Lecture 3: Abstract Quasiminimality

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Since I have and will use the term:

Definition 1 *A structure M is κ-sequence homogeneous if for any $\boldsymbol{a}, \mathbf{b} \in M$ of length less than κ, if $(M, \boldsymbol{a}) \equiv (M, \mathbf{b})$ then for every c, there exists d such that $(M, \boldsymbol{a}c) \equiv (M, \mathbf{b}d)$.*

An abstract quasiminimal class is a class of structures that satisfy the following five conditions, which we expound leisurely.

Assumption 2 (Condition I) *Let $\boldsymbol{K}$ be a class of L-structures which admit a monotone idempotent closure operation* cl *taking subsets of $M \in \boldsymbol{K}$ to substructures of M such that* cl *has finite character.*

Strictly speaking, we should write $\mathrm{cl}_M(X)$ rather than $\mathrm{cl}(X)$ but we will omit the M where it is clear from context.

Definition 3 *Let A be a subset of $H, H' \in \boldsymbol{K}$. A map from $X \subset H - A$ to $X' \subset H' - A$ is called a partial A-monomorphism if its union with the identity map on A preserves quantifier free formulas.*

Freqently, but not necessarily we will have $A = G$ which is in $\boldsymbol{K}$.

Definition 4 *Let $Ab \subset M$ and $M \in \boldsymbol{K}$. The (quantifier-free) type of b over A in M, written $\mathrm{tp}_{qf}(b/A; M)$ ($\mathrm{tp}_{qf}(b/A; M)$), is the set of (quantifier-free) first-order formulas with parameters from A true of b in M.*

Exercise 5 *Why is M a parameter in Definition 4?*

Exercise 6 *Let $Ab \subset M$, $Gb' \subset M'$ with $M, M' \in bK$. Show there is a partial A-monomorphism taking a' to b' if and only if $\mathrm{tp}_{qf}(b/A; M) = \mathrm{tp}_{qf}(b'/A; M')$.*

The next assumption connects the geometry with the structure on members of $\boldsymbol{K}$.

Assumption 7 (Condition II) *Let $G \subseteq H, H' \in \boldsymbol{K}$ with G empty or in $\boldsymbol{K}$.*

1. *If f is a bijection between X and X' which are* separately cl-*independent (over G) subsets of H and H' then f is a partial G-monomorphism.*

2. *If f is a partial G-monomorphism from H to H' taking $X \cup \{y\}$ to $X' \cup \{y'\}$ then $y \in \mathrm{cl}(XG)$ iff $y' \in \mathrm{cl}(X'G)$.*

Condition 7.2) has an *a priori* unlikely strength: quantifier free formulas determine the closure; in practice, the language is specifically expanded to guarantee this condition. Part 2 of Assumption 7 implies that each M with $G \subseteq M \in \boldsymbol{K}$ is finite sequence homogeneous over G.

Assumption 8 (Condition III : $\aleph_0$-homogeneity over models) *If f is a partial G-monomorphism from H to H' with finite domain X then for any $y \in H$ there is $y' \in H'$ such that $f \cup \{\langle y, y' \rangle\}$ extends f to a partial G-monomorphism.*

Question 9 *Let a, b be independent over the empty set. Suppose f_a, f_b map $\mathrm{cl}(a)$ $(\mathrm{cl}(b))$ into a $\boldsymbol{K}$-structure H. Is $f_a \cup f_b$ a monomorphism? (We prove below that the answer is yes assuming exchange; is exchange necessary?*

Definition 10 *We say a closure operation satisfies the* countable closure condition *if the closure of a countable set is countable.*

We easily see:

Lemma 11 *Suppose Assumptions I, II, and III are satisfied by* cl *on an uncountable structure $M \in \boldsymbol{K}$ that satisfies the countable closure condition.*

1. *For any finite set $X \subset M$, if $a, b \in M - \mathrm{cl}(X)$, a, b realize the same $L_{\omega_1,\omega}$ type over X.*

2. *Every $L_{\omega_1,\omega}$ definable subset of M is countable or cocountable. This implies that $a \in \mathrm{cl}(X)$ iff it satisfies some ϕ over X, which has only countably many solutions.*

Proof. Condition 1) follows directly from Conditions II and III (Assumption 7 and Assumption 8) by constructing a back and forth. To see condition 2), suppose both ϕ and $\neg\phi$ had uncountably many solutions with ϕ defined over X. Then there are a and b satisfying ϕ and $\neg\phi$ respectively and neither is in $\mathrm{cl}(X)$; this contradicts 1).

