Axiomatizing changing conceptions of the geometric continuum

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3 kinds of problems

- [philosophical] Characterize a ‘nice axiomatization’ of a given mathematical topic: modest descriptive axiomatization
- [mathematical] Prove that certain set of axioms provides modest descriptive axiomatization of specific ‘data sets’ in geometry
- [historical/philosophical] Analyze the meanings of certain collections of geometrical facts for various authors. Argue that certain axiomatizations are immodest.

Assignment of historical personages to particular positions should be taken with large grains of salt.
Corrections appreciated.

Changing conceptions of the geometric continuum

By the *geometric continuum* we mean the line situated in the context of the plane. Consider the following two propositions

(*) Euclid VI.1: Triangles and parallelograms which are under the same height are to one another as their bases.

Hilbert gives the area of a triangle by the following formula.
(**) Consider a triangle ABC having a right angle at A. The measure of the area of this triangle is expressed by the formula

$$F(ABC) = \frac{1}{2} AB \cdot AC.$$

What is the difference?

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1. Triangles and parallelgrams have the same statement in Euclid.
2. Hilbert specifies a proportionality constant.
3. Hilbert assigns a *number* (albeit with units) for area and length.

Hilbert's task

When formulating a new axiom set in the late 19th century Hilbert faced several challenges:

1. Identify and fill 'gaps' or remove 'extraneous hypotheses' in Euclid's reasoning.
2. Reformulate propositions such as VI.1 to reflect the 19th century understanding of real numbers as measuring both length and area.
3. Ground the geometry of Descartes.

Hilbert's view of his task

Hilbert begins the Grundlagen with:

> *The following investigation is a new attempt to choose for geometry a* simple *and* complete *set of* independent axioms *and to deduce from them the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance of the groups of axioms and the scope of the conclusions to be derived from the individual axioms.*

Hallet's gloss

Hallett presaged much of the intent of this article:

> *Thus completeness appears to mean [for Hilbert] 'deductive completeness with respect to the geometrical facts'. . . . In the case of Euclidean geometry there are various ways in which 'the facts before us' can be presented. If interpreted as 'the facts presented in school geometry' (or the initial stages of Euclid's geometry), then arguably the system of the original Festschrift [i.e. 1899 French version] is adequate. If, however, the facts are those given by geometrical intuition, then matters are less clear.*

Descriptive axiomatization

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If we use logical notions (such as quantifiers, connectives, etc) for the purpose of capturing a class of structures studied in a particular mathematical theory, we are pursuing the descriptive use of logic.

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Detlefsen

1. Axiomatization generally takes place against the background of a data set: the commonly accepted sentences pertaining to a given subject area.
2. The basic purpose of axiomatization is to deductively organize a data set.
3. To fully accomplish 2, the axioms of a proposed axiomatization must be descriptively complete that is, all elements of the data set must be deducible from the axioms.

Modest Axiomatization: a rough notion

Definition

A *modest* axiomatization of a data set is one we mean one implies all the data and not too much more. It doesn't show too much.

Not too much more

Most mathematics proves new theorems about old topics !

A axiom system which implies results that are not in the 'deductive closure' of the axiom set is immodest.

Example: Geometry

3 basic data sets

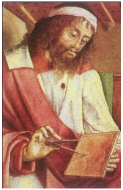
1. Polygonal geometry (most of books I-VI)
2. Euclid on circle
3. Descartes: higher degree polynomials

2 further sets

1. area and circumference of circle; arc length
2. continuity properties

The axioms

Euclid-Hilbert formalization 1900:

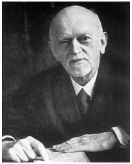
Euclid  Hilbert 

The Euclid-Hilbert (the Hilbert of the Grundlagen) framework has the notions of axioms, definitions, proofs and, with Hilbert, models.
But the arguments and statements take place in natural language.

Euclid uses diagrams essentially; Hilbert uses them only heuristically.

