ω-stable fields with infinite definable subsets *

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These notes shift back and forth between the consideration of Poizat's specific example and the general framework for building complicated ω-stable structures of finite rank.

The aim of the special case is the construction of a rank 2 field with a definable infinite subset.

The aim of the general study is to understand the family of constructions and in particular to generalize to the construction of a rank 2 field with a definable infinite subgroup of the multiplicative group – a bad field..

1 The Main Example 1

Fix a language L for fields along with an additional unary predicate B (for noir).

(There is one basic decision taken differently in the two sources for these lectures: Poizat works with finitely generated algebraicaly closed fields, Baldwin-Holland work with finitely generated fields. The link between these two conceptions may be confused at times.)

For any finite subset A of a field with unary predicate, the predimension $\delta(A)$ is defined to be the transcendence degree of the set A minus twice the number of black elements of A.

We call k is a *bicolored field* if k is algebraically closed and $\delta(A) \geq 0$ for all finite $A \subseteq k$. Note that the class of bicolored fields is definable by first order sentences (not universal because of algebraic closure but π_2 so closed under unions of chains).

2 Generalities 1

Fix a first order language L which may have function symbols. We begin with a theory T_0 whose class $\overline{\mathbf{K}}_0$ of models is closed under substructure. We let $\mathbf{K}_0$ denote the class of *finitely generated* (as structures) elements of $\overline{\mathbf{K}}_0$.

*These lectures are based on notes, Constructing ω-stable Structures by myself and Kitty Holland and on the paper, Le Carre de l'egalite, by Bruno Poizat

2.1 Notation. We will write $B \subseteq_\omega N$ to indicate B is a finitely generated substructure of N. If $A, B \subseteq N$ we write AB for $A \cup B$. For X a subset of N, $\langle X \rangle_N$ denotes the substructure of N generated by X. We will generally omit the subscript. For $A \subseteq B \in \mathbf{K}_0$, we write $\mathrm{Diag}(A)$ for the conjunction of the *quantifier-free* diagram of A. Note that when $A = \overline{a}$ is a finite sequence, $\mathrm{Diag}(\overline{a})$ is in general a type not a formula.

2.2 Weight Functions (Predimensions?). Throughout, δ is a function from (quantifier free diagrams of) finite subsets of elements of $\overline{\mathbf{K}}_0$ into the natural numbers.

Note that $\langle a \rangle = \langle b \rangle$ need not imply that $\delta(a) = \delta(b)$. For example in bicolored fields, if a is white and $2a$ is black, $\langle a \rangle = \langle b \rangle$ while $\delta(a) = 2$ and $\delta(2a) = 1$.

2.3 Definition. For $N \in \overline{\mathbf{K}}_0$ and X and Y finite subsets of N, we write

$$\delta(X/Y) = \delta(XY) - \delta(Y).$$

For U and V subsets of N with $U \subseteq V$, we say that U is a *strong* subset of V, and write $U \leq V$, if for every finite subset X of V, $\delta(X/X \cap U) \geq 0$. If $A \leq A\overline{a}$ we write $\delta(\overline{a}/A$ for $\min\{\delta(\overline{a}/B); \overline{a} \cap A \subseteq B \subseteq_\omega A$. ($B \subseteq_\omega A$ means B is a finite subset of A.

2.4 Definition. For $N \in \overline{\mathbf{K}}_0$ and X a finite subset of N, we define the *dimension* of X in N by

$$d_N(X) = min\{\delta(Y) : \ X \subseteq Y \subseteq_\omega N\},$$

and for Y also a finite subset of N, we define the *relative dimension* of X over Y in N by

$$d_N(X/Y) = d_N(XY) - d_N(Y).$$

Note that while $\delta(X/Y)$ may be negative, $d_N(X/Y)$ is always nonnegative. We may thus extend the notion of relative dimension to allow an infinite base by setting, for arbitrary Y,

$$d_N(X/Y) = inf\{d_N(X/Y_0) : \ Y_0 \subseteq_\omega Y\}.$$

2.5 Definition. $\mathrm{icl_B(X)}$ is the unique smallest set C with $X \subseteq C \leq B$. If $A \leq B$ and $\overline{b} \subseteq B$, then $\mathrm{icl_B(\overline{b}/A) = icl_B(A\overline{b}) - A}$.

