ARITHMETIC IN THE FORM

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This essay explores constructions of arithmetic as departures from the Calculus of Indications of Spencer-Brown. In the course of making these constructions, we discuss their relationship with the idea of contextual interpretations. In this way, the present paper is an investigation of the role of the observer in mathematical systems.

The purpose of this essay is to investigate a foundation for the arithmetic of the natural numbers. The next section contains a review of ideas related to *Laws of Form* (Spencer-Brown, 1969), making the present essay self-contained. In fact, we give a capsule summary of our reconstruction of arithmetic in the present introduction, followed by a discussion of ideas, intents, and a description of the contents of the paper.

We regard a natural number as a row of marks that is surrounded by a mark. For example $3 = \left|\overline{\lceil\;\lceil\;\lceil\;}\right.$ represents the number 3. This

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It gives the author great pleasure to thank James Flagg and Jack Engstrom for helpful conversations and the National Science Foundation for support of this research under grant DMS-9205277. The method of doing formal arithmetic explained herein is due to Spencer-Brown (1961). The interpretation of evaluation of arbitrary expressions (and consequent proof of relative consistency with respect to ordinary arithmetic) is due to the author of this paper. I particularly thank Jack Engstrom for introducing me to Spencer-Brown's 1961 manuscript and for numerous conversations about *Laws of Form*.

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is not at variance with the Russellian idea of a number as the class of all classes in one-to-one correspondence with a given class. We are, however, taking some care in the indication of one particular class that represents the number. The marks within the mark are distinguished by their position in the space demarcated by the containing mark, not by their intrinsic structure. The notation is irreducibly part of the mathematics that it represents.

In this system of numbers we will have a rule about boundaries (Crossing) that reads $A = \overline{\overline{A}}$ for any A. This rule allows the removal of boundaries and gives us the definition of addition in the form $A + B = \overline{\overline{A}\;\overline{B}}$. Multiplication demands a copying of one number by the form of the other, and this is done via (for example) $A \times 3 = A \times \overline{\overline{\ \ }\;\overline{\ \ }\;\overline{\ \ }} = \overline{\overline{A}\;\overline{A}\;\overline{A}}$. Once again the boundary is removed from A by putting a mark around it, and $A \times 3$ becomes a mark containing three times as many marks as A. This is a capsule summary of the formal arithmetic of this paper.

This arithmetic is a formalization of actions that are intimate to the person who performs the mathematics. This investigation places the mathematics in a boundary world between the general and the particular, between the notational and the conceptual. In this sense this essay is an essay on second-order cybernetics. It is an investigation of the role of the observer in a mathematical system. It is an investigation of that observer through an insistence that the observer participate in the mathematical action and through a quest for ways in which the actions of the observer can be shifted into inherent actions of the mathematics. There is much still to be done in this field. The Turing machine (Turing, 1936) is one of the fruits of such an investigation. Another is the network viewpoint that began with the work of McCulloch and Pitts (1943). We can no longer look at numbers just as absolutes because there is such a wealth of information in the ways that we produce them, and in those ways we can begin to learn about who it is that produces a number. To paraphrase McCulloch (1960)—what is a number that she may be known by an observer, and an observer that she may know a number? This is our enterprise.

To return to the formal arithmetic described so briefly above, we decide to look more closely. The number $1 = \overline{\overline{\ \ }}$ is a container containing one mark. But our rule of crossing tells us that these concatenated marks cancel each other so that 1 is equal to the void!

From the point of view of multiplication this is hardly a problem, since the insertion of a void inside the inner marks of a number will not change it, and 1 is the number that does not change another number under multiplication. That lack of change is seen to be the consequence of an absence of action, an absence that is occasioned by the presence of a void. The number $0 = \left|\overline{}\right.$ is an empty container, and $A \times 0 = 0$ for any A exactly because there are no marks in 0 to occasion an insertion.

Now $0 \times 0 = 0$ is written as $\left|\overline{}\right. \times \left|\overline{}\right. = \left|\overline{}\right.$, and this is perilously close to the equation $\left|\overline{}\right. \left|\overline{}\right. = \left|\overline{}\right.$ which would seem to topple our arithmetic by collapsing the contents of all those faithful marks to one or none. Can we not indicate multiplication by simple juxtaposition without introducing either new notations or paradox? This is the problem that we solve in this paper. It is a nontrivial problem for the following reason. If multiplication can be done without indicating a sign of multiplication, then multiplication becomes a property of the numbers themselves. By this divestment of a sign, we shift into a new domain where the operations of arithmetic and the arithmetic itself are inseparable.

