Categoricity
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Modern Model Theory Begins

Theorem (Morley 1965)

If a countable first order theory is categorical in one uncountable cardinal it is categorical in all uncountable cardinals.

Outline

Themes

1. Motivations for studying Categoricity
   1. (Shelah) Understanding classes of structures
   2. (Zilber) Understanding 'canonical' mathematical structures
2. Study of the infinitary case illuminates first order model theory.
3. The infinitary case raises mathematical and set theoretic problems.

Languages

- The first order language ($L_{\omega,\omega}$) associated with L is the least set of formulas containing the atomic L-formulas and closed under **finite** Boolean operations and quantification over finitely many individuals.
- The $L_{\omega_1,\omega}$ language associated with L is the least set of formulas containing the atomic L-formulas and closed under **countable** Boolean operations and quantification over finitely many individuals.

The Transition

'. . . what makes his paper seminal are its new techniques, which involve a systematic study of Stone spaces of Boolean algebras of definable sets, called type spaces. For the theories under consideration, these type spaces admit a Cantor Bendixson analysis, yielding the key notions of Morley rank and ω-stability.'

Citation awarding Michael Morley the 2003 Steele prize for seminal paper.

The 70's

Stability theory developed

1. abstractly with the stability classification
2. concretely by finding the stability class of important mathematical theories and using the techniques of the abstract theory.

The absoluteness of fundamental notions such as $\aleph_1$-categoricity and stability liberated first order model theory from set theory.

First Order Categorical Structures

I. $(\mathcal{C}, =)$

II. $(\mathcal{C}, +, =)$ vector spaces over Q.

III. $(\mathcal{C}^*, \times, =)$

IV. $(\mathcal{C}, +, \times, =)$ Algebraically closed fields - Steinitz

V. Simple algebraic groups over algebraically closed fields

Zilber's Precept

Fundamental canonical mathematical structures like I-IV should admit logical descriptions that are categorical in power.

Another Canonical Structure

COMPLEX EXPONENTIATION

Consider the structure $(C, +, \cdot, e^x, 0, 1)$.

The integers are defined as $\{a : e^{2a\pi i} = 1\}$.
This makes the first order theory unstable, provides a two cardinal model .... The theory is clearly not categorical.

Thus first order axiomatization can not determine categoricity.

ZILBER'S INSIGHT

Maybe Z is the source of all the difficulty.
Fix Z by adding the axiom:

$$(\forall x)e^x = 1 \rightarrow \bigvee_{n \in Z} x = 2n\pi i.$$

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In fact, the solution required other extensions of first order logic, which will be described later.

First order model theory

The main tool of first order model theory is the classification of complete theories by stability-like notions.

If complete theories have similar semi-syntactic theoretic properties:
$\aleph_1$-categorical, ω-stable, o-minimal, strictly stable,

then their class of models have similar algebraic properties:

number of models, existence of dimension functions, interpretability of groups, existence of generic elements,

The Standard Example

$\mathrm{Th}(M)$ for any M.

Leitmotif

Study a mathematical structure M by studying $\mathrm{Th}(M)$.

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Thus, Weil-style algebraic geometry is the model theory of $(\mathcal{C}, +, \cdot, 0, 1)$.

$L_{\omega_1,\omega}$-completeness

For Δ a fragment of $L_{\omega_1,\omega}$, a Δ-theory T is complete if for every Δ-sentence ϕ,

$T \models \phi$

or

$T \models \neg\phi$.

Löwenheim Skolem properties

Downward: Every consistent countable set of $L_{\omega_1,\omega}$-sentences has a countable model.

No upward: There are sentences with maximal models in each $\aleph_\alpha$ and each $\beth_\alpha$ – (that characterize $\aleph_\alpha$ or $\beth_\alpha$).

Vaught's test

Let T be a set of first order sentences with no finite models, in a countable language.

If T is κ-categorical for some $\kappa \geq \aleph_0$,
then T is complete.

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Fails for $L_{\omega_1,\omega}$

Small

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Definition

An $L_{\omega_1,\omega}$-sentence ϕ is Δ-small if there is a set X countable of complete Δ-types over the empty set and each model realizes some subset of X.

'small' means $\Delta = L_{\omega_1,\omega}$

Small implies complet(able)

Structures

If M is small then M satisfies a complete sentence.

Sentences

If ϕ is small then there is a complete sentence ψ_ϕ such that: $\phi \wedge \psi_\phi$ has a countable model.

So ψ_ϕ implies ϕ.

But ψ_ϕ may not be unique.

The $L_{\omega_1,\omega}$-Vaught test

Shelah If ϕ has an uncountable model M that is Δ-small for every countable Δ and ϕ is κ-categorical then ϕ is implied by a complete sentence ψ with a model of cardinality κ.