Since δ takes values in the natural numbers, for any finite sequence $\overline{b}$, $\mathrm{icl_B(\overline{b}/A)}$ is finite.

3 The Main Example 2

The result of the previous section apply directly to this example. For any finitely generated bicolored (n.b. algebraically closed) field k, we take $d(k)$ as $d(A)$ for any finite A containing a generating set for k which $\delta(X)$ for any finite X containing a generating set for k and all the black points.

3.1 Lemma. *The class of bicolored fields has the amalgamation property for strong substructures.*

Proof. Let K and L be strong extensions of k (so k is algebraically closed in each of them. Choose K and L to be linearly disjoint over k inside some monster acf. Let M be the algebraic closure of KL and make each element of $M - KL$ white. Now it is easy to check that M is a strong extension of both K and L as required.

This gives us a generic structure; the difficulty is to guarantee that it is ω-saturated.

4 Generalities 2

We restrict to the case where $\mathbf{K}_0$ has the amalgamation property with respect to strong extensions. As usual this produces a unique countable model which is homogenous with respect to strong extensions of finitely generated substructures and is a union of finitely generated strong substructures. We call this the generic model M. Henceforth, $T = \mathrm{Th}(\mathrm{M})$, $\mathcal{M}$ is a large saturated model of T and N, N' etc range over other models of T.

4.1 Constraints on δ. We require δ to satisfy the following conditions:

1. If $\mathrm{Diag}(\overline{a}) = \mathrm{Diag}(\overline{b})$ then $\delta(\overline{a}) = \delta(\overline{b})$.

2. For any integer k and $\overline{a}, \overline{b} \subseteq C \in \mathbf{K}_0$ with $\delta(\overline{a}/\overline{b}) = k$, there is a δ-*formula for* $\overline{a}$ *over* $\overline{b}$, $\varphi(\overline{x};\overline{y}) \in \mathrm{Diag}(\overline{a};\overline{b})$, such that the following hold:

   (a) for any $\overline{a}', \overline{b}', \overline{c}' \subseteq B \in \mathbf{K}_0$, with $\overline{c}' \cap \overline{a}' = \overline{c}' \cap \overline{b}'$, if $B \models \varphi(\overline{a}';\overline{b}')$, then $\delta(\overline{a}'/\overline{b}'\overline{c}') \leq k$. In particular, $\delta(\overline{a}/\overline{b}) \geq \delta(\overline{a}/\overline{b}\overline{c})$ whenever $\overline{c} \cap \overline{a} = \overline{c} \cap \overline{b}$.

   (b) If $\overline{b} \subseteq A \leq B \in \mathbf{K}_0$, $\overline{a} \cap A = \overline{a}' \cap A = \emptyset$, $B \models \varphi(\overline{a}';\overline{b})$, $A\overline{a} \leq \langle A\overline{a}\rangle$, $A\overline{a}' \leq \langle A\overline{a}'\rangle$, and $\delta(\overline{a}/A) = \delta(\overline{a}'/A) = k$, then $\langle A\overline{a}\rangle \simeq \langle A\overline{a}'\rangle$ by a map which is the identity on A and maps $\overline{a}$ to $\overline{a}'$.

3. If, for some n, $T_0 \vdash \forall\overline{y}\exists^{\leq n}\overline{x}\ \theta(\overline{x};\overline{y})$, then $\delta(\overline{a}/\overline{b}) \leq 0$ whenever $\theta(\overline{a};\overline{b})$ holds. In particular, for any $\overline{a} \subseteq \langle\overline{b}\rangle$, $\delta(\overline{a}/\overline{b}) \leq 0$.

Constraint 3) implies that if $\overline{a} \leq A \in \mathbf{K}_0$ then $\langle \overline{a} \rangle \leq A$.

Letting $\overline{b}$ be empty and A the structure generated by the empty set, we see from 2b) that if $\phi(\overline{x})$ is a δ-formula for $\overline{a}$, $\models \phi(\overline{a}')$, $A\overline{a} \leq \langle A\overline{a} \rangle$, $A\overline{a}' \leq \langle A\overline{a}' \rangle$, and $\delta(\overline{a}') = \delta(\overline{a})$ then $\langle \overline{Aa} \rangle \simeq \langle \overline{Aa}' \rangle$. Thus, there are only countably many isomorphism types in $\mathbf{K}_0$. Moreover, for each n, there are only countably many n-types of tuples which are strong in the structure they generate. Since for any finite $\overline{a}$ there are only countably many possibilities for $\mathrm{icl}_{\langle \overline{a} \rangle}(\overline{a})$, there are only countably quantifier free types of finite sets realized in members of $\mathbf{K}_0$.

