Lecture 4: Categoricity implies Completeness

John T. Baldwin
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago

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The logic $L_{\omega_1,\omega}$ is obtained by extending the formation rules of first order logic to allow countable conjunctions and disjunctions. A fragment of $L_{\omega_1,\omega}$ is a set of formulas closed under subformula and the finitary operations.

Definition 1 *A sentence ψ in $L_{\omega_1,\omega}$ is called complete if for every sentence ϕ in $L_{\omega_1,\omega}$, either $\psi \models \phi$ or $\psi \models \neg\phi$.*

In first order logic, the theory of a structure is a well-defined object; here such a theory is not so clearly specified. An infinite conjunction of first order sentences behaves very much like a single sentence; in particular it satisfies both the upward and downward Löwenheim Skolem theorems. In contrast, the conjunction of all $L_{\omega_1,\omega}$ true in an uncountable model may not have a countable model. In its strongest form Morley's theorem asserts: Let T be a first order theory having only infinite models. If T is categorical in some uncountable cardinal then T is complete and categorical in every uncountable cardinal. This strong form does not generalize to $L_{\omega_1,\omega}$; take the disjunction of a sentence which is categorical in all cardinalities with one that has models only up to, say, $\beth_2$. Since $L_{\omega_1,\omega}$ fails the upwards Löwenheim-Skolem theorem, the categoricity implies completeness argument that holds for first order sentences fails. However, if the $L_{\omega_1,\omega}$-sentence ψ is categorical in κ, then, applying the downwards Löwenheim-Skolem theorem, for every sentence ϕ either $\psi \rightarrow \phi$ or all models of ϕ have cardinality less than κ. So if ϕ and ψ are κ-categorical sentences with a common model of power κ they are equivalent. Such a sentence in necessarily $\aleph_0$-categorical (using downward Löwenheim-Skolem). Moreover, every countable structure is characterized by a complete sentence – its Scott sentence. So if a model satisfies a complete sentence, it is $L_{\infty,\omega}$-equivalent to a countable model.

For purposes of this chapter, one can think of $\tau = \tau'$ in the following. The greater generality will be used a bit later.

Definition 2 *Let $\tau \subseteq \tau'$.*

1. *A τ'-structure M is L^*-small for L^* a countable fragment of $L_{\omega_1,\omega}(\tau)$ if M realizes only countably many L^*-types.*
2. *A τ'-structure M is τ-small if realizes only countably many $L_{\omega_1,\omega}(\tau)$-types.*

Let M be the only model of power κ of an $L_{\omega_1,\omega}$-sentence ψ. We want to find sufficient conditions so that there is a complete sentence ψ' which implies ψ and is true in M. We will two such conditions: ψ has arbitrarily large models; ψ has few models of $\aleph_1$. One key tool for this analysis is a different representation of $L_{\omega_1,\omega}$-sentences.

It is quite easy to see:

Exercise 3 *If ψ is a complete sentence in $L_{\omega_1,\omega}$ in a countable language L then every model M of ψ realizes only countably many $L_{\omega_1,\omega}$-types.*

In general, an $L_{\omega_1,\omega}$-type may contain uncountably many formulas. But,

Exercise 4 *If the structure M realizes only countably many $L_{\omega_1,\omega}$-types, then for every tuple $\boldsymbol{a}$ in M there is a formula $\phi(\mathbf{x}) \in L_{\omega_1,\omega}$ such $M \models \phi(\mathbf{x}) \rightarrow \psi(\mathbf{x})$ for each $L_{\omega_1,\omega}$-formula true of $\boldsymbol{a}$.*

But we will give the short argument for the converse: small models have Scott sentences. A Scott sentence for a countable model M is a complete sentence satisfied by M; it characterizes M up to isomorphism among countable models. The Scott sentence for an uncountable small model is the Scott sentence a countable L^*-submodel of M, where L^* is the smallest fragment containing a formula for each type realized in M.

