Infinitary Model Theory: Covers of Abelian groups

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January 2, 2009

Short Exact Sequences

$$0 \to N \to V \stackrel{\exp}{\to} \mathbb{A} \to 1. \tag{1}$$

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3 cases

1. ${}^{\perp}N$. (Baldwin-Eklof-Trlifaj: APAL 07)
2. Axiomatize in $L_{\omega_1,\omega}$ to guarantee standard kernel: $N = \mathcal{Z}^n$.
   1. $\mathbb{A}$ is $\aleph_1$-free. Complicated examples. (Baldwin-Shelah: JSL 08)
   2. $\mathbb{A}$ is a commutative algebraic group. (Zilber et al). This talk.

Acknowledgements

This talk reports work of Shelah, Zilber, Gavrilovich, and Bays. Few proofs are new.

Goal:

What is the algebraic content of stability theoretic conditions in infinitary logic?

1. algebra $\Rrightarrow$ model theory.
2. model theory $\Rrightarrow$ algebra.

The 70's

Stability theory developed

1. abstractly with the stability classification
2. concretely by finding the stability class of important mathematical theories and using the techniques of the abstract theory.

What does the fo stability hierarchy mean?

1. (Macintyre) An infinite field is ω-stable if and only if it is algebraically closed.
2. (Cherlin) An infinite division ring is superstable if and only if it is algebraically closed.

Thus, an ω-stable field is categorical in all uncountable powers.

Hierarchy for $L_{\omega_1,\omega}$

1. complete
2. ω-stable
3. excellent

Superstable means ???

Completeness???

Vaught's test

Let T be a set of first order sentences with no finite models, in a countable first order language.

If T is κ-categorical for some $\kappa \geq \aleph_0$, then T is complete.

Needs upward and downward Lowenheim-Skolem theorem for theories

We search for a substitute in $L_{\omega_1,\omega}$.

Small

Let Δ be a fragment of $L_{\omega_1,\omega}$ that contains ϕ.

Definition

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Definition

An $L_{\omega_1,\omega}$-sentence ϕ is Δ-small if there is a set X countable of complete Δ-types over the empty set and each model realizes some subset of X.

'small' means $\Delta = L_{\omega_1,\omega}$

Small implies complet(able)

If M is small then M satisfies a complete sentence.

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If ϕ is small then there is a complete sentence ψ_ϕ such that:

$\phi \wedge \psi_\phi$ have a countable model.

So ψ_ϕ implies ϕ.

But ψ_ϕ is not in general unique (real examples).

The $L_{\omega_1,\omega}$-Vaught test

Shelah If ϕ has an uncountable model M that is Δ-small for every countable Δ and ϕ is κ-categorical then ϕ is implied by a complete sentence ψ with a model of cardinality κ.

Keisler If ϕ has $< 2^{\aleph_1}$ models of cardinality $\aleph_!$, then for every countable Δ, ϕ is Δ-not so big.
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So we effectively have Vaught's test.
But only in $\aleph_1$!
And only for completability!

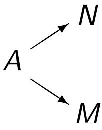
Reducing $L_{\omega_1,\omega}$ to 'first order'

The models of a complete sentence in $L_{\omega_1,\omega}$ can be represented as:
K is the class of atomic models (realize only principal types) of a first order theory (in an expanded language).

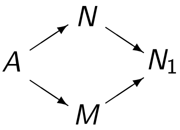
We study $S_{at}(A)$ where $A \subset M \in \mathbf{K}$ and
$p \in S_{at}(A)$ means Aa is atomic if a realizes p.

AMALGAMATION PROPERTY

The class **K** satisfies the *amalgamation property* if for any situation with $A, M, N \in \mathbf{K}$:

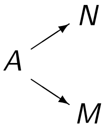


there exists an N_1 such that



SET AMALGAMATION PROPERTY

The class **K** satisfies the *set amalgamation property* if for any situation with $M, N \in \mathbf{K}$ and $A \subset M, A \subset N$:



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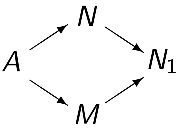

Is there a difference?