To understand these constraints note that in the *ab initio* case the δ-formula ϕ for $\overline{a}\overline{b}$ is just the diagram of $\overline{a}\overline{b}$ and any realization of ϕ is isomorphic to $\overline{a}\overline{b}$. Condition 2a) is in this case just a reformulation of the monotonicity condition $\delta(\overline{a}/\overline{b}\overline{c}) \leq \delta(\overline{a}/\overline{b})$ if $\overline{c} \cap \overline{a} \subseteq \overline{a}$. Condition 2b) translates the fact that in the *ab initio* case if $B \subseteq C$ and $\delta(A/C) = \delta(A/B)$ then A and C are freely amalgamated over B.

NEED similar paragraph for fusions.

In the case of bicolored fields, $\delta(\overline{a}, \overline{b})$ asserts the explicit algebraic dependencies among the elements and tells their colors.

4.2 Definition. A formula $\varphi(\overline{x}, \overline{y})$ is called a *complete δ-formula* for $\overline{a}$ over $\overline{m}$ if $\varphi(\overline{a}, \overline{m})$ holds in a model N in K_0 and for all $\overline{a}_1 \subseteq \overline{a}$, if $\overline{a}_1 = \overline{a}_2\overline{a}_3$ and $\delta(\overline{a}_3/\overline{a}_2\overline{m}) = k$, there is a formula $\varphi_1'(\overline{x_2}, \overline{x_3}; \overline{y})$ such that $T_0 \cup \{\varphi(\overline{x}; \overline{y})\} \vdash \varphi_1'(\overline{x_2}, \overline{x_3}; \overline{y})$ and $\varphi_1'(\overline{b_2}, \overline{b_3}; \overline{n})$ implies $\delta(\overline{b}_3/\overline{b}_2\overline{n}) \leq k$.

$\varphi(\overline{x}; \overline{y})$ is called a complete δ-formula for $\overline{a}$ over A based on $\overline{m} \in A$ if $\overline{m} \leq A$, $\varphi(\overline{x}; \overline{y})$ is a complete δ- formula for $\overline{a}$ over $\overline{m}$ and $\delta(\overline{a}/\overline{m}) = \delta(\overline{a}/A)$.

The following lemma just follows from iterating the conditions on δ-formulas.

4.3 Lemma. *For any $\overline{a}$ and $N \in K_0$, there are $\overline{m} \leq N$ and a* complete δ-formula *$\varphi(\overline{x}; \overline{y})$ for $\overline{a}$ over N based on $\overline{m}$.*

4.4 Definition. Let $J_{B,k}(\overline{x})$ be the partial type over B composed of formulas $\neg\phi(\overline{x}; \overline{b})$ where $\overline{b} \in B$ and $\phi(\overline{x}; \overline{y})$ is a δ-formula for a sequence $\overline{dc}$ with $\delta(\overline{d}/\overline{c}) < k$. If $\mathcal{M} \models J_{B,k}(\overline{a})$, $\delta(\overline{a}/B) \geq k$.

For any $B \in \mathbf{K}_0$, let $I_B(\overline{y})$ be the collection of all formulas $\forall\overline{x}\neg\varphi(\overline{x}; \overline{y}\overline{d})$ where $\overline{d} \in B$ and for some $\overline{a}, \overline{b}, \overline{c}$ such that $\delta(\overline{a}/\overline{b}\overline{c}) < 0$ but for every $\overline{a}'$ properly contained in $\overline{a}$, $\delta(\overline{a}/\overline{b}\overline{c}) \geq 0$, $\varphi(\overline{x}; \overline{yz})$ is a complete δ-formula for $\overline{a}$ over $\overline{b}\overline{c}$. If B is the empty set we just write $I(\overline{y})$. Note that $I_B(\overline{y})$ depends only on B and the length of $\overline{y}$.

Suppose $A \subseteq B$. If $B \models I_A(\overline{b})$, then $A\overline{b} \leq B$. If not, there is a minimal witness of the kind forbidden by I_A. This yields immediately.