Lemma 5 *Let M be a τ structure for some countable τ. If ψ is a sentence in $L_{\omega_1,\omega}$ and M is a model of ψ that realizes only countably many $L_{\omega_1,\omega}$-types then there is a complete $L_{\omega_1,\omega}$-sentence ψ' so that*

1. *$\psi' \models \psi$;*
2. *$M \models \psi'$.*

Proof. Let L^* be the smallest fragment of $L_{\omega_1,\omega}$ containing ψ and the conjunction of each countable type in $L_{\omega_1,\omega}$ type realized in M. Let N be a countable L^*-elementary submodel of M and let ψ' be a Scott sentence for N. Clearly ψ' is complete. By the choice of L^*, ψ' is in L^*; so $M \models \psi'$. $\square_5$

Confirm: By the choice of L^*, ψ' is in L^*.

Theorem 6 *Let ψ be a sentence in $L_{\omega_1,\omega}$ in a countable language L. Then there is a countable language L' extending L, a first order L'-theory T, and a collection of L'-types Γ such that reduct is a 1-1 map from the models of T which omit Γ onto the models of ψ.*

Proof. Expand L to L' by inductively adding a predicate $P_\phi(\mathbf{x})$ for each L^*-formula ϕ. Fix a model of ψ and expand it to an L' by interpreting the new predicates so that the new predicates represent each finite Boolean connective and quantification faithfully: E.g.

$$P_{\neg\phi(\mathbf{x})} \leftrightarrow \neg P_\phi(\mathbf{x}),$$

and

$$P_{(\forall \mathbf{x})\phi(\mathbf{x})} \leftrightarrow (\forall \mathbf{x}) P_\phi(\mathbf{x}),$$

and that, as far as first order logic can, the P_ϕ preserve the infinitary operations: for each i,

$$P_{\bigwedge_i \phi_i(\mathbf{x})} \rightarrow P_{\phi_i(\mathbf{x})}.$$

Let T be the first order theory of any such model and consider the set Γ of types

$$p_{\bigwedge_i \phi_i(\mathbf{x})} = \{\neg P_{\bigwedge_i \phi_i(\mathbf{x})}\} \cup \{P_{\phi_i(\mathbf{x})} : i < \omega\}.$$

Now if M is a model of T which omits all the types in Γ, $M|L \models \psi$ and each model of ψ has a *unique* expansion to a model of T which omits the types in Γ (since this is an expansion by definitions in $L_{\omega_1,\omega}$). $\square_6$

Since all the new predicates in the reduction described above are $L_{\omega_1,\omega}$-definable this is a natural extension of Morley's procedure of replacing each first order formula ϕ by a predicate symbol P_ϕ, thus guaranteeing amalgamation over sets for first order categorical T; the amalgamation does *not* follow in this case. In general, finite diagrams do not satisfy the upper Löwenheim-Skolem theorem.

Since there is a 1-1 correspondence between models of ψ and models of T that omit Γ, we can reduce spectrum considerations for sentences with arbitrarily large models to the study of $EC(T,\Gamma)$-classes. In addition, we have represented the models of ψ as a $PC\Gamma$ class in the following sense.

Definition 7 *A* $PC(T,\Gamma)$ class *is the class of reducts to $\tau \subset \tau'$ of models of a first order theory τ'-theory which omit all types from the specified collection Γ of types in finitely many variables over the empty set.*

We write $PC\Gamma$ to denote such a class without specifying either T or Γ. And we write $\boldsymbol{K}$ is $PC(\lambda,\mu)$ if $\boldsymbol{K}$ can be presented as $PC(T,\Gamma)$ with $|T| \leq \lambda$ and $|\Gamma| \leq \mu$. In the simplest case, we say $\boldsymbol{K}$ is λ-presented if $\boldsymbol{K}$ is $PC(\lambda,\lambda)$.

We have shown every $L_{\omega_1,\omega}$-sentence in a countable language is ω presented.

