Modern Model Theory
The impact on mathematics and philosophy

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Goals

Modern model theory has introduced many fundamental notions with philosophical significance that have masqueraded as technical mathematical notions.

I will describe some of the themes of this work.

Outline

What is model theory?

Definition for mathematicians

A model theorist is a self-conscious mathematician who uses formal languages and semantics to prove mathematical theorems.

Prelude: From Logic to Model Theory

The decades of the 50's and the 60's saw a transition from model theory as one aspect of the study of (first order) logic to the introduction of an independent subject of model theory.

This talk will expound the philosophical and mathematical impacts of that shift.

Model-Theoretic Logics

A vocabulary τ is a list of function, constant and relation symbols.

A τ-structure $\mathcal{A}$ is a set with an interpretation of each symbol in τ.

A logic $\mathcal{L}$ is a pair $(\mathcal{L}, \models_{\mathcal{L}})$ such that

$\mathcal{L}(\tau)$ is a collection of τ-sentences and for each $\phi \in \mathcal{L}(\tau)$,
$\mathcal{A} \models_{\mathcal{L}} \phi$
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(A natural variant allows for variables and formulas with free variables.)

Properties of first order logic (1915-1955):

1. Completeness and Compactness
2. Lowenheim-Skolem
3. syntactic characterization of preservation theorems
4. Interpolation

Theories

Axiomatic theories arise from two distinct motivations.

1. To understand a single significant structure such as $(N, +, \cdot)$ or $(R, +, \cdot)$.
2. To find the common characteristics of a number of structures; theories of the second sort include groups, rings, fields etc.

The Genesis of Modern Model Theory (1955-1970):

A. Robinson.
Complete Theories.
North Holland, Amsterdam, 1956.

A. Tarski and R.L. Vaught.
Arithmetical extensions of relational systems.
Compositio Mathematica, 13:81–102, 1956.

A. Ehrenfeucht and A. Mostowski.
Models of axiomatic theories admitting automophisms.
Fund. Math., 43:50–68, 1956.

B. Jónsson.
Homogeneous universal relational systems.
Mathematica Scandinavica, 8:137–142, 1960.

Categoricity in Power:

J. Łos.
On the categoricity in power of elementary deductive systems and related problems.
Colloquium Mathematicum, 3:58–62, 1954.

M. Morley.
Categoricity in power.
Transactions of the American Mathematical Society, 114:514–538, 1965.

New Paradigm

S. Shelah.
Stability, the f.c.p., and superstability; model theoretic properties of formulas in first-order theories.
Annals of Mathematical Logic, 3:271–362, 1970.

J.T. Baldwin and A.H. Lachlan.
On strongly minimal sets.
Journal of Symbolic Logic, 36:79–96, 1971.

B.I. Zil'ber.
Uncountably Categorical Theories.
Translations of the American Mathematical Society.
American Mathematical Society, 1991.
summary of earlier work.

Recording the shift:

S. Shelah.
Classification Theory and the Number of Nonisomorphic Models.
North-Holland, 1978.

foundationS of mathematics

Thesis:

Studying the model of different(complete first order) theories provides a framework for the understanding of the foundations of specific areas of mathematics.

This study cannot be carried out by interpreting the theory into an über theory such as ZFC; too much information is lost.

Categorical foundations may also be local in this sense.

Algebraic examples: complete theories

Algebraic Geometry

Algebraic geometry is the study of definable subsets of algebraically closed fields
Not quite: definable by positive formulas
REMEDY: Zariski Geometries

Chevalley-Tarski Theorem

Chevalley: The projection of a constructible set is constructible.
Tarski: Acf admits elimination of quantifiers.

More precisely, this describes 'Weil' style algebraic geometry. Macintyre and Hrushovski (App. to computing the Galois grp....) have contrasting views on the need for 'categorical foundations'.

Algebraic consequences for complete theories

1. Artin-Schreier theorem (A. Robinson)
2. Decidability and qe of the real field (Tarski)
3. Decidability and qe of the complex field (Tarski)
4. Decidability and model completeness of valued fields (Ax-Kochen-Ershov)
5. quantifier elimination for p-adic fields (Macintyre)
6. o-minimality of the real exponential field (Wilkie)

Why complete theories?

Studying elementary extensions is an important tool for studying definability in a particular structure.

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Studying elementary extensions is an important tool for studying definability in a particular structure.

Recursive axiomatizations of particular areas (e.g. algebraic geometry) restores the decidability sought by Hilbert (which Gödel shows is impossible for global foundations.)

Classification Theory

A pun on classify

1. A class of models of a theory admit a structure theory (can be classified) if there is a function assigning set theoretic invariants to each model which determine the model up to isomorphism.
2. The theories of a logic can be classified if there is a schema determining which theories admit a structure theory.

Dicta: Shelah

A powerful tool for classification is to establish dichotomies: a property Φ such that Φ implies non-structure and $\neg\Phi$ contributes to a structure result.

dichotomies must have absolute/syntactic/internal form and an external semantic form (Dzjamonja)

Classification Theory = Model Theory

In a review of **Classification Theory** Poizat wrote:

'Possibly, the editor found it too delicate to use a more ambitious term: this could give an impression of arrogance, or of a will to exclude people working in more traditional matters (...) . Those of us who are free from that kind of scruple will restore to this book the only name it deserves: Model theory.'

