Truth and Proof

John T. Baldwin

October 15, 2007

Reprise

1. structures and languages;
2. the compositional theory of truth;
3. defined the truth of a sentence in a structure.
4. discussed the properties of equality and equality axioms.

Truth and Validity

We have defined $M \models \phi$. ($M \models \phi$)
But what does it mean to say ϕ is true?!

Give an example of a sentence ϕ and models M_1 and M_2 such that $M_1 \models \phi$ and $M_2 \models \neg\phi$.

Truth and Validity

We have defined $M \models \phi$. ($M \models \phi$)
But what does it mean to say ϕ is true?!

Give an example of a sentence ϕ and models M_1 and M_2 such that $M_1 \models \phi$ and $M_2 \models \neg\phi$.

Validity

The sentence ϕ is valid if it is true in every structure.

For every M, $M \models \phi$.
Write a valid sentence.

Logical Implication

Let Γ be a set of first order sentences and ϕ a sentence.

Logical Implication

Let Γ be a set of first order sentences and ϕ a sentence.

Γ logically implies ϕ
(written $\Gamma \models \phi$) means

For every M,
If $M \models \gamma$ for each $\gamma \in \Gamma$ then
$M \models \phi$

Why proof

Why do we give proofs?

Why proof

Why do we give proofs?

1. to understand why!
2. to organize knowledge and make it easier to remember

Why proof

Why do we give proofs?

1. to understand why!
2. to organize knowledge and make it easier to remember
3. to obtain certainty

A proof system

Logical Axioms

1. Any tautology;
2. The equality axioms;
3. $(\forall x)\phi \to \phi^x_t$ (if t is substitutable for x in ϕ);
4. $(\forall x)(\phi \to \psi) \to [(\forall x)\phi \to (\forall x)\psi]$;
5. $\phi \to (\forall x)\phi(x)$ (if x not free in ϕ.

Inference rule

(Modus Ponens): From ϕ and $\phi \to \psi$, infer ψ.

formal proof

A formal proof from a set of axioms Γ
is a sequence of wff's such that each one

1. is a member of Γ
2. or is a logical axiom
3. or follows from earlier lines by a rule of inference

We write $\Gamma \vdash \phi$ if there is a proof of ϕ from the hypotheses Γ.

The completeness theorem

Gödel I

There is a proof of ψ if and only ψ is valid.

There is a proof of ψ from Φ if and only ψ is true in every structure that satisfies each member of Φ.

The incompleteness theorem

Gödel II

There is no effective way to decide whether a sentence ϕ is valid.

The inerrancy of mathematics

There is a procedure to check a proof is correct.
There is no procedure to check if a sentence is valid.
But the valid sentences are not interesting anyhow.
To actually encode mathematics, add nonlogical axioms:

Some important sets of axioms

1. axioms for arithmetic
2. Axioms for the real field $(\Re, +, \times, <, = 0, 1)$
3. axioms for set theory
4. axioms for geometry

Thus the 'inerrant' part of mathematics becomes the logical deductions. It is essential to make your hypotheses and conclusions explicit.

The Extended completeness theorem

Gödel Ia

$\Gamma \vdash \phi$ if and only $\Gamma \models \phi$

Independence

The set of sentences Γ is independent if for $\gamma \in \Gamma$,

$$\Gamma - \{\gamma\} \not\vdash \gamma.$$

The compactness Theorem

Gödel Ia

If for every finite $\Gamma_0 \subset \Gamma$, $\Gamma \cup \{\phi\}$ has a model then $\Gamma \cup \{\phi\}$ has a model.

Completeness Theorem proof

Should we do the proof in class?