Exercise 8 *Show that ψ is a sentence in $L_{\lambda^+,\omega}$ in a language of cardinality κ, ψ is μ-presented where μ is the larger of κ and λ.*

Exercise 9 *In general a $PC\Gamma$ class will not be an AEC class of τ structures. Why?*

Now, modify the proof of Theorem 6 to show:

Exercise 10 *Let ψ be a complete sentence in $L_{\omega_1,\omega}$ in a countable language L. Then there is a countable language L' extending L and a first order L'-theory T such that reduct is a* 1-1 *map from the atomic models of T onto the models of ψ. So in particular,* any complete sentence of $L_{\omega_1,\omega}$ can be replaced (for spectrum purposes) by considering the atomic models of a first order theory.

To show a categorical sentence with arbitrarily large models extends to a complete sentence we need the method of Ehrenfeucht-Mostowski models. 'Morley's method' (Section 7.2 of [**?**]) is a fundamental technique in first order model theory. It is essential for the foundations of simplicity theory and for the construction of indiscernibles in infinitary logic. We quote the first order version here; in Lemma **??**, we prove the analog for abstract elementary classes.

Notation 11

1. *For any linearly ordered set $X \subseteq M$ where M is a τ'-structure and $\tau' \supseteq \tau$, we write $\mathbf{D}_\tau(X)$ (diagram) for the set of τ-types of finite sequences (in the given order) from X. We will omit τ if it is clear from context.*
2. *Such a diagram of an order indiscernible set, $\mathbf{D}_\tau(X) = \Phi$, is called 'proper for linear orders'.*
3. *If X is a sequence of τ-indiscernibles with diagram $\Phi = \mathbf{D}_\tau(X)$ and any τ model of Φ has built in Skolem functions, then for any linear ordering I, $EM(I,\Phi)$ denotes the τ-hull of a sequence of order indiscernibles realizing Φ.*
4. *If $\tau_0 \subset \tau$, the reduct of $EM(I,\Phi)$ to τ_0 is denoted $EM_{\tau_0}(I,\Phi)$.*

Exercise 12 *Suppose* τ *'contains Skolem functions' and* $X \subset M$ *is sequence of order indiscernibles with diagram* ϕ*. Show that for any linearly ordered set* Z*,* $EM(Z, \Phi)$ *is a model that is* τ*-elementarily equivalent to* M*.*

Lemma 13 *If* $(X, <)$ *is a sufficiently long linearly ordered subset of a* τ*-structure* M*, for any* τ' *extending* τ *(the length needed for* X *depends on* $|\tau'|$*) there is a countable set* Y *of* τ'*-indiscernibles (and hence one of arbitrary order type) such that* $\mathbf{D}_\tau(Y) \subseteq \mathbf{D}_\tau(X)$*. This implies that the only (first order)* τ*-types realized in* $EM(X, \mathbf{D}_{\tau'}(Y))$ *were realized in* M*.*

We need a little background on orderings.

Definition 14 *A linear ordering* $(X, <)$ *is* k*-transitive if every map between increasing* k*-tuples extends to an order automorphism of* $(X, <)$*.*

Exercise 15 *Show any* 2*-transitive linear order is* k*-transitive for all finite* k*.*

Exercise 16 *Show there exist* 2*-transitive linear orders in every cardinal; hint: take the order type of an ordered field.*

Exercise 17 *If* $\boldsymbol{d}_\tau(Y)$ *is the diagram of a sequence of* τ*-order indiscernibles, show any order isomorphism of* Y *extends to an automorphism of the* τ*-structure* $EM(Y, \Phi)$*.*

Definition 18 *For any model* M *and* a, B *contained in* M*, the* Galois-type *of* a *over* B *in* M *is the orbit of* a *under the automorphisms of* M *which fix* B*.*

This notion of Galois type requires an ambient model M. We will speak indiscriminately of the number of Galois types in M as an upper bound on the number of Galois n-types over any finite n.

Exercise 19 *If* Y *is a* 2*-transitive linear ordering and then for any* τ *and* Φ *is proper for linear orders,* $EM(Y, \Phi)$ *has* $|\tau|$ *Galois types.*

Exercise 20 *For any reasonable logic* $\mathcal{L}$ *(i.e. a logic such that truth is preserved under isomorphism) and any model* M *the number of Galois types over the empty set in* M *is at most the number of* $\mathcal{L}$*-types over the empty set in* M*.*

Now we can make our first application of the omitting types theorem.