Properties of classes of theories (1970-present)

The stability hierarchy is defined by syntactical conditions on theories or (equivalently) by counting types over models of different cardinalities.

The Stability Hierarchy

Every complete first order theory falls into one of the following 4 classes.

1. ω-stable
2. superstable but not ω-stable
3. stable but not superstable
4. unstable

Two other classes, simple, and NIP (dependent) have been seen recently to have important structural consequences.

The stability hierarchy: examples

ω-stable

Algebraically closed fields (fixed characteristic), differentially closed fields, complex compact manifolds

strictly superstable

$(\mathcal{Z}, +)$, $(2^\omega, +) = (Z_2^\omega, H_i)_{i<\omega}$,

strictly stable

$(\mathcal{Z}, +)^\omega$, separably closed fields, random graphs with edge prob $n^{-\alpha}$ (α irrational

unstable

Arithmetic, Real closed fields, complex exponentiation, random graph

More than Taxonomy

This classification and the tools originally developed for the spectrum problem have had profound developments and consequences across mathematics:

1. Hrushovski's work on Mordell-Lang
2. Study of differential fields and difference fields
3. compact complex varieties are ω-stable.
4. o-minimality– solution of a problem of Hardy

Dimension Theory

The Van der Waerden theory of dimension assigns a dimension to certain structures,
those which admit a combinatorial geometry.

Examples:

- Vector Spaces,
- transcendence degree of algebraically closed fields,
- rank of certain modules.

(Families) of Combinatorial Geometries

A stable theory admits a Notion of Independence that

1. is weaker in some ways than a combinatorial geometry;
2. but uniformly studies dimensions over various subsets;
3. provides good dimension on sets definable by weight one types;
4. provides a means for connecting the dimensions on various types.

Consequences: Main Gap

Using these ideas of dimension Shelah proved:

Main Gap

For every first order theory T, either

1. Every model of T is decomposed into a tree of countable models with uniform bound on the depth of the tree, or
2. The theory T has the maximal number of models in all uncountable cardinalities.

What does this mean?

For 'classifiable theories' every model is determined by its countable submodels and a tree that serves as a skeleton.

Question

Is Shelah's claim:

the existence of the maximal number of models signals non-structure

correct?

What are the criteria for answering such a question?

Understanding Fundamental Structures

Real Exponentiation

An ordered structure is o-minimal if every definable subset is a Boolean combination of intervals.

Wilkie proved $\mathrm{Th}(\Re, +, \cdot, \exp)$ is o-minimal.

The study of o-minimal theories, another syntactically defined class, has substantial mathematical significance.

Zilber's Thesis

Fundamental structures are canonical

Fundamental mathematical structures can be characterized in an appropriate logic.

Conversely, characterizable structures are 'fundamental'.

Canonical Structures are Fundamental

Zilber Conjecture

Every strongly minimal first order theory is

1. disintegrated
2. group-like
3. field-like

Hrushovski refuted the precise conjecture; establishing a refined version of the underlying thesis is active work connecting model theory with complex analysis, algebraic geometry, and non-commutative geometry.

Logical Consequences from Mathematical Tools

Theorem: Zilber

There is no complete finitely axiomatizable first order theory T that is categorical in every uncountable power.

Ingredients

1. Deduce the existence of groups definable in the models of categorical first order theories.
2. Characterize those groups
3. Use the properties of those groups to analyze the fine structure of the model to show T has the finite model property.

Understanding Fundamental Structures

Complex Exponentiation

Find an axiomatization for $\mathrm{Th}(\mathcal{C}, +, \cdot, \exp)$.

The conjectured axiomatization is in $L_{\omega_1,\omega}(Q)$. The framework is a special case of our next topic.

Exploring Cantor's Paradise

Shelah's Attitude

"This work certainly reflects the authors preference to find something in the white part of the map, the 'terra incognita', rather than to understand perfectly what we have reasonably understood to begin with ..."

from preface to Classification Theory for Abstract Elementary Classes

Exploring Cantor's Paradise

Much of core mathematics is coarse:
It studies either

1. properties of particular structures of size at most the continuum
2. or makes assertions that are totally cardinal independent. E.g., if every element of a group has order two then the group is abelian.

Model theory, especially of infinitary logic recognizes distinct algebraic behavior at different cardinalities.

Other models matter

A reason to study complete theories is that non-standard, in particular, saturated models give information that is not available looking at one structure.

The stability hierarchy illustrates the interaction of the behavior of models in very different cardinalities.

Big models matter

Shelah's dicta:

There may be noise in small cardinalities that hides eventual uniformity.

1. first order categoricity begins at $|T|^+$.
2. superstability begins at $(2^{|T|})^+$.
3. Infinitary categoricity may begin at $\beth_{(2^{Ls(\mathbf{K})})^+}$.