For Euclid-Hilbert logic is a means of proof.

Hilbert-Gödel-Tarski-Vaught formalization 1918-1956:

Hilbert  Gödel  Tarski  Vaught 

In the Hilbert (the founder of proof theory)-Gödel-Tarski framework, logic is a mathematical subject.

There are explicit rules for defining a formal language and proof. Semantics is defined set-theoretically.

First order logic is complete. The theory of the real numbers is complete and easily axiomatized. The first order Peano axioms are not complete.

We work initially in the Euclid-Hilbert mode
but use the insights of model theory to study circles.

Vocabulary

The fundamental notions are:

1. two-sorted universe: points (P) and lines (L).
2. Binary relation $I(A, \ell)$:
   Read: a point is incident on a line;
3. Ternary relation $B(A, B, C)$:
   Read: B is between A and C (and A, B, C are collinear).
4. quaternary relation, $C(A, B, C, D)$:
   Read: two segments are congruent, in symbols $\overline{AB} \approx \overline{CD}$.
5. 6-ary relation $C'(A, B, C, A', B', C')$: Read: the two angles $\angle ABC$ and $\angle A'B'C'$ are congruent, in symbols $\angle ABC \approx \angle A'B'C'$.

τ is the vocabulary containing these symbols.

Note that I freely used defined terms: collinear, angle and segment, in giving the reading.

First order *fully* geometric Postulates

HP5 Hilbert:

1. Incidence postulates
2. the betweenness postulates (after Hilbert) (yield dense linear ordering of any line).
3. One congruence postulate
4. parallel postulate

EG Implicit in Euclid: circle-circle intersection

Theorem

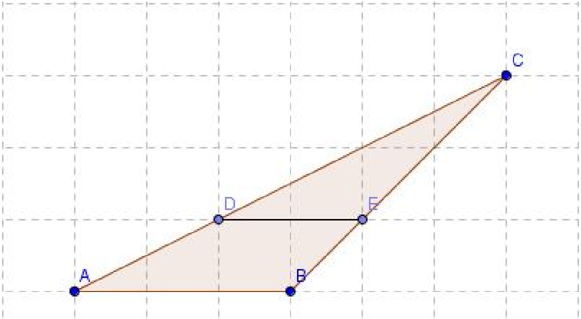
EG proves the geometric results of the first six books

Caveat: In chapter V on proportion Euclid implicitly uses the Axiom of Archimedes. But Hilbert shows it is not needed for the geometric application.

Side-splitter Theorem

Theorem: Euclid VI.2

If a line is drawn parallel to the base of triangle the corresponding sides of the two resulting triangles are proportional and conversely.



$$CD : CA :: CE : CB$$

What does proportional mean?

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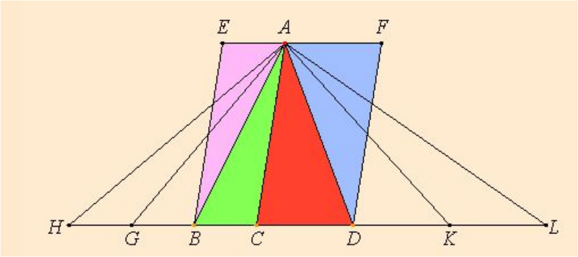
Euclid's proof

1. Uses Area
2. Use Eudoxus to deal with incommensurable side lengths.

How is proportionality used?



If, for example, BC, GB and HG are congruent segments then the area of ACH is triple that of ABC. But without assuming BC and BD are commensurable, Euclid calls on Definition V.5 of the proportionality chapter to assert that $ABD : ABC :: BD : BD$.

Definition V.5 basically asserts the Archimedean axiom.

Section 4: From Geometry to Numbers

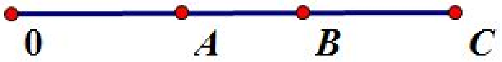
A short history of multiplication

- [Euclid] The product of two segments is an area.
- [Descartes] Based on the theory of proportion the product of two segments is a segment.
- [Hilbert] Using the product of two segments as a segment, define proportion.