With the paradoxes resolved, it becomes possible to regard the formal arithmetic as a direct generalization of the (two-valued) primary arithmetic of *Laws of Form*.

The key to resolving the paradox that zero zeros has value zero, $\left|\overline{}\right. \left|\overline{}\right. = \left|\overline{}\right.$, and that two is not equal to one, $\left|\overline{\left|\overline{}\right. \left|\overline{}\right.}\right. \neq \left|\overline{\left|\overline{}\right.}\right.$ lies in the spatial context of these numbers. The two marks inside the boundary of the 2 are in a different space (by one crossing) from the two marks in zero times zero. Two spatial contexts mean two kinds of void —the additive void and the multiplicative void. The value of the additive void is zero. The value of the multiplicative void is one. Add nothing and you do not change the number. You added zero. Multiply by nothing and you do not change the number. You multiplied by one. That is what you did and that is what we shall do. We shall follow the void and learn arithmetic all over again.

Every time a story of the creation of number is told, the story is told again but a mite differently from the time before. After a while, the author began to think that this itself might be a property of number, and so he has tried to illustrate it, but not completely, because completeness would entail a paper of infinite length or a Turing machine with an infinite tape. In the interests of finitude, then, here is a description of the contents of the paper.

The next section discusses the nature of mathematics seen as the articulation of a distinction, the structure of *Laws of Form* and, in particular, the way in which it is proved that the marked state and the void are distinct in the primary arithmetic of *Laws of Form*. This section introduces the notations of marks and of brackets that will be used in the rest of the paper. The third section is an informal introduction to the arithmetic system that we are investigating. This section and the next explain most of the properties of this arithmetic, giving a method for understanding any expression in the mark as an indicator of arithmetical value. The fourth section ends by giving a two-boundary description of this arithmetic. In two-boundary notation (compare James, 1993) the indicator of context is the mark itself and so one obtains two types of mark—the mark surrounding an additive space and the mark surrounding a multiplicative space. Our proof (which will be given fully) of the relative consistency of the formal arithmetic is also a proof of the consistency of this notational device. Exponentiation is treated in the two-boundary notation. The fifth section starts from the beginning and develops this arithmetic in a manner analogous to *Laws of Form* through to proving that every expression in the mark reduces to one of the numbers and that the ostensibly different numbers are indeed distinct in this formal context. The sixth section, entitled Coda, is a discussion of what we have done coupled with hints about extensions to transfinite ordinals and to combinatorics. The paper has three appendices. Appendix A is a concise exposition of another arithmetic using brackets and strings of symbols that is also a generalization of *Laws of Form*. This string arithmetic can be used in a digital computer and will eventually be quite useful in parallel computation. The string arithmetic naturally brings forth issues about recursion and fixed points that also apply to the formal arithmetic in the body of the paper. Appendix B is a very short description of the system of numbers invented by John Horton Conway. The Conway numbers are constructed by making distinctions in previously created sets of numbers, and they construct all reals, ordinals, and a vast array of infinitesimals and infinite numbers as well. The appendix ends with an exposition of the proof using ordinals that Goodstein sequences terminate in zero. Appendix C discusses yet another relationship between number and Boolean algebra. This time we view Boolean algebra with values zero and infinity. The operation of crossing is to take the reciprocal of a number, and so zero and infinity are the limiting pivot of a structure for all real values between them. This leads into electrical theory, the structural iconics of tangle categories, Dirac brackets, and the beginnings of knot theory.

INDICATIONAL SPACE

The foundation of mathematics is a blank sheet of paper! Another way of putting it: Mathematics studies that universal of which a blank sheet of paper is but a single instance. The fundamental mathematical domain is a space in which distinction can be drawn.

A space in which no distinctions have been drawn is called a void. A void may be the contents of given distinction, just as a blank sheet of paper is distinguished from its surroundings. Note that it is by our definition that we regard a given space as void. The average just-cleaned backboard is not devoid of marks, but it is devoid of marks considered significant by the mathematical observer. It is void of mathematics until drawn upon. To the extent that there is nothing to distinguish, all voids are identical. In practice, we distinguish multiplicities of voids just as we use a pad with many sheets of blank paper. As we shall see, it is precisely in the subject of arithmetic that the multiplicity of voids arises.