Keisler If ϕ has $< 2^{\aleph_1}$ models of cardinality $\aleph_!$, then for every countable Δ, ϕ is Δ not so big.
I.e. each model is Δ-small for every countable Δ.

So we effectively have Vaught's test.
But only in $\aleph_1$!
And only for completability!

Countable models I

Must an $\aleph_1$-categorical sentence have only countably many countable models?

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Must an $\aleph_1$-categorical sentence have only countably many countable models?

Trivially, no. Take the disjunction of a 'good' sentence with one that has $2^{\aleph_0}$-countable models and no uncountable models.

Countable models II

Is there a way to study the countable models of sufficiently nice incomplete sentences?

Must an $\aleph_1$-categorical sentence with the joint embedding property have only countably many countable models?

A direction: The Kesala-Hyttinen study of finitary abstract elementary classes.

Another direction: Kierstead's thesis using admissible model theory.

Two specific research questions

For ϕ a sentence in $L_{\omega_1,\omega}$:

Does categoricity in $\kappa > \aleph_1$ imply completeability?

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Does categoricity in $\kappa > \aleph_1$ imply completeability?

Is categoricity in $\aleph_1$ absolute (for ccc forcing)?

Prime Models

M is prime over A if every elementary embedding of A in to $N \models T$ extends to an elementary embedding of M into N.

κ-stability

K is κ-stable if for every $M \in \mathbf{K}$ with $|M| = \kappa$, $|S(M)| = \kappa$.

Categoricity implies stability

arbitrarily large models: : Ehrenfeucht-Mostowski models give stability below the categoricity cardinal in either logic.

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$2^{\aleph_0} < 2^{\aleph_1}$: For $L_{\omega_1,\omega}$, categoricity in $\aleph_1$ implies $\aleph_0$-stability.

But in the infinitary case we study $S_{at}(M)$.

Morley's Proof (1965)

Saturation means first order saturated.

Theorem

*If **K**, the class of models of a countable first order theory is categorical in one uncountable cardinal it is categorical in all uncountable cardinals.*

1. Saturated models of the same cardinality are isomorphic.
2. κ-categoricity implies $< \kappa$-stable. ω stable implies stability in all powers.
3. For any κ, κ-stable implies there is an $\aleph_1$-saturated model of cardinality κ.

Morley's Proof continued

4. ω-stable implies
   1. If there is a nonsaturated model there is a countable model M with a countable subset X such that:
      1. M contains an infinite set of indiscernibles over X;
      2. Some $p \in S(X)$ is omitted in M.
5. Taking prime models over sequences of indiscernibles Item 4) implies:
   If there is a nonsaturated model, then there is a model in every cardinal that is not $\aleph_1$-saturated.

Key Ideas - Morley

1. ω-stability
2. Ehrenfeucht-Mostowski Models
3. saturation
4. omitting types
5. indiscernibles
6. prime models

Another Approach (1971)

Keisler, Chudnovsky and Shelah: $L_{\omega_1,\omega}$

Replace the omitting types argument (4-5 of Morley's proof) by Morley's omitting types theorem and two cardinal theorem for cardinals far apart.

Problem

But if one restricts to types that are realized in models of **K**, the uniqueness of 'saturated' models fails.

Three Solutions:

1. (Keisler) Assume all models are $\aleph_1$-homogenous: a precursor of homogenous model theory.
2. (Shelah) AEC and Galois Types
3. (Shelah/ later Zilber) Excellence

ABSTRACT ELEMENTARY CLASSES

Generalizing Bjarni Jónsson:

A class of L-structures, $(\mathbf{K}, \prec_{\mathbf{K}})$, is said to be an *abstract elementary class: AEC* if both $\mathbf{K}$ and the binary relation $\prec_{\mathbf{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$;

Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

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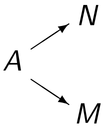
2. Closure under direct limits of $\prec_{\mathbf{K}}$-chains;
3. Downward Löwenheim-Skolem.

Examples

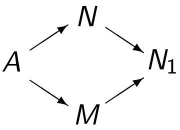
First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

AMALGAMATION PROPERTY

The class **K** satisfies the *amalgamation property* if for any situation with $A, M, N \in \mathbf{K}$:



there exists an N_1 such that



Jónsson AEC

If an aec has

1. arbitrarily large models
2. amalgamation
3. joint embedding

we call it a Jónsson AEC.