Corollary 21

1. *If an* $L_{\omega_1,\omega}(\tau)$*-sentence* ψ *has arbitrarily large models then in every infinite cardinality* ψ *has a model which realizes only countably many* $L_{\omega_1,\omega}(\tau)$*-types over the empty set.*
2. *Thus, if* ψ *is categorical in some cardinal,* ψ *is implied by a consistent complete sentence* ψ'*.*

Proof. By Theorem 6, we can extend τ to τ' and choose a first order theory T and a countable set of types Γ such $\mathrm{mod}(\psi) = PC_\tau(T, \Gamma)$. Since ψ has arbitrarily large models we can apply Theorem 13 to find τ''-indiscernibles for a Skolemization of T in an extended language τ''. Now take an Ehrenfeucht-Mostowski τ''-model M for the Skolemization of T over a set of indiscernibles ordered by a 2-transitive dense linear order. Then for every n, M has only countably many orbits of n-tuples and so realizes only countably many types in any logic where truth is preserved by automorphism – in particular in $L_{\omega_1,\omega}$. So the τ-reduct of M realizes only countably many $L_{\omega_1,\omega}(\tau)$-types. If ψ is κ-categorical, let ψ' be the Scott sentence of this Ehrenfeucht-Mostowski model with cardinality κ. $\square_{21}$

The countability of the language is crucial for this result.

For the second case, $I(\aleph_1, \psi) < 2^{\aleph_1}$, we require some further definitions and must quote some hard results. For our present purposes it would suffice to prove the next theorems for $L_{\omega_1,\omega}$ but for future applications, and since the proofs need few changes we work with 'pseudo-elementary' classes. We will apply Theorem 12 and Theorem 28 from [**?**].

Theorem 22 (Lopez-Escobar, Morley) *Let ψ be an $L_{\omega_1,\omega}(\tau)$-sentence and suppose $U, <$ are a unary and a binary relation in τ. Suppose that for each $\alpha < \omega_1$, there is a model M_α of ψ such that $<$ is linear orders $P(M_\alpha)$ and α imbeds into $(P(M_\alpha), <)$. Then there is a (countable) model M of ψ such that $(P(M), <)$ contains a copy of the rationals.*

If N is linearly ordered, N is an *end extension* of M if every element of N comes before every element of $N - M$.

Theorem 23 *Let L^* be a countable fragment of $L_{\omega_1,\omega}$. If a countable model M has a proper L^*-elementary end extension, then it has one with cardinality $\aleph_1$.*

Theorem 24 *If the τ' $L_{\omega_1,\omega}$-sentence ψ has a model of cardinality $\aleph_1$ which is L^*-small for every countable τ-fragment L^* of $L_{\omega_1,\omega}$, then ψ has a τ-small model of cardinality $\aleph_1$.*

Proof. Add to τ' a binary relation $<$, interpreted as a linear order of M with order type ω_1. Using that M realizes only countably many types in any τ-fragment, write $L_{\omega_1,\omega}(\tau)$ as a continuous increasing chain of fragments L_α such that each type in L_α realized in M is a formula in $L_{\alpha+1}$. Extend the similarity type to τ'' by adding new $2n+1$-ary predicates $E_n(x, \mathbf{y}, \mathbf{z})$ and $n+1$-ary functions f_n. Let M satisfy $E_n(\alpha, \boldsymbol{a}, \mathbf{b})$ if and only if $\boldsymbol{a}$ and $\mathbf{b}$ realize the same L_α-type and let f_n map M^{n+1} into the initial ω elements of the order, so that $E_n(\alpha, \boldsymbol{a}, \mathbf{b})$ implies $f_n(\alpha, \boldsymbol{a}) = f_n(\alpha, \mathbf{b})$. Note:

