Quadratics

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Here is the assignment. The following two pages were handed out on Monday night and can be reprinted if you need them. They contain an algorithm for finding the factors of a binomial when the leading coefficient is not 1. For credit you must justify each answer.

1. What is an algorithm? How does it differ from a proof?
2. Make up an example and work it to to show that you understand the algorithm given on the next page.
3. What is the difference between finding the roots of a quadratic and factoring the quadratic?
4. What is the relation between the roots of $ax^2 + bx + c$ and those of $x^2 + bx + c/a$?
5. What is the relation between the roots of $ax^2 + bx + c$ and those of $x^2 + (b/a)x + c/a$?
6. Does the algorithm provided on the next page provide a factorization of $ax^2 + bx + c$? Justify your answer. (You may assume that a, b, c have no common factor. You may want to explore what happens when they do have a common factor.) Partial credit will be given so explain as much as you can.
7. This question is not needed for the main problem, but will give you practice in work with parameters. Prove a different quadratic formula: The roots of $ax^2 + bx + c$ are $\frac{2c}{-b \pm \sqrt{(b^2 - 4ac)}}$.

Many high school students can sail through factoring binomials where the coefficient of x^2 is 1, but flounder on

$$6x^2 + 13x + 6.$$

A group of teachers proposed the following algorithm.

Multiply the constant term by the coefficient of the leading term to change to the problem.

$$x^2 + 13x + 36.$$

Factor this problem as $(x + 4)(x + 9)$. Divide the constant terms of the factors by the original coefficient of the leading term to get:$(x + \frac{4}{6})(x + \frac{9}{6}))$ which is $(x + \frac{2}{3})(x + \frac{3}{2}))$. Now we use the 'bottoms up method'; just move the denominator into the numerator to get: $(3x + 2)(2x + 3)$.

Does this method always work? That is if the binomial factors into a product with integer coefficients will this method get the right answer? If so, why? In particular, what is happening in each step of the algorithm?

2. Here is an approach to understanding problem 1: factor

$$6x^2 + 13x + 6.$$

$$6x^2 + 13x + 6 =$$

$$\frac{1}{6}(6^2x^2 + 13 \cdot 6 \cdot x + 36) =$$

$$\frac{1}{6}((6x)^2 + 13 \cdot (6x) + 36) =$$

$$\frac{1}{6}(6x + 4)(6x + 9) =$$

$$\frac{1}{6}2(3x + 2) \cdot 3(2x + 3) =$$

$$(3x + 2)(2x + 3).$$

Note that we certainly have the right roots. Why? But is it clear that the last step will always work; that we will be able to factor the '6'. The first part is a little clearer if we avoid the particular numbers. We have

$$ax^2 + bx + c =$$

$$\frac{1}{a}(a^2x^2 + abx + ac) =$$

$$\frac{1}{a}(a^2x^2 + b \cdot a \cdot x + ac) =$$

$$\frac{1}{a}((ax)^2 + b \cdot (ax) + ac) =$$

$$\frac{1}{a}(ax - s_1)(ax - s_2)$$

if s_1 and s_2 are the solutions of $t^2 + bt + ac = 0$. So the two roots of the original equation are s_1/a and s_2/a. But we still don't understand 'bottoms up'.