Small models matter

Sufficiently nice behavior on small cardinalities propagates to all cardinals.

Categoricity can fail (for $L_{\omega_1,\omega}$) at any point below $\aleph_\omega$.

Theorem: Shelah ZFC

Excellence (n-dimensional amalgamation in $\aleph_0$ for all n) implies amalgamation in all cardinalities (and in particular the existence of arbitrarily large models).

Theorem: Shelah $2^{\aleph_n} < 2^{\aleph_{n+1}}$

For $L_{\omega_1,\omega}$, Categoricity below $\aleph_\omega$ implies excellence.

The role of Set Theory

In 1970 model theory seemed to be intimately involved with problems in axiomatic set theory.

The stability hierarchy established the absoluteness of most important concepts and theorems in first order model theory.

But for infinitary logic, weak extensions of ZFC seem to be needed.

The role of Syntax

What is syntax?

- vocabulary and structures
- definable relations
- proof

David Pierce on Mathematicians vrs Model Theorists on vocabulary

pierce

Similar examples appear in Emily Grosholz on Productive Ambiguity.

The Choice of Vocabulary

Model Theoretic Convenience

1. Morley's 'quantifier elimination' by force.
2. Shelah's transformation from sentences in $L_{\omega_1,\omega}$ to atomic models of first order theories.
3. C^{eq}.

Mathematical Significance

Model theoretic algebra seeks a sufficiently powerful vocabulary but insists that the basic relations must be 'meaningful'.

Frege's concept analysis.

Positive Applications of Syntax

Much of model theory concerns first order definable sets.

1. The stability hierarchy is syntactic.
2. O-minimality allows the capture of topological properties of the real numbers by first order logic.
3. Work of Denef, Van den Dries et al essentially replaces 'blowing up of singularities' by induction on quantifier rank.
4. Continued work on motivic integration
5. Bays, Gavrilovich, Zilber analysis of covers of abelian varieties.

Detail on covers

Model Theory without logic

Model theory is about classes of structures of a fixed similarity type.

Must these classes be defined in a logic?

1. Zariski Structures
2. Zilber's theory of quasiminimal excellence.
3. Shelah's theory of abstract elementary classes (aka accessible categories).

When do mathematical conditions on classes generate a logic?

These approaches recall Tarski's contrast of logical and mathematical characterizations in the early 1950's. Kennedy calls this move a species of 'formalism freeness'.

The role of Syntax

What is a syntactical property?

A first order theory is small if for each n, there are only countably many n-types over the emptyset.

Vaught (1958) writes:

"A little thought convinces one that a notion of 'purely syntactical condition' wide enough to include 'small theory' would be so broad as to be pointless."

'small' is an anachronism. Vaught defined the notion but didn't name it.

The role of model theory?

The Conference manifesto reads

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The technical developments of the 60's and 70's created specialized subareas of mathematical logic. We have argued that Modern Model Theory:

1. has profound mathematical consequences
2. provides both questions and tools for philosophical investigations.

Next slide more topics.

Other Topics

1. Why is second order categoricity uninformative?
2. When is categoricity in power the right notion of canonicity? Reals vrs complexes
3. transfer of ideas used to study uncountable structures and vocabularies to the countable.
4. Conjectures in AEC: the diversity of infinity.
5. algebraic vrs combinatorial properties
6. Monster Models and Grothendieck universes.
7. completions (existential, Morley, eq)

$\mathcal{Z}$-Covers of Algebraic Groups

Definition A $\mathcal{Z}$-cover of a commutative algebraic group $\mathbb{A}(\mathcal{C})$ is a short exact sequence

$$0 \to Z^N \to V \stackrel{\exp}{\to} \mathbb{A}(\mathcal{C}) \to 1. \tag{1}$$

where V is a $\mathbb{Q}$ vector space and $\mathbb{A}$ is an algebraic group, defined over k_0 with the full structure imposed by $(\mathcal{C}, +, \cdot)$ and so interdefinable with the field.

$T_A + \Lambda = \mathcal{Z}^N$ denotes an $L_{\omega_1,\omega}$ axiomatization of such a structure.

Categoricity Problem

Is $T_A + \Lambda = \mathcal{Z}^N$ categorical in uncountable powers?

paraphrasing Zilber:

> *Categoricity would mean the short exact sequence is a reasonable 'algebraic' substitute for the classical complex universal cover.*

Return

Pierce on Spivak

Spivak's Calculus book asserts the following are equivalent condition on $\mathbb{N}$.

1 induction ($1 \in X$ and $k \in X$ implies $k+1 \in X$) implies $X = N$.
2 well-ordered Every non-empty subset has a least element.
3 strong induction ($1 \in X$ and for every $m < k$, $m \in X$ implies $k \in X$) implies $X = N$.
4 recursive definition For every unary f and element a there is a unique function h with $h(1) = a$ and $h(k+1) = f(h(k))$.

But this doesn't make sense:
1 and 4 are properties of unary algebras
2 is a property of ordered sets (and doesn't imply others)
3 is a property of ordered algebras.

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