1. $E_n(\beta, \mathbf{y}, \mathbf{z})$ refines $E_n(\alpha, \mathbf{y}, \mathbf{z})$ if $\beta > \alpha$;
2. $E_n(0, \boldsymbol{a}, \mathbf{b})$ implies $\boldsymbol{a}$ and $\mathbf{b}$ satisfy the same quantifier free τ-formulas;
3. If $\beta > \alpha$ and $E_n(\beta, \boldsymbol{a}, \mathbf{b})$, then for every c_1 there exists c_2 such that
   (a) $E_{n+1}(\alpha, x\boldsymbol{a}, y\mathbf{b})$ and
   (b) if there are uncountably many c such that $E_{n+1}(\alpha, c\boldsymbol{a}, c_1\boldsymbol{a})$ then there are uncountably many c such that $E_{n+1}(\alpha, c\mathbf{b}, c_2\mathbf{b})$.
4. f_n witnesses that for any $a \in M$ each equivalence relation $E_n(a, \mathbf{y}, \mathbf{z})$ has only countably many classes.

All these assertions can be expressed by an $L_{\omega_1,\omega}(\tau'')$ sentence ϕ. Let L^* be the smallest τ''-fragment containing $\phi \wedge \psi$. Now add a unary predicate symbol P and a sentence χ which asserts M is an *end* extension of $P(M)$. For every $\alpha < \omega_1$ there is a model M_α of $\phi \wedge \psi \wedge \chi$ with order type of $(P(M), <)$ greater than α. (Start with P as α and alternately take an L^*-elementary submodel and close down under $<$. After ω steps we have the P for M_α.) Now by Theorem 22 there is countable structure $(N_0, P(N_0))$ such that $P(N_0)$ contains a copy of $(Q, <)$ and N_0 is an end extension of $P(N_0)$. By Theorem 23, N_0 has an L^* elementary extension N_1 of cardinality $\aleph_1$. Fix an infinite decreasing sequence $d_0 > d_1 > \ldots$ in N_1. For each n, define $E^+_n(\mathbf{x}, \mathbf{y})$ if for some i, $E_n(d_i, \mathbf{x}, \mathbf{y})$.

Now using i), ii) and iii) prove by induction on the quantifier rank of ϕ that $N \models E_n^+(\boldsymbol{a}, \mathbf{b})$ implies $N \models \phi(\boldsymbol{a})$ if and only if $N \models \phi(\mathbf{b})$ for every $L_{\omega_1,\omega}(\tau)$-formula ϕ. For each n, $E_n(d_0, \mathbf{x}, \mathbf{y})$ refines $E_n^+(\mathbf{x}, \mathbf{y})$ and by iv) $E_n(d_0, \mathbf{x}, \mathbf{y})$ has only countably many classes; so N is small. $\square_{24}$

Now we show that sentences of $L_{\omega_1,\omega}$ that have few models can be extended to complete sentences. We rely on the following result of Keisler [Theorem 45 of [**?**]].

Theorem 25 *For any $L_{\omega_1,\omega}$-sentence ψ and any fragment L^* containing ψ, if ψ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then for any $M \models \psi$ of cardinality $\aleph_1$, M realizes only countably many L^*-types over the empty set.*

Theorem 26 *If an $L_{\omega_1,\omega}$-sentence ψ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then there is a complete $L_{\omega_1,\omega}$-sentence ψ' that implies ψ and has a model of cardinality $\aleph_1$.*

Proof. By Theorem 25 every model of ψ of cardinality $\aleph_1$ is L^*-small for every countable fragment L^*. By Theorem 24 ψ has a model of cardinality $\aleph_1$ which is small. By Lemma 5, we finish. $\square_{26}$

So to study categoricity of $L_{\omega_1,\omega}$-sentence ψ, we have established the following reduction. If ψ has arbitrarily large sentences, without loss of generality, ψ is complete. If ψ has few models of power $\aleph_1$, we can study a subclass of the models of ψ defined by a complete $L_{\omega_1,\omega}$-sentence ψ'. We will in fact prove sufficiently strong results about ψ' to deduce a nice theorem for ψ. Note that since ψ' is complete, the models of ψ' form an $EC(T, Atomic)$-class in an extended similarity type τ'.

Remark 27 *To extend this result to $L_{\omega_1,\omega}(Q)$ we need to have Theorems 22 and 23 for $L_{\omega_1,\omega}(Q)$.*