Defining addition I

Adding line segments

The sum of the line segments OA and OB is the segment OC obtained by extending OB to a straight line and then choose C on OB extended (on the other side of B from A) so that $OB \cong AC$.



Addition is clearly associative and has identity 0. But on line segments it is not a group.

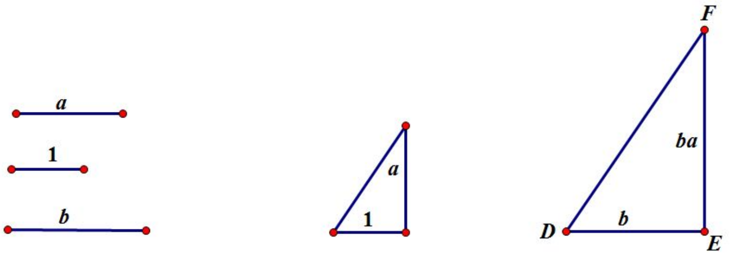
Defining Multiplication

Consider two segment classes *a* and *b*. To define their product, define a right triangle[1] with legs of length *a* and *b*. Denote the angle between the hypoteneuse and the side of length *a* by α.

Now construct another right triangle with base of length *b* with the angle between the hypoteneuse and the side of length *b* congruent to α. The length of the vertical leg of the triangle is *ab*.

[1] The right triangle is just for simplicity; we really just need to make the two triangles similar.

Defining segment Multiplication diagram



Note that we must appeal to the parallel postulate to guarantee the existence of the point F.

Parallel Postulate

Is multiplication just repeated addition?

On the one hand, we can think of laying 3 segments of length a end to end.

On the other, we can perform the segment multiplication of a segment of length 3 (i.e. 3 segments of length 1 laid end to end) by the segment of length a.

Only the second has a natural multiplicative inverse *on segments*.

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The theory of $(\omega, +, \times)$ is essentially undecidable.

The theory of $(\Re^+, +, \times)$ is decidable and proved consistent in systems of low proof theoretic strength (EFA).

Obtaining the field properties

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Addition and multiplication are associative and commutative.
There are additive and multiplicative units and inverses.
Multiplication distributes over addition.

Consequences

For any model M of the listed postulates: similar triangles have proportional sides.
There is no assumption that the field is Archimedean.

There is no appeal to approximation or limits.

It is easy to check that the multiplication is exactly the usual multiplication on the reals because they agree on the rationals.

The multiplication gives a good theory of area for polygons.

Archimedes and Circles

A 4th century AD view of arc length

Eutocius, in Archimedis Opera Omnia cum commentariis Eutociis, vol. 3, p. 266.

Even if it seemed not yet possible to produce a straight line equal to the circumference of the circle, nevertheless, the fact that there exists some straight line by nature equal to it is deemed by no one to be a matter of investigation.

Davide Crippa (Sphere, UMR 7219, Universit Paris Diderot) Reflexive knowledge in mathematics: the case of impossibility

Adding π

The field over the maximal quadratic field is a model of these postulates.

But Archimedes could compute the circumference and area of a circle. If a radius of a circle is in the model then the circumference is not.

Describing π

Add to the vocabulary τ a new constant symbol π. Let i_n (c_n) be length of a side of a regular $3 \cdot 2^n$-gon inscribed (circumscribed) in a circle of radius 1.
Add for each n,

$$i_n < 2\pi < c_n$$

to give a collection of sentences $\Sigma(\pi)$.

The theory with π

Definition

EG_π denotes the following set of axioms in the vocabulary τ along with the constant symbols $0, 1, \pi$.

1. the postulates *EG* of a Euclidean plane.
2. A family of sentences declaring every odd-degree polynomial has a root.
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The axioms are consistent by the compactness theorem (or looking inside $\Re$).

