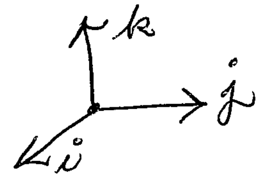
# Math 435 — Problems Due Tuesday April ~~15~~ 22, 2014

| Do Problems 1 → 6 | Skip Problem #2 |

First recall the quaternions and some facts about them:

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

$$\begin{cases} i^2 = j^2 = k^2 = -1 \\ ij = k,\ jk = i,\ ki = j \\ ji = -k,\ kj = -i,\ ik = -j \end{cases}$$




We regard $\mathbb{R}^3 = \{ri + sj + tk \mid r, s, t \in \mathbb{R}\}$

$$S^3 = \{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\}$$

These are the unit quaternions.

An element of $\mathbb{R}^3$ is called a pure quaternion.

If $w = a + ib + jc + kd$, the conjugate
define $\overline{w} = a - ib - jc - kd$, real numbers
quaternion. (Note that ... $ai = ia$.)
commute with $i, j, k$ so that

If $w = a + ib + jc + kd$ is a quaternion, define $\overline{w} = a - ib - jc - kd$, the conjugate quaternion. (Note that real numbers commute with $i, j, k$ so that $ai = ia$.)

1°. (a) Prove that if $w$ and $z$ are two quaternions $\in \mathbb{H}$, then
$$\overline{wz} = \overline{z}\,\overline{w}.$$

(b) Prove that if $w = a + bi + cj + dk$, then $w\overline{w} = a^2 + b^2 + c^2 + d^2$.

(c) Prove that if $u, v \in \mathbb{R}^3$ are pure quaternions, then
$$uv = -u \cdot v + u \times v$$
where (next page)

If $u = u_1 i + u_2 j + u_3 k$
$v = v_1 i + v_2 j + v_3 k$

then $u \times v \underset{\text{def}}{=} \mathrm{Det}\begin{bmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

is the vector cross-product in $\mathbb{R}^3$.

Note that you have shown that if $u \in \mathbb{R}^3$ then $uu = -u \cdot u + 0$.
Thus when $u \cdot u = 1$ ($u$ has unit length in $\mathbb{R}^3$), then $u^2 = -1$ in the quaternions.

2° Let $u, v, w \in \mathbb{R}^3$ be pure quaternions.
Show that $(uv)w = u(vw)$.
You can use the following formulas about the vector cross product in your proof.

$$(u \times v) \times w = -(v \cdot w)u + (u \cdot w)v$$
$$u \times (v \times w) = -(u \cdot v)w + (u \cdot w)v$$

(You do not have to prove these identities. We'll discuss them in class.)

3° Assume that quaternion multiplication is associative and that you only know that

$$i^2 = j^2 = k^2 = ijk = -1.$$

Prove the rest of the identities about $i, j$ and $k$ just from these assumptions.

4° Let $SU(2) = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \;\middle|\; \begin{matrix} z = a+ib & \text{complex} \\ w = c+id & \text{numbers} \\ \text{and } z\bar{z} + w\bar{w} = 1 & \end{matrix} \right\}$

This is a set of $2\times 2$ matrices with entries in the complex numbers as indicated above.

(a) Prove that $SU(2)$ is a group under matrix multiplication. (Note that you need to show closure under multiplication. For inverses note that if $U = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ and $U^* = \begin{pmatrix} \bar{z} & -w \\ \bar{w} & z \end{pmatrix}$ (conjugate transpose) then $UU^* = U^*U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.)

(b) Show $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$

Let $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

(continue on next page)

$E=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $I=\begin{pmatrix}i&0\\0&-i\end{pmatrix}$, $J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $K=\begin{pmatrix}0&i\\i&0\end{pmatrix}$.

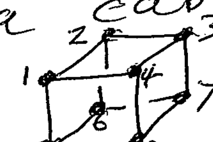
Show that $I^2=J^2=K^2=IJK=-E$.

Since you can think of $E$ as $\mathbb{1}$, this shows that $\{E, I, J, K\}$ generate a matrix model for the quaternions and that $su(2)$ is a matrix model for the unit quaternions.

5.° We have seen that if
$g=e^{u\theta}=\cos(\theta)+u\sin(\theta)$
for $u$ a unit vector in $\mathbb{R}^3$,
then $v \overset{R}{\longmapsto} gv\bar{g}$
gives us a rotation by $2\theta$ around the axis $v$ in $\mathbb{R}^3$.

Work-out the result of $S \circ R$ where
$R = 180°$ rotation about $k$.
$S = 90°$ rotation about $i$.

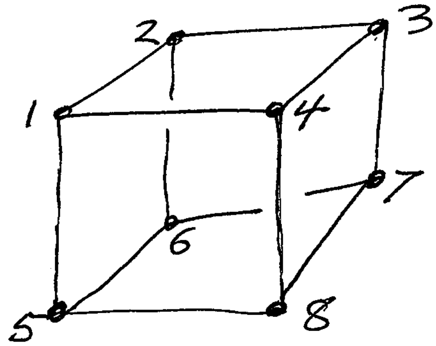
(a) algebraically using quaternions.
(b) geometrically using a cube.



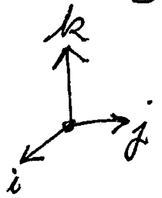
5.° (continued)

Compare this problem with what is done on pages 423 → 425 of the quaternion notes. On those pages we illustrate what happens in composing rotations of $\pi/2$ about $k$ and then $\pi/2$ about $j$. Note how Figure 10.7 shows the geometry.

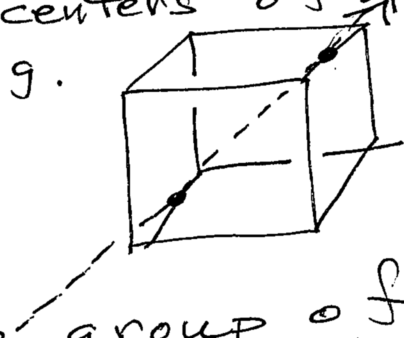
(c)




Rotational symmetries of a cube are generated by




1. The identity transformation.
2. $\frac{\pi}{2}$ rotations about $i$, $j$ or $k$.
3. $\frac{2\pi}{3}$ rotations about the cube's diagonals ($\overline{28}$, $\overline{35}$, $\overline{17}$, $\overline{46}$).
4. $\pi$ rotations about axes connecting centers of opposite edges. e.g.



Show that the group of the rotational symmetries of the cube has 24 elements.

Explain in which of the above categories are the following quaternions (next page):

$$g = \frac{\sqrt{2}}{2}(1+i)$$

$$h = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{i+j+k}{\sqrt{3}}\right)$$

$$f = \frac{k+j}{\sqrt{2}}$$

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Form the products
$gh$, $gf$, $hf$
and explain what rotations of the cube they correspond to.

6. Describe the group of rotational symmetries of a regular tetrahedron. Tell as much as you can without looking things up. Then compare with the discussions in Goodman.