The void is a highly flexible and reactive domain (just like our sheet of paper). One can enfold the void in such a way that a distinction is formed. (Draw a circle on a sheet of paper. Fold a cloth to form a crease.) All forms are mutable; the distinction can be erased as well as formed. Nevertheless, we envision a space of such capacity that it can act as a memory record for the distinctions that have been drawn within it.

Each distinction divides the space in which it is written into two parts. Each distinction is seen as joining its two parts into the one original space. The concept of distinction is at once the concept of joining and the concept of separation.

Call the space in which distinctions are drawn the *indicational space*. Whenever a distinction is drawn in the indicational space, that space divides into two new indicational spaces. These spaces will be called the *divisions* of the original indicational space. Each division is itself an indicational space in which the further distinctions can be made. If desired, this process can be taken to infinity, but we shall at first concentrate on spaces with a finite number of divisions. For purposes of visualization it may be of interest to contemplate the topological character of these divisions of space. Imagine that the original space is a Euclidean space of dimension three and that the distinctions are formed by describing two-dimensional spherical boundaries in the space. A single sphere divides the three-dimensional space into two spaces. One space, the interior of the sphere, is bounded. The other space, the exterior of the sphere, is unbounded. If we make a further distinction in the interior space, then the resulting spaces consist of a bounded ball (the deepest interior space), an annular region, and the unbounded region. Thus spaces created in the process of distinguishing may have different topological characteristics. We take these extra characteristics as artifacts of the specific process of distinguishing and look for what is universal.

The fact that a distinction in an unbounded Euclidean space creates one bounded region and one unbounded region is useful as an implicit index of the distinction itself. This same characteristic is true for distinctions in the plane and in the line.

If we draw distinctions in the plane by delineating circles, then we shall say that circle A is *inside* another circle B if A is inside the bounded region created by B. Otherwise A is outside B. Then in any complex pattern of nonintersecting circles (i.e., the boundaries do not intersect) it is possible to say, of any two circles in the pattern, whether one circle is inside or outside the other.

Ordinary English or mathematical text can be regarded as composed within an indicational space of dimension two, where the amount of freedom allotted in the vertical dimension is considerably less than that allotted in the horizontal direction. To compensate for this restriction, the conventions of subscripting and superscripting create additional (implicit) spaces related to any sign in a given typographical space. In this form we are accustomed to delineating distinctions via the use of parentheses and by symbols such as the square root sign $\sqrt{}$. Let us consider the square root sign first, as it constitutes a single connected sign that can indicate a distinction in typographical space. (I shall refer to the text space as *typographical space* and call the space of a given line of writing the *line space*.) The square root sign is read as enclosing the part of the line space to the right of the vertical part of the sign that is underneath the horizontal extension of the sign. Thus the square root sign makes a distinction in line space by using the already prepared distinctions of left-right and up-down that are given in the typographical space.

Typographical space is not a topological space in the usual mathematical sense of the word. Rather, it is composed of the distinctions of left and right, up and down combined with the use of characters that can be individually delineated. This means that the contents of a typographical space can be systematically encoded as a string of characters plus instructions for their recombination into the given visual format. These extra instructions constitute operators that indicate ends of lines, indentations, super- and subscripting, and, in the case of the square root sign, the beginning and end of the overline. This is a description of typographical space as it is understood by a word processor engaged in recording and displaying the contents of the space. In this regard, the square root sign is an artifact of display conventions convenient for visual processing of the contents of the typographical space. These remarks about the sequential character of the information in the typographical space apply particularly to the printed word.

For the written word, it has long been possible to regard a symbol such as the square root sign as a primitive character. As such, the sign is inherently two-dimensional, and its act of enclosure is given in the process of writing. In mathematics it has become commonplace to regard a subject as formalized if it is (or can be) written in a form that is inherently typographical. This implicit view of formalization has the base language encoded in sequential character form. From this point of view the square root sign is informal, and if an author introduces a new sign that partakes of two-dimensionality then this sign is regarded as informal until it is explained how to replace it by a satisfactory linear character format.

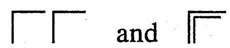
Of course, in the case of the square root, this is accomplished by the notation $\sqrt{A} = \mathrm{SQRT}(A)$, where SQRT denotes the square root operation. Nevertheless, any user of mathematical notation would not hesitate to regard the square root sign as part of the formalism, and he or she might even be surprised to hear that it is actually informal!

