ so that N_{i+1} is primary over the $N_i b_i$ where b_i realizes the non-splitting extension of q to $S_{\mathrm{at}}(N_i)$. Fix finite C contained in M so that q does not split over C. We prove by induction that each $\phi(N_i) = \phi(M)$. Suppose this holds for i, but there is an $e \in \phi(N_{i+1}) - \phi(M)$. Fix $\mathbf{m} \in M_i$ and $\theta(x, \mathbf{z}, y)$ such that $\theta(b_i, \mathbf{m}, y)$ isolates $\mathrm{tp}(e/M_i)$. We will obtain a contradiction.

For every $\mathbf{n} \in N$, if $(\exists y)(\theta(b, \mathbf{n}, y) \wedge \phi(y)$ then for some $d \in M$, $\theta(b, \mathbf{n}, d) \wedge \phi(d)$ holds. Thus,

$$(\forall \mathbf{z})[(d_q x)(\exists y)\theta(x, \mathbf{z}, y) \wedge \phi(y))[\mathbf{z}, \mathbf{c}] \rightarrow (\exists y)\phi(y) \wedge (d_q x)\theta(x, \mathbf{z}, y)[\mathbf{z}, y, \mathbf{c}]].$$

We have $\theta(b_i, \mathbf{m}, e)$, so $M_i \models (d_q x)((\exists y)\theta(x, \mathbf{z}, y) \wedge \phi(y)))[\mathbf{m}, \mathbf{c}]$. Thus by the displayed formula $M_i \models (\exists y)\phi(y) \wedge (d_q x)(\theta(x, \mathbf{z}, y))[\mathbf{m}, y, \mathbf{c}]$. That is, for some $d \in M$, $M_i \models (d_q x)(\theta(x, \mathbf{z}, y))[\mathbf{m}, d, \mathbf{c}]$. Since $\mathrm{tp}(b_i/M_i$ is defined by d_q, we have $\theta(\mathbf{m}, d, \mathbf{c})$. But this contradicts the fact that $\theta(b_i, \mathbf{m}, y)$ isolates $\mathrm{tp}(e/M_i)$.

Thus, we have constructed a model M_μ of M_μ power μ where ϕ is satisfied only countably many times. To construct a (κ', λ') model, iteratively realize the non-splitting extension of ϕ, λ' times.

$\square_{25}$

Now we conclude that categoricity transfers.

Theorem 26 *Suppose* $\boldsymbol{K}$ *is *-excellent. The following are equivalent.*

1. $\boldsymbol{K}$ *is categorical in some uncountable cardinality.*
2. $\boldsymbol{K}$ *has no two cardinal models.*
3. $\boldsymbol{K}$ *is categorical in every uncountable cardinal.*

Proof. We first show 1) implies 2). By Theorem 25, if $\boldsymbol{K}$ has a two-cardinal then it has a $(\lambda, \aleph_0)$-model for every λ. But by Theorem **??**, if it categorical there is a full model in the categoricity cardinal and every big definable subset of a full model has the same cardinality as the model.

3) implies 2) is obvious; it remains to show 2) implies 3). Let M_0 be the unique countable model. By Lemma 19, there is a quasiminimal formula $\phi(x, \mathbf{c})$ with parameters from M_0. For any λ, by Theorem **??** for any model $M \in \boldsymbol{K}$, there is a full model N of $\boldsymbol{K}$ extending M with cardinality λ. By Lemma 22, cl is a pregeometry on $\phi(N)$. Note that $\phi(M)$ is closed since by definition any element a of $\mathrm{cl}(\phi(M))$ both satisfies ϕ and is in every model containing $\phi(M)$, including M. Thus we can choose a basis X for $\phi(M)$. By Lemma ref NEED THIS in next section, there is a prime model $M_{|X|}$ over MX. But $X \subset \phi(M_{|X|}) \subset \phi(M)$ so $\phi(M_{|X|}) = \phi(M)$; whence by Lemma 25, $M_{|X|} = M$ and M is prime and minimal over MX.

Now we show categoricity in any uncountable cardinality. If M, N are models of power λ, they are each prime and minimal over X, a basis for $\phi(M)$ and Y, a basis for $\phi(N)$, respectively. Now any bijection between X and Y is elementary by the moreover clause in Lemma 22. It extends to a map from M into N by primeness and it must be onto; otherwise there is a two cardinal model.