Assignment 9 due Nov. 6

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We first define addition and multiplication. These are operations on equivalence classes of segments. So I write $[a]$ for the set of all segments a' such that $a \cong a'$. Sometimes segments are denoted by a single small letter and sometimes by their endpoints XY.

I describe the operation on congruence classes of segments a, b; if $a \cong a'$ (congruent) and $b \cong b'$ then $a \oplus b \cong a' \oplus b'$ and similarly for $\otimes$.

Thus in the language of the previous assignment I am thinking of 'magnitude' as naming an equivalence class. That is a and b have the same magnitude if and only if $a \cong b$, i.e $[a] = [b]$.

I made the equivalence relation of congruence explicit to clarify the difference between 'magnitude' and 'segment'.

Retraction I made a mistake in class last week. I had not noticed that Hilbert neither defined subtraction nor claimed that there was an additive inverse. In fact, the operations below correspond to addition and multiplication on a set of positive numbers (certainly including all positive rationals, perhaps more). But we will have to think more about subtraction. In particular, the structure we have below $(X, \oplus, \otimes, [1])$ is *not* a field because the addition is not a group –although the multiplication is.

Definition 1

1. *Given magnitudes $[a]$, $[b]$, choose a segment $AB \in [a]$ and on the ray AB choose an element C so that $BC \in [b]$ and C is not between A and C and A is not between C and B. Then $[a] \oplus [b] = AC$.*

2. *Given magnitudes $[a]$, $[b]$, fix a magnitude $[1]$ by fiat to be the unit. Choose a segment $AB \in [a]$ and on the ray AB choose an element C so that $AC \in [1]$. Construct a segment AD on a different ray through A with $AD \in [b]$. Draw the line DC and then draw a line through B parallel to DC. Let the intersection of that line with the ray AD be E. Then $[b] \otimes [a] = AE$.*

Assignment

You may use below Hilbert's axioms in Groups I-IV and any theorems we have proved from them. Note that the ability to construct a segment a' congruent to given a from any point A is Hilbert's IV.1. (We proved it as Proposition 3 of Euclid.) You may also use Euclid's common notions to reason about equality.

1. Prove that if $[a] \oplus [b] = [a] \oplus [c]$ then $[b] = [c]$.
2. Prove $[a] \otimes ([b] \oplus [c]) = ([a] \otimes [b]) \oplus ([a] \otimes [c])$.
3. Prove $[a] \otimes ([b] \otimes [c]) = ([a] \otimes ([b]) \otimes [c]$.

Number 2 is pretty easy; number three is fairly complicated, using the Pascal (Pappus) theorem.

4. Read the Usiskin article on Teacher's mathematics. Consider especially his analysis of parallelism and his various formulas for the area of a triangle.