The ω-homogeneity yields by an easy induction:

Lemma 12 *Suppose Assumptions I II and III hold. Let $G \in \boldsymbol{K}$ be countable and suppose $G \subset H_1, H_2 \in \boldsymbol{K}$.*

1. *If $X \subset H - G$, $X' \subset H - G$ are finite and f is a G-partial monomorphism from X to X' then f extends to a G-partial monomorphism from $\mathrm{cl}_H(GX)$ to $\mathrm{cl}_{H'}(GX')$.*

2. *If X is independent set of cardinality at most $\aleph_1$, and f is a G-partial monomorphism from X to X' then f extends to a G-partial monomorphism from $\mathrm{cl}_H(GX)$ to $\mathrm{cl}_{H'}(GX')$.*

Proof. The first statement is immediate from homogeneity. The second follows by induction from the first (by replacing G by $\mathrm{cl}(GX_0)$ for X_0 a countable initial segment of X). $\square_{12}$

For algebraic closure the cardinality restriction on X is unnecessary. We will have to add Assumption 8 to remove the restriction in excellent classes.

Lemma 13 *Suppose Assumptions I II and III hold. Suppose further that $\boldsymbol{K}$ is defined by a sentence of $L_{\omega_1,\omega}$ and so is the relation $x \in \mathrm{cl}(\mathbf{y})$. If there is an $H \in \boldsymbol{K}$ which contains an infinite* cl*-independent set, then there are members of $\boldsymbol{K}$ of arbitrary cardinality which satisfy the countable chain condition.*

Proof. Let X be a countable independent set of H and let $H_0 = \mathrm{cl}_H(X)$. Let L^* be a countable fragment of $L_{\omega_1,\omega}$ containing the sentence axiomatizing $\boldsymbol{K}$. Note that $H_0 \prec L^* H_1$ where is H_1 is the closure of Xa where a is independent from X. Continuing, one can construct a continuous L^*-elementary increasing chain of members of $\boldsymbol{K}$ for any length α. Since the chain is L^*-elementary, each $H_\alpha \in \boldsymbol{K}$. But the closure of any countable set is contained in the union of a countable subset of this basis which is countable. Thus, each H_α has countable closures.

Assumption 14 (Condition IV) cl *satisfies the exchange axiom: $y \in \mathrm{cl}(Xx) - \mathrm{cl}(X)$ implies $x \in \mathrm{cl}(Xy)$.*

Zilber omits exchange in the fundamental definition but it arises in the natural contexts he considers so we make it part of quasiminimal excellence. Note however that the examples of first order theories with finite Morley rank greater than 1 (e.g. [**?**]) fail exchange.

In the following definition it is essential that $\subset$ be understood as *proper* subset.

Definition 15

1. *For any Y, $\mathrm{cl}^-(Y) = \bigcup_{X \subset Y} \mathrm{cl}(X)$.*
2. *We call C (the union of)* an n-dimensional cl-independent system *if $C = \mathrm{cl}^-(Z)$ and Z is an independent set of cardinality n.*

To visualize a 3-dimensional independent system think of a cube with the empty set at one corner A and each of the independent elements z_0, z_1, z_2 at the corners connected to A. Then each of $\mathrm{cl}(z_i, z_j)$ for $i < j < 3$ determines a side of the cube: $\mathrm{cl}^-(Z)$ is the union of these three sides; $\mathrm{cl}(Z)$ is the entire cube.