Examples

1. Complete first order theories
2. Homogeneous model theory
3. excellent classes - quasiminimal excellent classes
4. Covers of Abelian algebraic groups
5. the Hart-Shelah examples
6. the class of modules: $(N^{\perp}, \prec)$

The Monster Model

If an Abstract Elementary Class has the amalgamation property and the joint embedding property then we can work inside a monster model (universal domain) $\mathcal{M}$ that is $|\mathcal{M}|$-model homegeneous. That is,

If $N \prec_{\mathbf{K}} \mathcal{M}$ and $N \prec_{\mathbf{K}} M \in \mathbf{K}$ and $|M| < |\mathcal{M}|$ there is embedding of M into $\mathcal{M}$ over N.

Galois Types

Fix a monster model $\mathbb{M}$ for $\mathbf{K}$.

Definition

Let $A \subseteq M \prec_{\mathbf{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The *Galois type* of a over M $(\in \mathbb{M})$ is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.

The set of Galois types over A is denoted $\mathcal{S}(A)$.

Galois Saturation

Definition

The model M is μ-*Galois saturated* if for every $N \prec_{\mathbf{K}} M$ with $|N| < \mu$ and every Galois type p over N, p is realized in M.

Theorem

For $\lambda > \mathrm{LS}(\mathbf{K})$, *If* M, N *are* λ-*Galois saturated with cardinality* λ *then* $M \approx N$.

Galois vrs Syntactic Types

Syntactic types have certain natural locality properties.

- **locality** Any increasing chain of types has at most one upper bound;
- **tameness** two distinct types differ on a finite set;
- **compactness** an increasing chain of types has a realization.

The translations of these conditions to Galois types do not hold in general.

Tameness

Grossberg and VanDieren focused on the idea of studying 'tame' abstract elementary classes:

Distinct Galois types differ on a small submodel.

Definition

We say $\mathbf{K}$ is (χ, μ)-*tame* if for any $N \in \mathbf{K}$ with $|N| = \mu$ if $p, q, \in \mathcal{S}(N)$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ then $q = p$.

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

If for every $N \prec_{\mathbf{K}} M$ with $|N| \leq \chi$ there exists $\alpha \in \mathrm{aut}_{\mathrm{N}}(\mathcal{M})$ with $\alpha(a) = b$,

then there exists $\alpha \in \mathrm{aut}_{\mathrm{M}}(\mathcal{M})$ with $\alpha(a) = b$.

Consequences of Tameness

Suppose $(\mathbf{K}, \prec_{\mathbf{K}})$ is a Jónsson AEC.

Theorem (Grossberg-Vandieren: 2006)

If $\lambda > \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ is λ^+-categorical and $(\lambda, < \infty)$-tame then $\mathbf{K}$ is categorical in all $\theta \geq \lambda^+$.

Theorem (Lessmann)

If $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ is $\aleph_1$-categorical and $(\aleph_0, \infty)$-tame then $\mathbf{K}$ is categorical in all uncountable cardinals

Key Ideas - Jónsson AEC

Prove by induction that

1. every model above a successor categoricity cardinal is saturated.
2. (and a fortiori) Galois stability in every cardinal.

1. Galois stability and saturation saturation
2. Ehrenfeucht-Mostowski Models
3. tameness
4. splitting

There is no use of prime models or indiscernibles.

Specializes to first order.

Shelah 1999: AEC downward-categoricity

Theorem

$(\mathbf{K}, \prec_{\mathbf{K}})$ *is a Jónsson AEC. If* $\mathbf{K}$ *is categorical in some* λ^+ *above* H_2, $\mathbf{K}$ *is categorical on* $[H_2, \lambda^+]$.

Tools

1. All previous AEC tools
2. Morley omitting types theorem/ two cardinal theorem,
3. models of set theory and absoluteness of Galois saturation below a categoricity cardinal.

Conclusions on Jónsson AEC

Categoricity on a proper class of successor cardinals implies eventual categoricity.

Categoricity on a proper class of limit cardinals remains open. First order categoricity transfers upwards from $|T|^{++}$ by these proofs. (From $\aleph_1$ for countable language.)

GEOMETRIES

Definition. A pregeometry is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.
If points are closed the structure is called a geometry.

Key Ideas - B-L

Theme: The geometry on a strongly minimal set determines the model.

Theorem A countable first order theory is categorical in all uncountable powers iff it has no two-cardinal models and is ω-stable.

1. ω-stability
2. Ehrenfeucht-Mostowski Models
3. strongly minimal sets and dimension
4. two-cardinal models
5. prime models

Prime models are essential for both the upwards and downwards arguments. Saturation is not used.