For a complete first order theory, Morley taught us:

There is no difference.

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There is no difference.

Tweak the language and we obtain set amalgamation.

(Tweak: put predicates for every definable set in the language)

There is a difference!

Zilber's examples of quasiminimal excellent classes have amalgamation over models but the interesting examples do not have set amalgamation.

ψ is categorical in all infinite cardinalities but no model is $\aleph_1$-homogeneous because there is a countably infinite maximal indiscernible set.

Quasiminimality

A class $(\mathbf{K}, \mathrm{cl})$ is *quasiminimal* if cl is a combinatorial geometry which satisfies on each $M \in \mathbf{K}$:

1. there is a unique type of a basis,
2. a technical homogeneity condition: $\aleph_0$-homogeneity over $\emptyset$ and over models.
3. Closure of countable sets is countable

Theorem

A quasiminimal class is $\aleph_1$-categorical.

$L_{\omega_1,\omega}$: The General Case

Quasiminimality is the 'rank one' case

Any geometry has a notion of independent n-system.

In the more general setting

Splitting gives an analogous notion of independent n-system.
And thus a more general notion of excellence.

ω-stability

Definition

ϕ is ω-stable if for every countable model of ϕ, there are only countably many types over M that are realized in models of ϕ (i.e. $|S_{at}(M)| = \aleph_0$).

Essence of Excellence

Let **K** be the class of models of a sentence of $L_{\omega_1,\omega}$.

K is excellent

K is ω-stable and any of the following equivalent conditions hold.

For any finite independent system of countable models with union C:

1. $S_{at}(C)$ is countable.
2. There is a unique primary model over C.
3. The isolated types are dense in $S_{at}(C)$.

Quasiminimal Excellence

means
Quasiminimal and excellent.

QM EXCELLENCE IMPLIES CATEGORICITY EVERYWHERE

QM Excellence implies by a direct limit argument:

Lemma

An isomorphism between independent X and Y extends to an isomorphism of $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$.

This gives categoricity in all uncountable powers if the closure of finite sets is countable.

What excellence buys

Theorem: Shelah

If an atomic class **K** is excellent and has an uncountable model then

1. **K** has models of arbitrarily large cardinality;
2. Categoricity in one uncountable power implies categoricity in all uncountable powers.

Covers of Algebraic Groups

Definition A cover of a commutative algebraic group $\mathbb{A}(\mathcal{C})$ is a short exact sequence

$$0 \to Z^N \to V \stackrel{\exp}{\to} \mathbb{A}(\mathcal{C}) \to 1. \qquad (2)$$

where V is a $\mathbb{Q}$ vector space and $\mathbb{A}$ is an algebraic group, defined over k_0 with the full structure imposed by $(\mathcal{C}, +, \cdot)$ and so interdefinable with the field.

Axiomatizing Covers: first order

Let $\mathbb{A}$ be a commutative algebraic group over an algebraically closed field F.

Let T_A be the first order theory asserting:

1. $(V, +, f_q)_{q \in \mathbb{Q}}$ is a $\mathbb{Q}$-vector space.
2. The complete first order theory of $\mathbb{A}(F)$ in a language with a symbol for each k_0-definable variety (where k_0 is the field of definition of $\mathbb{A}$).
3. exp is a group homomorphism from $(V, +)$ to $(\mathbb{A}(F), \cdot)$.

Axiomatizing Covers: $L_{\omega_1,\omega}$

Add to T_A
$\Lambda = \mathcal{Z}^N$ asserting the kernel of exp is standard.

$$(\exists \mathbf{x} \in (\exp^{-1}(1))^N)(\forall y)[\exp(y) = 1 \rightarrow \bigvee_{\mathbf{m} \in \mathcal{Z}^N} \Sigma_{i<N} m_i x_i = y]$$

Finitary AEC

For any $\mathbb{A}$:

$T_A + \Lambda = \mathcal{Z}^N$

1. has arbitrarily large models
2. has the amalgamation property

Thus, the rudiments of Geometric stability theory for finitary AEC developed by Hyttinen and Kesala apply.