Why do we act out this contortion into linearity? Some reasons have been given already—printing presses and word processors need their characters and codes. Mathematical logic needs its Gödel numbers. The Gödel numbers are integer codes for elements in the formal system that are essentially derived from the sequential form of the contents. A deeper reason may be the attitude that all spaces other than a discrete linear space are higher-order constructions and not to be taken as fundamental.

It is a purpose of this discussion to dispel the idea that an indicational space is any less fundamental than a typographical space. The contents of an indicational space are not inherently linear. Nevertheless, they are easily coded in linear fashion, and we shall discuss this aspect of indicational space now.

Let the symbol $\overline{\vert\;\;\;}$ be called the (left) mark. The mark is regarded as a form of parenthesis. Its syntax is identical to that of the square root sign except that we do not take it to indicate square root; rather it denotes enclosure only. The mark makes a distinction in the line space.

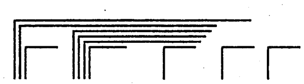
Concatenated distinctions can occur in two forms:



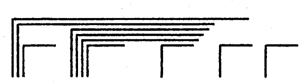
In the first instance, each mark is outside the other mark. In the second instance one mark is inside the other mark.

The linear version of the mark can be denoted by a pair of brackets. Thus $\overline{\vert\;\;\;} = \langle\,\rangle$ and $\overline{\vert\;\;\;}\,\overline{\vert\;\;\;} = \langle\,\rangle\langle\,\rangle$ while $\overline{\vert\overline{\vert\;\;\;}} = \langle\langle\,\rangle\rangle$.

An *expression* involving the mark is any disjoint collection of marks in the line space such that each mark is definitely inside or outside any other mark. Each mark in the expression makes a distinction between inside and outside. The simplest expressions are a single mark and the absence of a mark (empty expression). A typical complex expression is



In brackets this expression becomes

 $= \langle\langle\langle\,\rangle\langle\langle\langle\langle\,\rangle\langle\,\rangle\rangle\rangle\rangle\rangle\langle\,\rangle\rangle\langle\,\rangle$

In marks, an expression is well formed if it is drawn so that any mark is either inside or outside any other mark. Note again that an expression is inside a given mark if it is to the right of the vertical line in that mark and underneath the horizontal line.

It is not hard to see that the following recursive rules describe the construction of all expressions in the mark:

1. The mark and the absence of the mark are expressions.
2. If A and B are expressions, then so are AB and $\overline{|A\;}$ expressions.

Advantages of using marks rather than brackets, aside from readability, are the following facts:

Fact1: If $\overline{|A\;}$ is an expression, then A is an expression.
Fact2: If AB is an expression, then A and B are expressions.

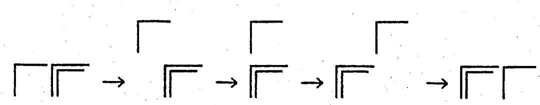
Here it is understood that A and B are composed only of marks and that A and B occupy disjoint regions in the line space. The two facts then follow immediately from our definition of an expression as a disjoint collection of marks such that any given mark is either inside or outside any other given mark.

These facts are false for expressions rendered into brackets. That is, it does not follow that A is an expression if $\langle A \rangle$ is an expression. The simplest counterexample is $\langle\, \rangle \langle\, \rangle$. It does not follow that A and B are expressions if AB is an expression. The simplest counterexample is $\langle\, \rangle$. Both of these examples derive from the fact that a distinction docketed by brackets requires a pair of brackets. A string of brackets (left and right) represents an expression if and only if every left bracket is paired with a right bracket. (The expression must begin with a left bracket. Move from left to right counting -1 for each left bracket and $+1$ for each right bracket. When the sum is zero you have just passed the right bracket that is paired with the original left bracket.) The rules for pairing brackets are a linearization of the overhang of the mark.

Conceptually, the syntax of the square root sign, or of the mark, is simpler and prior to the syntax of parentheses and brackets. This is the most elementary instance of the use of indicational space as a background for typographical space. Everything that we say from now on in the paper will be written in indicational space using the mark and expressions in the mark (plus the usual signs and symbols of English and standard mathematics). It is to be understood that this is a formal background with a perfect translation into the fully linearized formalisms of bracketing and parentheses.