4.5 Lemma. *If $A, B \in \mathbf{K}_0$ and A is an elementary submodel of B, then $A \leq B$.*

Let $I^*(\overline{b})$ be union of $\mathrm{tp}_{\mathrm{qf}}(\overline{b})$ and $I(\overline{b})$. Thus, $I^*(\overline{b})$ is realized in a model N just if $\overline{b} \leq \langle \overline{b} \rangle$ and there is a strong embedding of $\langle \overline{b} \rangle$ into N. Now suppose $\overline{a} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle$ and $\overline{a} \leq M$, the generic. Then there is a strong embedding $\overline{b}$ into

M over $\overline{a}$. If M is saturated then in $T = \mathrm{Th(M)}$, for each $\phi(\overline{x}, \overline{y}) \in I^*(\overline{a}, \overline{b})$, $T \cup I^*(\overline{a}) \vdash (\exists \overline{y})\phi(\overline{a}, \overline{y})$. Saturation of the generic is necessary for this to hold; otherwise there may be instances of $\overline{a}$ in other models which cannot be so extended. See [?].

We want to find a checkable condition on $(\mathbf{K}_0, \delta)$ which ensures that the generic structure is ω-saturated. We use the converse to our last remark.

4.6 Definition. If $A \subseteq B \in \mathbf{K}_0$ and $\overline{b} \subseteq B$, then $\overline{b}$ is *minimal strong* over A if $\overline{b} \not\subseteq A$, $A \leq A\overline{b}$ and for any nonnull, proper subtuple $\overline{c}$ from $\overline{b}$, $A\overline{c} \not\leq A\overline{b}$.

4.7 Definition. We say that T has *syntactic amalgamation* if for any $\overline{a} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \mathbf{K}_0$ with $\overline{b}$ minimal strong over $\langle \overline{a} \rangle$, for each $\phi(\overline{x}, \overline{y}) \in I^*(\overline{a}, \overline{b})$, $T \cup I^*(\overline{a}) \vdash (\exists \overline{y})\phi(\overline{a}, \overline{y})$.

The significance of this definition is given in the following theorem.

4.8 Theorem. *If T has syntactic amalgamation then the generic M is saturated.*

Proof. Fix an ω-saturated model N of $Th(M)$. We show M is $L_{\infty,\omega}$-equivalent to N by constructing a back and forth; thus M is saturated. It suffices to show that whenever $\overline{a} \leq M$ and $\overline{a}' \leq N$ with $\mathrm{Diag}(\overline{a}) = \mathrm{Diag}(\overline{a}')$,

1. If $\overline{a}\overline{b} \leq M$ with $\overline{b}$ minimal strong over $\overline{a}$ then there is $\overline{b}'$ with $\overline{a}'\overline{b}' \leq N$ and $diag(\overline{a}\overline{b}) = \mathrm{Diag}(\overline{a}'\overline{b}')$;
2. if $\overline{a}'\overline{b}' \leq N$ with $\overline{b}'$ minimal strong over $\overline{a}'$ then there is $\overline{b}$ with $\overline{a}\overline{b} \leq M$ and $\mathrm{Diag}(\overline{a}'\overline{b}') = \mathrm{Diag}(\overline{a}\overline{b})$.

The second condition is immediate from genericity of M, so we prove the first only. Suppose that $\overline{a}\overline{b} \leq M$. Since T has syntactic amalgamation and $I^*(\overline{a}) \subseteq tp_N(\overline{a}')$, we have that $N \models (\exists \overline{y})\theta(\overline{a}', \overline{y}))$ for every $\theta(\overline{x}, \overline{y})$ from $I^*(\overline{a}, \overline{b})$. In the ω-saturated N there is a $\overline{b}'$ such that $\overline{a}', \overline{b}'$ realize $I^*(\overline{a}, \overline{b})$ as required. □

We can immediately reformulate syntactic amalgamation as follows. For any formula $\tau((\overline{x}; \overline{y})$ in $I^*(\overline{a}, \overline{b})$ there are formulas $\sigma(\overline{x}) \in I(\overline{x})$ and $\mu(\overline{x}) \in \mathrm{Diag}(\overline{a})$ such that

$$M \models \forall \overline{x}((\sigma \wedge \mu)(\overline{x}) \rightarrow \exists \overline{y}\tau(\overline{x}; \overline{y})).$$

Now we provide an equivalent property of $(\mathbf{K}_0, \delta)$.

