Parallels, Similarity, Proportion

John T. Baldwin

October 29, 2007

Context

We are working in the situation with Hilbert's axiom groups I, II, III (incidence, order, and congruence).

We have proved SAS, ASA, and ASA.

Basic properties of lines and transversal

Euclid I.27,I.28: I

If a transversal crosses a pair lines and:

1. corresponding angles are equal, or
2. alternate interior angles are equal, or
3. the sum of two interior angles on the same side is equal to two right angles

then the lines are parallel.

This does not require the parallel postulate; the actual argument is I.16; an exterior angle of a triangle is greater than either of the other two angles.

Exercise

Show using basic properties (including all straight angles are equal) that the three conditions of Euclid I.27,28 are equivalent.

Basic properties of lines and transversal, II

Euclid I.27,I.28

If a transversal crosses a pair lines and the lines are parallel then:

1. corresponding angles are equal, and
2. alternate interior angles are equal, and
3. the sum of two interior angles on the same side equal to two right angles

This does require the parallel postulate.
Note that Euclid's version of the 5th postulate just says 3) implies not parallel.

Parallelograms

Euclid I.34

In any parallelogram the opposite sides and angles are equal. Moreover the diagonal splits the parallelogram into two congruent triangles.

Immediate from our results on parallelogram and the congruence theorems.

Area of Parallelograms and triangles

Euclid I.35, I.38

Parallelograms on the same base and in the same parallels have the same area.
Triangles on the same base and in the same parallels have the same area.

Commensurability

We say two segments AB and CD are Commensurable if there is a third segment EF and two natural numbers n and m such that AB can be covered by n disjoint copies of EF and CD can be covered by m disjoint copies of EF.
We write $\frac{AB}{CD} = \frac{m}{n}$.

Commensurable Triangles

Euclid VI.1

Triangles under the same height are to each other as their bases.

The proof requires that the areas of the triangles be commensurable.

Proportionality: base case

Euclid VI.2

If a line is drawn parallel to the base of triangle the corresponding sides of the two resulting triangles are proportional and conversely.

This uses the previous result and so needs that the two parts of each side are commensurable. Euclid does not recognize this explicitly.

Assumptions

to find any triangle with area 12 and fixed base of length 6

1. can call any side the base
2. triangles can have one obtuse angle
3. length is always positive (to allow placement on $x = -4$
4. Euclidean geometry
5. properties of the real numbers
6. area defined as usual
7. "important to consider acute, right, isosceles, equilateral and obtuse triangles."
8. $A - bh/2$

Not mentioned:

1. the parallel postulate
2. commensurability