Commutativity

A mark makes a distinction in indicational space. The left-to-right ordering of distinctions is (initially) an artifact of drawing distinctions along a line. Thus we see a difference in ordering between $\overline{|\overline{|\;\;}}\ \overline{|\;\;}$ and $\overline{|\;\;}\ \overline{|\overline{|\;\;}}$. As patterns of distinction in the plane these forms represent identical relations (a distinction within a distinction and a distinction outside that distinction). In fact, if we were writing in two dimensions, we could move continuously from one form to the other by shifting one mark up to the next line, sliding it along the top of the nested marks, and sliding it down into its own line again on the other side:



All such movements must respect the distinction drawn by the expression. In this sense it is not necessary to state explicitly the commutative law, $AB = BA$, for formal expressions unless we regard them as confined to the single dimension of a typographical line space. From now on we take commutativity as implicit in the definition of these forms.

Recalling Laws of Form

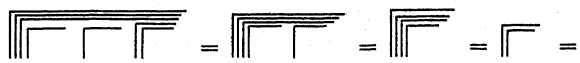
The first mathematical system discussed in *Laws of Form* (Spencer-Brown, 1969) is the *calculus of indications*, here abbreviated CI. Formally, CI is the study of the set of finite expressions in the mark $\overline{|\;\;}$ under the following two transformations:

1. (Calling) $\overline{|\;\;}\ \overline{|\;\;} = \overline{|\;\;}$
2. (Crossing) $\overline{|\overline{|\;\;}} =$

In calling, two empty marks in the same space are replaced by a single mark. In crossing, a mark containing a single empty mark is replaced by void. Both of these operations can occur in either the direction of simplification (fewer marks) or that of complexity (more marks). In crossing, a pair of nested marks can be written in a void. In calling, a copy of an empty mark can be written in the same space as that mark.

Both calling and crossing can occur in larger expressions that contain these forms.

For example, we find



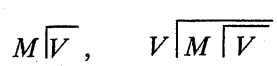
The first is accomplished by crossing, then calling, then crossing and once more crossing, reducing the expression to void.

It is not hard to see that any finite expression can be reduced by calling and crossing either to the mark or to void. It is a bit more difficult to prove that there is no sequence of applications of calling and or crossing (possibly in the direction of complexity) taking the mark to void. We shall recall the proof of this fact briefly:

Theorem: The mark, , cannot be transformed by calling and crossing to the void, .

Proof: Let the symbols M and V stand, respectively for marked and void. Adopt the following rules:

1. If V labels the inside of a mark, then M will be placed just outside it; if M labels the inside of a mark, then V will be placed just outside it. For example,

2. If a space has many copies of the symbols M and V, it is regarded as labeled by M if there is an M in the space. Otherwise it is labeled by V.

In a given expression E, label all the empty spaces (i.e., the insides of empty marks) by V. Having labeled the empty spaces by V, the above rules determine labelings of all the remaining spaces in the expression. In particular, they determine a labeling of the shallowest space in the expression. Call this label the *value* of the expression. It is now easy to see that the value of an expression is unchanged under applications of the transformations of calling and crossing in the directions of simplicity and complexity. But the mark has value M and the void has value V.

Therefore there is no sequence of transformations of the mark to the void. Q.E.D.

Example: To underscore the need for this proof, consider the system in left and right brackets generated by the transformations

1. $\rangle\langle = \quad$.
2. $\langle\langle\rangle\rangle = \quad$.

We see that calling is a consequence since $\langle\rangle\langle\rangle = \langle\quad\rangle$ by 1. Unfortunately, $\langle\rangle = \langle\langle\rangle\rangle\langle\rangle = \langle\langle\rangle\quad\rangle = \langle\langle\rangle\rangle = \quad$. Thus in this system the mark and the void are not distinguished. The theorem eliminates any worry that such muggery might go on in the calculus of indications.

This proof is the cornerstone of the calculus of indications. It shows that the system really indicates two distinct values. This allows the investigation of the properties of CI and in particular of the algebraic identities that are satisfied by expressions where equality means that one expression can be transformed to the other via calling and crossing. For example, it is not hard to see that the following equations are always true:

1. $P\,\overline{\vert P} = \overline{\vert\quad}$ for any expression P.

2. $\overline{\vert\,\overline{\vert P}} = P$ for any expression P.