Categoricity Transfer in $L_{\omega_1,\omega}$

1. Zilber (Geometry explicit): Upwards only (ZFC)
2. Shelah (Geometry in background): Upwards from Categoricity below $\aleph_\omega$ (VWGCH)

Quasiminimality

A class $(\mathbf{K}, \mathrm{cl})$ is *quasiminimal* if cl is a combinatorial geometry which satisfies on each $M \in \mathbf{K}$:

1. there is a unique type of a basis,
2. a technical homogeneity condition: $\aleph_0$-homogeneity over $\emptyset$ and over models.
3. Closure of countable sets is countable

Theorem

A quasiminimal class is $\aleph_1$-categorical.

$L_{\omega_1,\omega}$: The General Case

Quasiminimality is the 'rank one' case

Any geometry has a notion of independent n-system.

Reducing $L_{\omega_1,\omega}$ to 'first order'

The models of a complete sentence in $L_{\omega_1,\omega}$ can be represented as:
$\mathbf{K}$ is the class of atomic models (realize only principal types) of a first order theory (in an expanded language).

We study $S_{at}(A)$ where $A \subset M \in \mathbf{K}$ and
$p \in S_{at}(A)$ means Aa is atomic if a realizes p.

ω-stability

Definition

ϕ is ω-stable if for every countable model of ϕ, there are only countably many types over M that are realized in models of ϕ (i.e. $|S_{at}(M)| = \aleph_0$).

Essence of Excellence

Let **K** be the class of models of a sentence of $L_{\omega_1,\omega}$.

K is excellent

K is ω-stable and any of the following equivalent conditions hold.

For any finite independent system of countable models with union C:

1. $S_{at}(C)$ is countable.
2. There is a unique primary model over C.
3. The isolated types are dense in $S_{at}(C)$.

Quasiminimal Excellence

means

Quasiminimal and excellent.

Quasiminimal Excellence implies Categoricity

Quasiminimal excellence implies by a direct limit argument:

Lemma

An isomorphism between independent X and Y extends to an isomorphism of $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$.

This gives categoricity in all uncountable powers if the closure of each finite set is countable.

Categoricity for Quasiminimal classes

Theorem Suppose the quasiminimal excellent class **K** is axiomatized by a sentence Σ of $L_{\omega_1,\omega}$, and the relations $y \in \mathrm{cl}(x_1, \ldots x_n)$ are $L_{\omega_1,\omega}$-definable.

Then, for any infinite κ there is a unique structure in **K** of cardinality κ which satisfies the countable closure property.

NOTE BENE: The categorical class could be axiomatized in $L_{\omega_1,\omega}(Q)$. But, the categoricity result does not depend on any such axiomatization.

ZILBER'S PROGRAM FOR $(\mathcal{C}, +, \cdot, \exp)$

Goal: Realize $(\mathcal{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}(Q)$-sentence discovered by the Hrushovski construction.

Done

A. Expand $(\mathcal{C}, +, \cdot)$ by a unary function which behaves like exponentiation using a Hrushovski-like dimension function. Prove some $L_{\omega_1,\omega}(Q)$-sentence Σ is categorical and has quantifier elimination.

Very open

B. Prove $(\mathcal{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

There raises many intriguing problems in number theory and complex analysis.

The Weak Generalized Continuum Hypothesis

Setting

ZFC is the base theory throughout.

Axiom: WGCH

For every cardinal λ, $2^\lambda < 2^{\lambda^+}$.

Axiom: VWGCH

For every $n < \omega$, $2^{\aleph_n} < 2^{\aleph_{n+1}}$.

ω-stability I

Definition

The atomic class **K** is λ-stable if for every $M \in \mathbf{K}$ of cardinality λ, $|S_{\mathrm{at}}(M)| = \lambda$.

Theorem (Keisler-Shelah)

If **K** is $\aleph_1$-categorical and $2^{\aleph_0} < 2^{\aleph_1}$ then **K** is ω-stable.

This proof uses CH directly and also WCH.

Is CH is necessary?

Does MA + ¬ CH imply there is a sentence of $L_{\omega_1,\omega}$ that is $\aleph_1$ categorical but

a. is not ω-stable

b. does not satisfy amalgamation even for countable models.

There is such an example in $L_{\omega_1,\omega}(Q)$ but Laskowski showed the example proposed for $L_{\omega_1,\omega}$ by Shelah (and me) fails.

Categoricity Transfer in $L_{\omega_1,\omega}$

ZFC: Shelah 1983

If **K** is an excellent $EC(T, Atomic)$-class then if it is categorical in one uncountable cardinal, it is categorical in all uncountable cardinals.

VWGCH: Shelah 1983

If an $EC(T, Atomic)$-class **K** is categorical in $\aleph_n$ for all $n < \omega$, then it is excellent.

References

1. Shelah: Classification for Abstract Elementary Classes Amazon $< \$30$
2. Baldwin: Categoricity, To appear AMS, on website. `http://www2.math.uic.edu/~jbaldwin/org/res.html`