Assumption 16 (Condition V) *Let $G \subseteq H, H' \in \boldsymbol{K}$ with G empty or in $\boldsymbol{K}$. Suppose $Z \subset H - G$ is an n-dimensional independent system, $C = \mathrm{cl}^-(Z)$, and X is a finite subset of $\mathrm{cl}(Z)$. Then there is a finite C_0 contained in C such that: for every G-partial monomorphism f mapping X into H', for every G-partial monomorphism f_1 mapping C into H', if $f \cup (f_1 \restriction C_0)$ is a G-partial monomorphism, $f \cup f_1$ is also a G-partial monomorphism.*

We can rephrase the conclusion as for any $\boldsymbol{a} \in \mathrm{cl}(Z)$ there is a finite $C_{\boldsymbol{a}} \subseteq C$ such that $\mathrm{tp}_{qf}(\boldsymbol{a}/C_{\boldsymbol{a}}; H)$ implies $\mathrm{tp}_{qf}(\boldsymbol{a}/C; H)$.

Thus Condition IV, which is the central point of excellence, asserts (for dimension 3) that the type of any element in the cube over the union of the three given sides is determined by the type over a finite subset of the sides. The 'thumbtack lemma' of Subsection **??** verifies this condition in a specific algebraic context. Here is less syntactic version of Assumption 8

Definition 17 *We say $M \in \boldsymbol{K}$ is* prime *over the set $X \subset M$ if every partial monomorphism of X into $N \in \boldsymbol{K}$ extends to a partial monomorphism of M into N.*

Remark 18 *Note that in first order logic this corresponds to 'algebraically prime' rather than 'elementarily prime'. In the first order context algebraically prime is a notoriously unstable (in a nontechnical sense) concept.*

Lemma 19 *Let $G \subseteq H, H' \in \boldsymbol{K}$ with G empty or in $\boldsymbol{K}$ and countable. Suppose $Z \subset H - G$ is an n-dimensional independent system, $C = \mathrm{cl}^{-}(Z)$, then $\mathrm{cl}(Z) \subseteq H$ is prime over C.*

Proof. Fix an embedding f from C into H' containing G. We must extend f to $\hat{f}$ mapping $\mathrm{cl}(Z)$ into H'. We can enumerate $\mathrm{cl}(X)$ as $a_i : i < \omega$. Let A_n denote $\{a_i : i < n\}$. Now define by induction an increasing sequence of finite sets $B_n : n < \omega$ such that $\mathrm{tp}_{qf}(A_n/B_n; H)$ implies $\mathrm{tp}_{qf}(A_n/C; H)$ and $\bigcup_{n<\omega} B_n = C$. Now, using only part of Lemma 12 1), construct an increasing family of maps f_n with the domain of $f_n = A_n \cup B_n$. Then the union of these functions is the required embedding. $\square_{19}$

Theorem 20 *Let $\boldsymbol{K}$ be a quasiminimal excellent class and suppose $H, H' \in \boldsymbol{K}$ satisfy the countable closure condition. Let $\mathcal{A}, \mathcal{A}'$ be cl-independent subsets of H, H' with $\mathrm{cl}(\mathcal{A}) = H$, $\mathrm{cl}(\mathcal{A}') = H'$, respectively, and ψ a bijection between $\mathcal{A}$ and $\mathcal{A}'$. Then ψ extends to an isomorphism of H and H'.*

Suppose further, that some model of $\boldsymbol{K}$ contains an infinite cl-independent set. Then the class of $\boldsymbol{K}$-structures which satisfy the countable closure condition is categorical in every uncountable cardinality.