3. $P\,\overline{\vert\,\overline{\vert Q}\ \overline{\vert R}} = \overline{\vert\,\overline{\vert PQ}\ \overline{\vert PR}}$ for any expressions P, Q, R.

Because any expression is equivalent to either the mark or the void, we check equations by imagining the possible cases. Thus if P is void in 1, it reads $\overline{\vert\quad} = \overline{\vert\quad}$. If P is marked, then we have $\overline{\vert\quad}\ \overline{\vert\,\overline{\vert\quad}} = \overline{\vert\quad}$ as desired. The other equations can be checked similarly. Laws of form then goes on to give a pair of equations from which all other true equations about CI can be derived. Then there is an excursion to infinity and self-reference, circuits that count, and a wealth of notes and commentary. The algebra of laws of form translates to Boolean algebra, giving a natural foundation for this subject and for the logic of propositions. The calculus of indications is also called the primary arithmetic. It is the arithmetic whose algebra gives rise to Boolean algebra.

In *Laws of Form*, Spencer-Brown remarked that one could obtain a model for the ordinary arithmetic of counting, adding, and multiplying by suitably restricting the law of calling. If we do not allow calling, then it is possible for $\overline{|\;\;}$, $\overline{|\;\;}\;\overline{|\;\;}$, $\overline{|\;\;}\;\overline{|\;\;}\;\overline{|\;\;}$,... all to be distinct forms each representing a successive number. In an unpublished manuscript, Spencer-Brown (1961) outlined the details of such a construction. The purpose of the next three sections of this paper is to give an account of arithmetic that is faithful to this original idea of Spencer-Brown. The next section explains the idea and shows the logical difficulties that arise in carrying out the idea directly. The following two sections show our happy solution to these difficulties.

ARITHMETIC

Here is a first pass through the domain of doing natural number arithmetic with indicational forms.

Let $\overline{|\;\;}$ denote the number zero (0). Let $\overline{|\,\overline{|\;\;}}$ denote the number one (1), $\overline{|\,\overline{|\;\;}\;\overline{|\;\;}}$ the number two (2), and generally $\overline{|\,\overline{|\;\;}\;\overline{|\;\;}\cdots\overline{|\;\;}}$ denotes the number n if there are n crosses enclosed by the outer mark.

$$\overline{|\;\;}$$
$$\overline{|\,\overline{|\;\;}}$$
$$\overline{|\,\overline{|\;\;}\;\overline{|\;\;}}$$
$$\overline{|\,\overline{|\;\;}\;\overline{|\;\;}\;\overline{|\;\;}}$$
$$\overline{|\,\overline{|\;\;}\;\overline{|\;\;}\;\overline{|\;\;}\;\overline{|\;\;}}$$
$$\cdots$$

We regard a natural number as a row of marks that is surrounded by a mark. The marks within the mark are distinguished by their position in the space demarcated by the containing mark, not by their intrinsic structure.

In this system of numbers we adopt a rule about boundaries (crossing) that reads $A = \overline{|\,\overline{|A}}$ for any A. This rule allows the removal of boundaries and it gives us the definition of addition in the form $A + B = \overline{|\,\overline{|A}\;\overline{|B}}$. Multiplication demands a copying of one number by the form of the other, and this is done via (for example) $A * 3 =$

$A * \overline{|\,\overline{|\ }\ \overline{|\ }\ \overline{|\ }} = \overline{|\,\overline{|A}\ \overline{|A}\ \overline{|A}}$. Once again the boundary is removed from A by putting a mark around it, and $A \times 3$ becomes a mark containing three times as many marks as A.

These relations are summarized in the following two axioms and definitions:

I. $A * \overline{|\,\overline{|B}\ \overline{|C}} = \overline{|\,\overline{|A * B}\ \overline{|A * C}}$
II. $\overline{|\,\overline{|A}} = A$

Definition of Multiplication: $A * \overline{|\,\overline{|\ }\ \overline{|\ }\cdots\overline{|\ }} = \overline{|\,\overline{|A}\ \overline{|A}\ \cdots\overline{|A}}$.

Definition of Addition: $A + B = \overline{|\,\overline{|A}\ \overline{|B}}$.