4.9 Definition. We say $(\mathbf{K}_0, \delta)$ *has syntactic amalgamation* if for any $\overline{a} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \mathbf{K}_0$ with $\overline{b}$ minimal strong over $\langle \overline{a} \rangle$, the following holds: For any formula $\tau((\overline{x}; \overline{y})$ in $I^*(\overline{a}, \overline{b})$ there are formulas $\sigma(\overline{x}) \in I(\overline{x})$ and $\mu(\overline{x}) \in \mathrm{Diag}(\overline{a})$ such that whenever $\overline{a}' \subseteq C \in \mathbf{K}_0$ and $C \models (\sigma \wedge \mu)(\overline{a}')$, there is $D \in \mathbf{K}_0$ with $C \leq D$ and $\overline{b}' \in D$ such that $D \models \tau(\overline{a}'; \overline{b}')$.

4.10 Lemma. *Let M be the countable generic for $(\mathbf{K}_0, \delta)$. The following are equivalent:*

1. $(\mathbf{K}_0, \delta)$ *has syntactic amalgamation.*

2. T *has syntactic amalgamation.*

Proof. Suppose 1 holds. Fix $\overline{a} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \mathbf{K}_0$ with $\overline{b}$ minimal strong over $\langle \overline{a} \rangle$ and suppose $\tau((\overline{x}; \overline{y})$ in $I^*(\overline{a}, \overline{b})$. Choose $\sigma(\overline{x})$ and $\mu(\overline{x})$ by syntactic amalgamation for $(\mathbf{K}_0, \delta)$. Suppose $\overline{a}' \in M$ and $M \models (\sigma \wedge \mu)(\overline{a}')$. Let C be the substructure of M generated by some finite, strong subset of M containing $\overline{a}'$; then $C \in \mathbf{K}_0$ and $C \leq M$. Since $\sigma \wedge \mu$ is universal, $C \models (\sigma \wedge \mu)(\overline{a}')$. Since $(\mathbf{K}_0, \delta)$ has syntactic amalgamation, there is $D \in \mathbf{K}_0$ with $C \leq D$ and $\overline{b}' \in D$ with $D \models \tau(\overline{a}'; \overline{b}')$. By the definition of generic, D can be embedded strongly into M over C, so that $M \models \tau(\overline{a}'; \overline{b}')$ as well. Thus, $M \models \forall \overline{x}((\sigma \wedge \mu)(\overline{x}) \rightarrow \exists \overline{y}(\tau)(\overline{x}; \overline{y}))$, as desired.

Now assume that 2 holds. Fix $\overline{a}, \overline{b}$ and τ as in the definition of syntactic amalgamation for $(\mathbf{K}_0, \delta)$. Using the reformulation of syntactic amalgamation for T choose σ and μ. Let $\overline{a}' \subseteq C \in \mathbf{K}_0$ with $C \models (\sigma \wedge \mu)(\overline{a}')$. We must find $D \in \mathbf{K}_0$ with $C \leq D$ and $\overline{b}' \in D$ such that $D \models \tau(\overline{a}'; \overline{b}')$. Without loss of generality, $C \leq M$. Since μ is quantifier free, $M \models \mu(\overline{a}')$. Suppose $M \models \neg\sigma(\overline{a}')$. Now $\neg\sigma(\overline{a}')$ has the form $(\exists \overline{y})\sigma'(\overline{a}', \overline{y})$ where $\sigma'(\overline{x}, \overline{y})$ is complete δ-formula for some $\overline{e}$ over $\overline{a}$ which is minimal with $\delta(\overline{e}/\overline{a}) < 0$. Thus, there is an $\overline{e}' \in M$ such that for every partition $\overline{e}'_1\overline{e}'_2$ of $\overline{e}'$ with $\overline{e}'_2$ nonempty, $\delta(\overline{e}'_2/\overline{e}'_1\overline{a}') < 0$. In particular, if $\overline{e}' \cap C = \overline{e}'_1$ is a proper subset of $\overline{e}'$ we contradict $C \leq M$. But $\overline{e}' \in C$ contradicts $C \models \sigma(\overline{a}')$. So $M \models (\sigma \wedge \mu)(\overline{a}')$. Then, by hypothesis, $M \models \tau(\overline{a}'; \overline{b}')$ for some $\overline{b}'$. Take D to be any strong substructure of M containing $C\overline{b}'$. $\square$