The remainder of this section is devoted to the proof of Theorem 20

Notation 21 *Fix a countable subset W and write $\mathcal{A}$ as the disjoint union of $\mathcal{A}_0$ and a set $\mathcal{A}_1$; without loss of generality, we can assume ψ is the identity on $\mathrm{cl}_H(\mathcal{A}_0)$ and work over $G = \mathrm{cl}_H(\mathcal{A}_0)$. We may write $\mathrm{cl}^*(X)$ to abbreviate $\mathrm{cl}(\mathcal{A}_0 X)$.*

Lemma 22 *Suppose X, Y are subsets of $\mathcal{A}_1$. Suppose $\mathbf{b} \in \mathrm{cl}(\mathcal{A}_0 X)$ and $\mathbf{c} \in \mathrm{cl}(\mathcal{A}_0 Y)$ and $p(\mathbf{b}, \mathbf{c}, \mathbf{g})$ for some quantifier-free type q (over $\mathbf{g} \in G$). Then there is a map π into $\mathrm{cl}(\mathcal{A}_0 Y)$ whose domain includes $\mathbf{bcg}$, that fixes $\mathbf{cg}$, and such that $p(\mathbf{b}^{\pi}\mathbf{cg})$ holds.*

Proof. Choose finite $A^* \subset \mathcal{A}_0$ such that $\mathbf{g} \in \mathrm{cl}(A^*)$, $\mathbf{b} \in \mathrm{cl}(A^* X)$, and $\mathbf{c} \in \mathrm{cl}(A^* Y)$. Let $G_0 = \mathrm{cl}(A^* Y)$. Extend the identity map on G_0 to π_1 with domain $G_0 X$ by mapping $X - Y$ into $\mathcal{A}_0 - (A^* Y)$. By Assumption 7 .1, π_1 is a partial G_0-monomorphism. By Lemma 12 π_1 extends to a partial G_0-monomorphism π from $\mathrm{cl}(A^* XY)$ into $\mathrm{cl}^*(Y)$. Clearly π has the required property. $\square_{22}$

Lemma 23 *Suppose X, Y are subsets of $\mathcal{A}_1$ and that ψ_X and ψ_Y are each partial G-monomorphisms from H into H' with $\mathrm{dom}\,\psi_X = \mathrm{cl}(\mathcal{A}_0 X)$ and $\mathrm{dom}\,\psi_Y = \mathrm{cl}(\mathcal{A}_0 Y)$ that agree on $\mathrm{cl}(\mathcal{A}_0 X) \cap \mathrm{cl}(\mathcal{A}_0 Y)$. Then $\psi_X \cup \psi_Y$ is a partial G-monomorphism.*

Proof. By Lemma 12 there is a partial G-monomorphism ψ_{XY} which extends ψ_X and maps $\mathrm{cl}^*(XY)$ into H'. Another partial map on $\mathrm{cl}^*(X) \cup \mathrm{cl}^*(Y)$ is given by $\psi_X \cup \psi_Y$. It suffices to show that for any $\mathbf{b} \in \mathrm{cl}^*(X)$, $\mathbf{g} \in G$ and $\mathbf{c} \in \mathrm{cl}^*(Y)$ and any quantifier free R, $H' \models R(\psi_X(\mathbf{b}), \psi_{XY}(\mathbf{c}), \mathbf{g})$ if and only if $H' \models R(\psi_X(\mathbf{b}), \psi_Y(\mathbf{c}), \mathbf{g})$. So we are finished if we apply the following lemma to H'. To apply the Lemma, note that $\psi_{XY} \circ \psi_Y^{-1}$ is a partial G-monomorphism taking $\psi_Y(\mathbf{c})$ to $\psi_{XY}(\mathbf{c})$.

Lemma 24 *Let $X, Y, Y' \subseteq \mathcal{A}_1$. Let $\mathbf{b} \in \mathrm{cl}(\mathcal{A}_0 X)$, $\mathbf{c} \in \mathrm{cl}(\mathcal{A}_0 Y)$, and $\mathbf{c}' \in \mathrm{cl}(\mathcal{A}_0 Y')$. Suppose f is a partial G-monomorphism taking $\mathbf{c}$ to $\mathbf{c}'$, then f is a partial $\mathrm{cl}^*(X)$ monomorphism.*

Proof. If not there exists $\mathbf{b} \in \mathrm{cl}^*(X)$ and $\mathbf{g} \in G$ and a quantifier free R such that $R(\mathbf{b}, \mathbf{c}, \mathbf{g}) \wedge \neg R(\mathbf{b}, \mathbf{c}', \mathbf{g})$. Now apply Lemma 22, to obtain a map π into $\mathrm{cl}(\mathcal{A}_0 Y Y')$ which fixes $\mathbf{c}, \mathbf{c}', \mathbf{g}$ and such that $R(\mathbf{b}^{\pi}, \mathbf{c}, \mathbf{g}) \wedge \neg R(\mathbf{b}^{\pi}, \mathbf{c}', \mathbf{g})$. This contradicts that $\mathbf{c}, \mathbf{c}'$ are partially isomorphic over G. $\square_{24}$

$\square_{23}$

We have by straightforward induction.

