What is π?

I quote the following story from Wigner's: Unreasonable Effectiveness of Mathematics.

'There story about two friends, who were classmates in high school, talking about their jobs. One of them became a statistician and was working on population trends. He showed a reprint to his former classmate. The reprint started, as usual, with the Gaussian distribution and the statistician explained to his former classmate the meaning of the symbols for the actual population, for the average population, and so on. His classmate was a bit incredulous and was not quite sure whether the statistician was pulling his leg. "How can you know that?" was his query. "And what is this symbol here?" "Oh," said the statistician, "this is pi." "What is that?" "The ratio of the circumference of the circle to its diameter." "Well, now you are pushing your joke too far," said the classmate, "surely the population has nothing to do with the circumference of the circle."'

Recall that we have defined the measure of an angle as being an equivalence class of congruent angles. Strictly speaking this is defined only for angles formed by intersection of two distinct lines. Moreover, we defined the addition of angles only when the 'sum' was less than a 'straight angle'. We defined *ad hoc* the notion of the sum of the sum of a supplementary sequence of angles. (page 286)

We extend the notion of angle.

Definition 1. *A* straight angle *is* $\overrightarrow{AB} \cup \overrightarrow{AC}$ *where* A, B, C *are collinear.*

We already proved:

Theorem 2. *All straight angles are congruent.*

Definition 3. π *is the congruence class of straight angles.*

Now we can talk about the sum of two angles being a straight angle (or in Euclid's parlance) - the sum is the same as the sum of two right angles.

Theorem 4. *The sum of the interior angles of a triangle is* π*.*

Proof. The angles are a supplementary sequence. (13.56).

Theorem 5. *The sum of the interior angles of a convex* n*-gon is* $(n-2)\pi$*.*

Proof. There are $n-2$ triangle formed by connecting a given vertex to the nonadjacent vertices of the triangle. The sum of the angles of these triangles is the sum of the interior angles of the polygon.

Corollary 6. *The sum of the central angles of a convex polygon (circle) is* 2π*.*

Proof. Any point in the interior defines n-triangles (with bases the sides of the polygons). The sum of all the angles is $n\pi$. From Theorem 5 the sum of the interior angles of the polygon is $(n-2)\pi$, leaving 2π as the sum of the central angles.

Definition 7.

1. *$\mathcal{C}$ is a circle with radius* **b** *and center P if for each $X \in C$, $|XP| = \mathbf{r}$.*
2. int $\mathcal{C}$ *is the collection of points P with $|XP| \leq \mathbf{r}$.*
3. *For any points A, B on $\mathcal{C}$ the chord AB is the $AB \cap$* int $\mathcal{C}$*.*
4. *A* central angle *is an angle $\angle CPB$ with vertex the center P of a circle and C, B points on the circle. We insist that* int $\angle$CPB *is convex.*

We follow Hilbert by saying that such definition determines some collection of points (possibly empty). There are theorems to say that the circle $\mathcal{C}$ is non-empty. Indeed it is clear that every line through P contains two points that are on $\mathcal{C}$. Thus, unlike Euclid, we don't have an axiom declaring the existence of a circle with a given radius and center.

A priori there are two angles determined by every pair of lines with vertex P; we have chosen the smaller one as the central angle. The next remark is evident from Definition 3. This is what is meant by saying there are 2π radians (or 360^o) in a circle.

Lemma 8. *For any finite sequence of points $A_1, A_2, \ldots A_n$ on a circle $\mathcal{C}$, determining angles $\angle A_i P A_{i+1}$, the sum of the angles is 2π.*

The proof of Archimedes
`http://www-groups.dcs.st-and.ac.uk history/HistTopics/Pi_through_the_ages.html`
that the circumference of a unit circle has length 2π is based on the definition that the measure of a straight angle is π. By proportionality, we can extend to prove the measure of an arc is the same as the measure of the central angle.

Definition 9. *1. The segment $\overline{AB}$ is a* diameter *of the circle C with center P if $A, B \in C$ and $P \in AB$.*

2. An angle $\angle ABC$ is said to be circumscribed in a semicircle *if all A, B, C are points on a circle and A, C are the ends of a diameter.*

Theorem 10. *An angle circumscribed in a semicircle is a right angle.*

Proof. Let P be the center, AB be a diameter and $C \in \mathcal{C}$. Consider the points $C, C_1 = \Gamma_{AB}(C), C_2 = \Gamma_P(C)$.

Note $\overline{CA} \approx \overline{C_2B}$ and $\overline{CB} \approx \overline{C_2A}$ and $CA \parallel C_2B$ (under Γ_P). So by SSS, $\triangle ACB \approx \triangle BC_2A$. Similarly, $\triangle ACC_2 \approx BC_2C$. Note that since $C \in \mathcal{C}$, and $|PC| = |PC_2|$, C_2 is on the circle and $|AB| = |CC_2|$. By alternate interior angles, $\angle CAB \approx \angle C_2BA$. So by SAS, $\triangle ABC \approx \triangle ACC_2$. Now the four angles $\angle ACB, \angle CBC_2, \angle BC_2$ and $\angle CAC_2$ are all congruent. But the sum of the four angles is 2π by Theorem 5 and so each has measure $\pi/2$ and is therefore a right angle by Definition 3 and the definition of a right angle. $\Box_{10}$

Theorem 11. *If $\overline{AB}$ is a diameter of C (with center P and radius* **r***).*

1. PA, PB have length **r**.

2. Γ_{AB} fixes X setwise.

Proof. 1) is clear. For 2), let $D \in \mathcal{C}$ and $D' = \Gamma_{AB}(D)$. Now $PD \approx PD'$, so $D' \in \mathcal{C}$.

We want a converse.

Theorem 12. *If CD is a chord of $\mathcal{C}$ and Γ_{CD} fixes $\mathcal{C}$ setwise then CD is a diameter.*

There were definite gaps in my argument for this and I have not had time to work out the solution. My argument was to go through a proof that each circle is convex. Here are two possibilities in that direction.

Lemma 13. *For any circle $\mathcal{C}$,* int *$\mathcal{C}$ is convex.*

Proof.

Proof 1. (Weinzweig's suggestion) For any finite sequence of points on the circle the tangent to the circle at that point (i.e. perpendicular to the radius) defines a half-plane which includes the circle. The intersection of these half-planes is the circle and since the intersection of convex sets is convex, the circle is convex.

Proof 2. It seems we could get a more 'basic' proof if we could prove: given $\triangle ABC$. Choose D on $\overline{BC}$. Then $|\overline{AD}| < |\overline{AB}|$. This ought to be a variant on 13.61 but I haven't seen how to do it yet.