Descending Chain Condition

Let K be a field and $W \subset K^r$ a variety defined over K.

Definition

$T_A + \Lambda = \mathcal{Z}^N$ has the *dcc over* K if
there is no infinite sequence of varieties $W^{1/m}$ such that:

$(W^{1/mk})^k = W^{1/m}$,
each $W^{1/m}$ is a minimal K-variety
and a $\mathbf{c} \in V$ such that
$\exp(\mathbf{c}/m) \in W^{1/m}$ and such that

$$W^{1/m}(\exp(\mathbf{y}/m))$$

is a proper descending chain.

Smallness Implies

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is small,

1. the dcc over K if K is finitely generated over $\mathbb{Q}$;
2. every model of $T_A + \Lambda(V) = \mathcal{Z}^N$ is atomic in L^*;
3. $T_A + \Lambda(V) = \mathcal{Z}^N$ admits elimination of quantifiers in L^*;
4. every countable model of $T_A + \Lambda(V) = \mathcal{Z}^N$ + 'infinite dimension' is ω-homogeneous.

JB $T_A + \Lambda(V) = \mathcal{Z}^N$ has a finite number of completions that have uncountable models.

Aside: Characteristic p

[Bays, Zilber] Consider

$$0 \to Z[1/p] \to V \to F_p^* \to 0.$$

where $Z[1/p]$ is the localization at p and F_p^* is an infinite dimensional algebraically closed field of characteristic p.

$T_A + \Lambda(V) = \mathcal{Z}^N$ is not small. There are $2^{\aleph_0}$ completions - distinct minimal models.

The theories must be analyzed separately; each is quasiminimal excellent.

ω-stable Implies

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is ω-stable

1. $T_A + \Lambda(V) = \mathcal{Z}^N$ admits elimination of quantifiers in L^*.
2. Every countable model of $T_A + \Lambda(V) = \mathcal{Z}^N$ + 'infinite dimension' is ω-homogeneous over elementary submodels
3. the dcc over K if K is a countable acf.

ω-stability: Kummer theory

F is a countable algebraically closed field and $b_1, \ldots b_k$ are multiplicatively independent over F.

For any m, let $F_m = F(b_1^{\frac{1}{m}}, \ldots, b_k^{\frac{1}{m}})$.
For fixed ℓ, let $G_m = \mathrm{Gal}(F_{m \cdot \ell}/F_\ell)$.

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is ω-stable.

$$G_m \approx \mathbb{A}_m(F)^k \approx (\mathcal{Z}/m\mathcal{Z})^{Nk}$$

Group automorphisms are field automorphisms

Algebraic Formulations of Excellence

Let $\mathcal{S} = \{F_s : s \subset n\}$ be an independent n-system of algebraically closed fields contained in a suitable monster $\mathcal{M}$. Denote the subfield of $\mathcal{M}$ generated by $(\bigcup_{s \subset n} F_s)$ as k.

Canonical completions

$$\mathcal{A}(k) = A^n \oplus \prod_{s \subset n} \mathcal{A}(F_s)$$

where A^n is a free Abelian group.

Almost Quasiminimal Excellence

Let **K** be a class of L-structures which admit a function cl_M mapping $X \subseteq M$ to $\mathrm{cl}_M(X) \subseteq M$ with a distinguished sort U.
K is quasiminimal if:

1. cl_M satisfies is a monotone idempotent closure operator with $\mathrm{cl}_M(X) \in \mathbf{K}$
2. For $X, Y \subset U$, $\mathrm{cl}(X) \cap \mathrm{cl}(Y) = \mathrm{cl}(X \cap Y)$.
3. cl_M satisfies exchange on U.
4. $M = \mathrm{cl}_M(U)$.
5. The usual homogeneity conditions are satisfied.