4.11 Corollary. *If* $(\mathbf{K}_0, \delta)$ *has syntactic amalgamation, then the generic* M *is saturated.*

5 Main Example 3

5.1 Theorem. *The theory of bicolored fields has the syntactic amalgamation property.*

Proof. Consider $\overline{a} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \mathbf{K}_0$ with $\overline{b}$ minimal strong over $\langle \overline{a} \rangle$, and $\phi(\overline{x}, \overline{y}) \in I^*(\overline{a}, \overline{b})$. Let $\beta(\overline{x}, \overline{y})$ be a complete δ-formula for $\overline{b}$ over $\overline{a}$. Now, $\phi(\overline{x}, \overline{y})$ has the form $(\forall \overline{z})\neg\tau(\overline{x}, \overline{y}; \overline{z})$ where τ is a complete δ-formula for some $\overline{c}$ with $\overline{c}$ a minimal intrinsic extension over $\overline{a}\overline{b}$. Note that all elements of $\overline{c}$ must be black (by minimality). τ is the conjunction of a quantifier free field formula $\tau'(\overline{x}, \overline{y}; \overline{z})$ and a declaration of which of $\overline{x}, \overline{y}; \overline{z}$ are black. Thus there is a natural way to restrict τ to $\overline{x}$. We consider two cases.

If the $\delta(\overline{c}/\overline{a}) < 0$, then the transendence degree of $\overline{c}$ over $\overline{a}$ is less than one half the length of $\overline{c}$. There is a field formula $\theta(\overline{a}, \overline{c})$ which describes exactly the way $\overline{c}$ is in the algebraic closure of a proper subsequence $\overline{c}_1$. Let $\theta'(\overline{x}, \overline{z})$

be the conjunction θ and a formula asserting the z's are all black. Then, the conjunction of the restriction of $\tau(\overline{x}, \overline{y}, \overline{z})$ to $\overline{x}$ and $(\forall \overline{z})\neg\theta'(\overline{x}, \overline{z})$ is the required formula from $I^*(\overline{a})$.

If $\delta(\overline{c}/\overline{a}) \geq 0$, restriction of $\tau(\overline{x}, \overline{y}, \overline{z})$ to $\overline{x}$ suffices as σ and there is no μ. For, whenever $\overline{a}' \subseteq C \in \mathbf{K}_0$ and $C \models \sigma(\overline{a}')$, take any $\overline{b}'$ such that $\overline{a}'\overline{b}'$ realizes $\beta(\overline{a}', \overline{b}')$. Let $D \in \mathbf{K}_0$ be $\mathrm{acl}(\overline{a}'\overline{b}')$ in the sense of fields and color no additional points black. Then $C \leq D$ and $D \models \tau(\overline{a}'; \overline{b}')$.

6 The Finite Rank case:

6.1 Definition. $\overline{a}$ is *primitive* over A if it is minimal strong over A with $\delta(\overline{a}/A) = 0$.

To find a rank two, indeed $\aleph_1$-categorical, bicolored field we want to modify the construction so that a complete δ-formula for a primitive is algebraic. There are two ways to see the $\aleph_1$- categoricity. If $\delta(a/\emptyset) = 0$ then $a \in \mathrm{acl}(\emptyset)$; all types of a with $\delta(a/\emptyset) = 1$ are nonorthogonal. Alternatively, one can vary the construction to guarantee that the bicolored field is in the algebraic closure of the black points and prove the set of black points is strongly minimal.

Let X be a set of 2n black points and suppose that X is primitive over Y. Then the transcendence degree of X over Y is $|X|/2$. Let $\phi_1(\overline{x}, \overline{y})$ be a complete δ-formula for X/Y. Note that ϕ_1 is the conjunction the assertion that all $x's$ are black with a field formula ϕ_1' which asserts for each partition $\overline{x} = \overline{x}_1\overline{x}_2$, the transcendence degree of $\overline{x}_2$ over $\overline{x}_1\overline{y}$ is at most $|\overline{x}_2|/2$. (This assertion specifies a particular way the algebraicity takes place.) Using the definability of multiplicity, we can assume without loss of generality that for any $\overline{a}$, $\phi_1'(\overline{x}; \overline{a})$ has multiplicity one. (We will specify the apparent use of this below.) By elimination of imaginaries, we can replace $\phi_1(\overline{x}, \overline{y})$ by $\phi(\overline{x}, \overline{y})$ where $\overline{y}$ is a canonical parameter for $\phi(\overline{x}, \overline{y})$.