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'Any fool can omit a type, it takes a model theorist to omit one'

Gerald Sacks

Arc Length

arcs and segments

Let $\mathcal{S}$ be the set (of equivalence classes of) straight line segments. Let $\mathcal{C}$ be the set (of equivalence classes) of arcs on a circle of a given radius

Ordering arcs and segments

For $s \in \mathcal{S}$ and $c \in \mathcal{C}$

1. The segment $s < c$ if and only if there is a chord XY of a circular segment $AB \in c$ such that $XY \in s$.
2. The segment $s > c$ if and only if there is an approximant $X_1 \ldots X_n$ to c with $[X_1 \ldots X_n] > c$.

Circumference function

Write formulas describing that X_1, X_n are the vertices on the circle of an inscribed $n - gon$ (or the midpoints of a circumscribed n-gon).

Theorem

In EG_π, $C(r) = 2\pi r$ is between the perimeter of any inscribed n-gon and circumscribed n-gon.

A similar argument will give a theory of π for the area function: $A(r) = \pi r^2$.

From Descartes to Tarski

Descartes

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Tarski

- a Gödel complete axiomatization $\mathcal{E}^2$ of first order geometry

  Arguably that of Descartes.
- But points are now elements of lines
- Binterpretability of RCF and geometry
- quantifier eliminability and o-minimality of RCF

Properties of $\mathcal{E}^2_\pi$

Theorem

$\mathcal{E}^2_\pi$ defined as $\mathcal{E}^2 + \Sigma(\pi)$ satisfies:

1. complete decidable First order, o-minimal
2. has non-Archimedean models
3. provably consistent in EFA (and thus primitive recursive arithmetic)

A first order theory for a vocabulary including a binary relation $<$ is *o-minimal* if every 1-ary formula is equivalent to a Boolean combination of equalities and inequalities.

Area and angle measure

Further, one can similarly give first order specifications for an angle measure so that.

Angle measure

1. Angle measure behaves on Archimedean models.
2. Every countable model is contained in a countable model where every angle has a measure.

Query

Perhaps the continuum is just a line in a recursively saturated model of $\mathcal{E}$.

Summary

Against Axiom group V

Hilbert's Axiom group are the Archimedean axiom and a disguised version of Dedekind (better Veronese) completeness.
They yield an immodest axiomatization of geometry because

1. They are never used in the Grundlagen to prove geometric theorems.
2. The requirement that there be a straight-line segment measuring any circular arc is clearly contrary to the intent of geometers through Descartes.
3. Dedekind's postulate is not part of the data set but rather an external limitative principle.
4. Proofs from Dedekind's postulate obscure the true geometric reason for certain theorems.
5. The use of second order logic undermines a key proof method – informal proof, which is licensed by the first order completeness theorem.

Modest Axiomatization: Examples and non-examples

Examples

1. Hilbert's first order axioms are a Modest Axiomatization of polygonal geometry.
2. Tarski's $\mathcal{E}^2$ (RCF) is a Modest Axiomatization of Cartesian geometry (the geometry over real closed fields).
3. $\mathcal{E}^2_\pi$ is a Modest Axiomatization of Cartesian geometry with formulas for area and circumference of circles.

non-Examples

1. $\mathcal{E}^2_\pi$ is a *Immodest* Axiomatization of Cartesian geometry.
2. Hilbert's full axiom set including Archimedes and 'completeness' is an *Immodest Axiomatization* of geometry.
3. Birkhoff geometry, by reference to the reals, is an *Immodest Axiomatization* of geometry.

Why Group V: A wider notion of Geometry

Hilbert expressly says the reason for group V is to enable the embedding of the line in the reals.

Although the bulk of the Grundlagen is dedicated to the foundations of 'Euclidean' and 'non-Euclidean' geometry, Hilbert was in fact concerned with the foundations of all geometry, (metric, differential, projective etc. etc.) as well as analysis.

Slides and paper

paper:
http://homepages.math.uic.edu/~jbaldwin/pub/geomalgoct2014.pdf