If in addition, the excellence condition holds for special subsets of U, the class is almost quasiminimal excellent.

ω-stable

Every ω-stable cover is $\aleph_1$-categorical.
But, unlike the first order case,
this doesn't automatically imply categoricity in all cardinals
-not even the continuum.

Excellence

The following are equivalent under VWGCH ($2^{\aleph_n} < 2^{\aleph_{n+1}}$)

1. The cover of $\mathbb{A}$ is categorical in all uncountable κ.
2. The cover of $\mathbb{A}$ is categorical in all $\aleph_n$ for $n < \omega$.
3. The cover of $\mathbb{A}$ is almost quasiminimal excellent.
4. The cover of $\mathbb{A}$ is almost quasiminimal excellent and $\mathbb{A}$ satisfies the algebraic conditions for excellence: has canonical completions.

Where did the set theory come from?

VWGCH: $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$.

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VWGCH: Shelah 1983

An atomic class **K** that has at least one uncountable model and is categorical in $\aleph_n$ for each $n < \omega$ is excellent.

What is true?

From the algebraic side,
If $\mathbb{A}$ is

1. $(\mathcal{C}, \cdot)$: quasiminimal excellent (Zilber)
2. $(\tilde{F}_p, \cdot)$: not small. (Bays-Zilber)
3. elliptic curve w/o cm: ω-stable (Gavrilovich/Bays)
4. elliptic curve w cm: open (ω-stable as an $\mathrm{End}(E)$-module (G))
5. higher dimensional: open

Relies on number theoretic results of Serre, Bashmakov

GALOIS TYPES: Algebraic Form

Suppose **K** has the amalgamation property. Then there is a monster model $\mathcal{M}$.

Definition

Let $M \in \mathbf{K}$, $M \prec_{\mathbf{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The Galois type of a over M is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.

We say a Galois type p over M is realized in N with $M \prec_{\mathbf{K}} N \prec_{\mathbf{K}} \mathbb{M}$ if $p \cap N \neq \emptyset$.

Tameness

tame is short for $(\aleph_0, \infty)$ tame:

Distinct Galois types differ on a countable submodel.

Grossberg and VanDieren focused on the idea of studying 'tame' abstract elementary classes.

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

If for every $N \prec_{\mathbf{K}} M$ with $|N| \leq \chi$ there exists $\alpha \in \mathrm{aut}_{\mathrm{N}}(\mathcal{M})$ with $\alpha(a) = b$,

then there exists $\alpha \in \mathrm{aut}_{\mathrm{M}}(\mathcal{M})$ with $\alpha(a) = b$.

Consequences of Tameness

Suppose **K** has arbitrarily large models and amalgamation.

Theorem (Grossberg-Vandieren)

If $\lambda > \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ is λ^+-categorical and $(\lambda, < \infty)$-tame then $\mathbf{K}$ is categorical in all $\theta \geq \lambda^+$.

Theorem (Lessmann)

If $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ is $\aleph_1$-categorical and $(\aleph_0, \infty)$-tame then $\mathbf{K}$ is categorical in all uncountable cardinals

AQE and covers

Assume: $T_A + \Lambda = \mathcal{Z}^N$ is ω-stable (VWGCH)

The following are equivalent

1. $T_A + \Lambda = \mathcal{Z}^N$ is $(\aleph_0, \infty)$-tame.
2. $T_A + \Lambda = \mathcal{Z}^N$ is almost quasiminimal excellent.
3. $T_A + \Lambda = \mathcal{Z}^N$ is categorical in all uncountable cardinalities.

Are there $\mathbb{A}$ whose covers are ω-stable but not excellent?
There are ϕ that are $\aleph_1$-categorical but not tame (Baldwin-Kolesnikov).