Fix $\overline{a} \in k$ and $\overline{b} \in K$ with $\phi(\overline{b}, \overline{a})$. If $\overline{b}_i$ are sequence of distinct tuples satisfying $\phi(\overline{b}_i, \overline{a})$, they must be disjoint by primitivity and they are forking independent. Thus there is a number b such that $\overline{a}$ is algebraic in $\langle \overline{b}_i : i < b\rangle$. Fix a field formula $\alpha(\overline{x}_1 \ldots \overline{x}_b, \overline{y})$ which witnesses this algebraicity.

For every type p of a primitive $\overline{ab}$, choose a constant $m = \mu(p)$, (greater than the b defined in the last paragraph. In fact, for b = 0, take $m \geq 1$, for $b = 1$, $m \geq n + 1$ and for $b \geq 2$, $m \geq (n + 2)b - 2$. We will justify the choices shortly.)

Fix $\overline{x}_i$ a $2n$-tuple of variables. Let $\sigma(\overline{x}_1 \ldots \overline{x}_m, \overline{y})$ express the following:

1. the $\overline{x}_i$ are black and disjoint,
2. $\phi(\overline{x}_i, \overline{y})$,
3. $\alpha(\overline{x}_{i_l} \ldots \overline{x}_{i_b}, \overline{y})$ for $\leq i_1 \leq \ldots i_b \leq m$.

We say a bicolored field omits $\sigma_{\mu(p)}$ if it satisfies $(\forall \overline{x}_1, \ldots \overline{x}_m, \overline{y})\neg\sigma(\overline{x}_1 \ldots \overline{x}_m, \overline{y})$.

Let μ be a function from types of primitive pairs $\overline{b}/\overline{a}$ into the integers as above. Let T^μ be the theory of bicolored fields augmented by the assertion that $\sigma(\mu(p))$ is omitted for each p.

6.2 Definition. A strong amalgamation of K and L over k is called *economical* if K and L are freely amalgamated over their intersection k and there are no intermediate fields K_1 and L_1 which are strong over k and strong in K, L respectively and with K_1 and L_1 isomorphic.

6.3 Lemma. *T^μ has the amalgamation property.*

Proof. We show in fact that if k has strong extensions K, L and all three satisfy T^μ then they have an economical amalgam which also models T^μ. Given K, L which are both strong over their intersection we can identify the largest strong extension of k in K which a copy in L with that copy and the form the free amalgamation k^*. Suppose for contradiction that some X, Y contained in k^* witnesses that k^* is not a model of T^μ. Then $X = X_1 \ldots X_m$ and we write U_i for $X_i \cup (K - k)$, V_i for $X_i \cup (L - k)$, and W_i for $X_i \cup k$.

First suppose Y is not contained in either K or L. This implies that at most $b-1$ of the X_i are in K or in L; for the others U_i and V_i are both nonempty. By renumbering we may assume that the first $b-1$ are in K, that $V_b \neq \emptyset$, and that for $b+1 \leq i \leq (n+1)b$, U_i and V_i are both nonempty. We chose m large enough to make this possible. Since $V_b \neq \emptyset$, there are less than nb elements in the union of $U_1, \ldots U_b$. For $nb \leq i \leq (n+1)b$, the transcendence degree of U_i over YW_iV_i is less than the degree of transcendence of U_i over $X_1 \ldots X_b W_i V_i$ which is again less than the degree of transcendence of U_i over $k(U_1 \ldots U_b)$. This degree of transcence is less than $|U_i|/2$, so we have shown each $\delta(U_i/k(U_1 \ldots U_b) < 0$ for each i with $nb \leq i \leq (n+1)b$. Since $\delta(k(U_1 \ldots U_b)/k) < nb$, this contradicts $k \leq K$.

Now suppose Y is contained in one of the fields, say L; we will show $Y \subset k$. Since L omits σ, one of the X_i is not contained in L whence $X_i \subset (K - k)$. Now we can choose an independent sequence extending $\mathrm{tp(X_i/k)}$ and independent from L over k. Since Y must be in the algebraic closure of the first b of these, $Y \subset K$. Thus, $Y \subset k$. (Note the multiplicity one does not seem essential here????)

But if $Y \subset k$, there must be $X_i \subset (K - k)$ and $X_j \subset (L - k)$. But then they can be identified contradicting economical amalgamation.