Corollary 25 *Suppose $\langle X_i : i < m\rangle$ are subsets of A and that each ψ_{X_i} is a partial G-monomorphisms from H into H' with $\operatorname{dom}\psi_{X_i} = \mathrm{cl}(\mathcal{A}_0 X_i)$ and that for any i, j, ψ_{X_i} and ψ_{X_j} agree whenever both are defined. Then $\psi_X \cup \psi_Y$ is a partial G-monomorphism.*

Proof of Theorem 20. Note that $H = \lim_{X\subset\mathcal{A};|X|<\aleph_0} \mathrm{cl}(X)$. We have the obvious directed system on $\{\mathrm{cl}(X) : X \subset \mathcal{A}; |X| < \aleph_0\}$. So the theorem follows immediately if for each finite X we can choose $\psi_X : \mathrm{cl}(X) \to H'$ so that $X \subset Y$ implies $\psi_X \subset \psi_Y$. We prove this by induction on $|X|$. Suppose $|Y| = n+1$ and we have appropriate ψ_X for $|X| < n+1$. We will prove two statements by induction.

1. $\psi_Y^- : \mathrm{cl}^-(Y) \to H'$ defined by $\psi_Y^- = \bigcup_{X\subset Y} \psi_X$ is a monomorphism.
2. ψ_Y^- extends to ψ_Y defined on $\mathrm{cl}(Y)$.

The first step is done by induction using Corollary 25 with the $X \subset Y$ with $|X| = n$ as the X_i. The exchange axiom is used to guarantee that the maps ψ_X agree where more than one is defined. The second step follows by Lemma 19.

We have shown that the isomorphism type of a structure in $\boldsymbol{K}$ with countable closures is determined by the cardinality of a basis for the geometry. If M is an uncountable model in $\boldsymbol{K}$ that satisfies the countable closure condition, the size of M is the same as its dimension so there is at most one model in each uncountable cardinality which has countable closures. But there is at least one since we can build a model with any dimension κ. (Build by induction on $\alpha < \kappa$ structures M_α with cl-basis $\langle x_\gamma : \gamma < \alpha\rangle$. Note that each M_α has countable closures since the closure of any countable set is contained in a model isomorphic to M_ω.) $\square_{20}$

Note that a sentence ψ can define a quasiminimal excellent class without being $\aleph_0$-categorical. But we could extend ψ to ψ' - the Scott sentence of the model with countably infinite cl-dimension and attain $\aleph_0$-categoricity.

The following corollary seems to rely on the categoricity argument. The key is Condition II (**??**) and for countable G it follows from Lemma 12. But in general we use Theorem 20.

Corollary 26 *Let $\boldsymbol{K}$ be a quasiminimal excellent class, with $G \subset H, H'$ all in $\boldsymbol{K}$. If $a \in H, a' \in H'$ realize the same quantifier free type over G (i.e. there is a G-monomorphism taking a to a') then there is a $\boldsymbol{K}$-isomorphism from $\mathrm{cl}(Ga)$ onto $\mathrm{cl}(Ga')$.*

Thus (G, a, H) and (G, a', H') realize the same Galois type.

Exercise 27 *Define a notion of almost quasiminimality analogous to almost strong minimality and prove that almost quasiminimal classes are categorical in all powers ([**?**]).*

Question 28 *Zilber's proof of Theorem 20 is considerably more complicated. I think this is because he does not assume exchange. How would you modify the argument here to avoid the use of exchange?*