Mordell-Weil Theorem

For $\mathbb{A}$ a smooth elliptic curve,
If k is a finitely generated extension of $\mathbb{Q}$, $\mathbb{A}(k)$ is a finitely generated abelian group.

Smallness and Mordell-Weil

$\mathbb{A}_\ell(k)$ is the k-rational points of order ℓ.
$\mathbb{A}_{\rm tor}(k)$ is the k-rational points of any finite order.

For *any* commutative algebraic group $\mathbb{A}$:

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is small.

If k is finitely generated over $\mathbb{Q}$, $\mathbb{A}_{\rm tor}(k)$ is finite.

Pseudo-generating sequences

Let $\mathbb{V} = (V, A) \models T_A$. $\langle \tau_1, \ldots, \tau_N \rangle \in V$ is a *pseudogenerating tuple* of $\Lambda(V)$ if for each $m \in \mathcal{Z}$:

$$n_1\tau_1 + \ldots, +n_N\tau_N \in m\Lambda \text{ iff } \gcd(n_1, \ldots, n_N) \in m\mathcal{Z}.$$

We write $\mathrm{PG}^N(\tau_1, \ldots, \tau_N)$.

smallness implies finite torsion: boundedness

Definition

The algebraic group $\mathbb{A}$ is bounded if for every finitely generated extension k of the field of definition k_0 of $\mathbb{A}$, there is a d such that for every ℓ the Galois group of $\mathrm{Gal}(\tilde{k}, k)$ has only d-orbits on the set

$$X_\ell = \{\langle a_1, \ldots, a_N\rangle \in \mathbb{A}_\ell^N(\tilde{k}) : (\exists \mathbf{b})[\mathbf{a} = \exp(\mathbf{b}/\ell) \wedge \mathrm{PG}^N(\mathbf{b})]\ \}.$$

smallness implies bounded

Lemma

If $T_A + \Lambda(V)$ is small, then $\mathbb{A}$ is bounded.

Proof. Every sequence over k associated with the type $p = \mathrm{PG}^N(\mathbf{x})$ stabilizes.
Thus, there are only finitely many extensions of p to complete types over $(V(K), A(K))$ and by the homogeneity over the empty set we have a bound d on the number of orbits of pseudogenerating sets.
But since each automorphism of $\mathbb{V}$ induces an automorphism of $\mathbb{A}_\ell(\tilde{k})$ for each ℓ, we have the same bound in X_ℓ.

smallness implies finite torsion

Lemma

If $\mathbb{A}$ is bounded, then for every finitely generated extension k of the field of definition k_0, $\mathbb{A}_{\rm tors}(k)$ is finite.

Proof. We show that if $\phi(\ell) > d$, no element of $\mathbb{A}(k)$ has order ℓ.

Suppose $a \in \mathbb{A}(k)$ is a counterexample. Then a can be taken as the first element in an N-tuple $\mathbf{a}$ from $\mathbb{A}_\ell(\tilde{k})$ with $\mathbf{a} = \exp(\mathbf{b}/\ell)$ and $\mathrm{PG}^N(\mathbf{b})$ For any m that is coprime to ℓ, a^m also has order ℓ and can be extended to a sequence $\mathbf{a}_m$, so that $\mathbf{a}_m = \exp(\mathbf{b}_m/\ell)$ with $\mathrm{PG}^N(\mathbf{b}_m)$.

Thus the sequences $\mathbf{a}_m$ for $m < \ell$ and $(m, \ell) = 1$ represent distinct orbits in X_ℓ under $\mathrm{Gal}(\tilde{k}, k)$ (the first elements of the sequences are distinct elements of k). So if $\phi(\ell) > d$, we have a contradiction.

Infinitary logic

1. Difficulties
   1. No upward LS; downward LS restricted
   2. Model homogeneity not set homogeneity
   3. Galois types not types
   4. some weak set theory used
2. Advantages
   1. Ability to fix countable obstructions
   2. A further tool to understand the relation between model theoretic and number theoretic notions.