Equations I and II are to hold even if the expressions A, B, C are empty. Note that I is a tautology if A is empty as long as it is understood that multiplication by an empty word leaves an expression intact. II says that the number one ($\overline{|\,\overline{|\ }}$) can be replaced by a blank. This is true for one as a multiplicative unit. It also works in the definition of addition, for note:

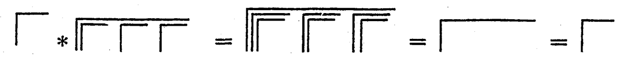
$$\overline{|A} + 1 = \overline{|A} + \overline{|\,\overline{|\ }} = \overline{|\,\overline{|\,\overline{|A}}\ \overline{|\,\overline{|\,\overline{|\ }}}} = \overline{|A\,\overline{|\ }}$$

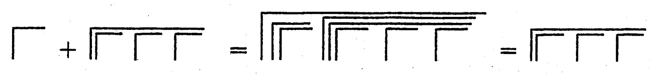
This is the desired result in this notation. The cancellation of the twice-nested cross is instrumental to the entire structure.

Addition in this system formalizes the simplest concept of the combination of collections. A number is a container filled with "that many" objects (marks). To add two numbers, cancel their containers and put the contents together into a single container. Cancellation is accomplished (through the second initial) by enclosing the form within a mark. Hence the definition of addition is precisely a transcription of the italicized statement into symbols.

The first initial expresses the distributivity of multiplication over addition: $A * (B + C) = A * \overline{|\,\overline{|B}\ \overline{|C}} = \overline{|\,\overline{|A * B}\ \overline{|A * C}} = A * B + A * C$. The definition of multiplication is actually equivalent to I in the presence of II, but we wish to emphasize it directly because it embodies the idea of multiplication. It is assumed in these axioms that A, B, and C are all of the form $\overline{|\,\overline{|\ }\ \overline{|\ }\cdots\overline{|\ }}$.

Note that the empty mark, $\ulcorner$, acts correctly as zero both for multiplication and for addition.

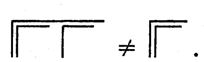




A Hint of Paradox

In formulating this arithmetic, we have used a special symbol $(*)$ for the operation of multiplication, while there is an intrinsic formulation of addition in the form $A + B = \overline{|\overline{|A}\;\overline{|B}}$. Can we drop the sign of multiplication and just denote it by the juxtaposition of forms? Consider the consequences of writing $A * B = AB$.

The situation is stranger than it seems! For zero $*$ zero $=$ zero, and hence when the mark is interpreted as zero we shall have the equation for the law of calling, $\ulcorner\ulcorner = \ulcorner$. But calling is prohibited in the interior of an expression for a number:



There is no paradox when we write $\ulcorner * \ulcorner = \ulcorner$, for this equation is distinct from the law of calling as long as multiplication is indicated by an explicit sign.

If we do eliminate the multiplication sign, we must also determine a way to evaluate any expression in the mark, not just the ones in the form of a container for a row of marks. There should be an unambiguous way to assign an integer value to any well-formed expression in the mark that generalizes our rules of calculation and allows the removal of the multiplication sign. It is the purpose of the rest of this essay to show that such a theory does indeed exist.

The resulting *arithmetical theory of forms* is a departure from *Laws of Form* but retains its spirit. The departure consists of regarding arithmetic as a contextual theory in which every division of a space has either a multiplicative or an additive context. In particular, there are two contexts for the void. Zero is the value of the additive void. One is the value of the multiplicative void.

The next section explains this point of view formally, and the following section gives an exposition of this theory of arithmetic in the manner of *Laws of Form*.

ARITHMETICAL EVALUATION OF FORMS

In the preceding section we took the representation of the numbers so that $1 = \left|\overline{\left|\overline{\;\;}\right.}\right.$, $2 = \left|\overline{\left|\overline{\;\;}\right.\;\left|\overline{\;\;}\right.}\right.$, and in general the natural number n is represented by a mark with a row of n marks inside it. We shall refer to this as a crossed row of marks. It is interesting that in order to construct this system of arithmetic, we need to have the concept of number, and certainly we are using an implicit representation of numbers as rows of strokes as in

1: |

2: | |

3: | | |

4: | | | |

...

In fact, we could also have begun with a representation where an integer is represented by an uncrossed row of marks as in

$$1 = \left|\overline{\;\;}\right., \quad 2 = \left|\overline{\;\;}\right.\left|\overline{\;\;}\right., \quad 3 = \left|\overline{\;\;}\right.\left|\overline{\;\;}\right.\left|\overline{\;\;}\right., \ldots$$

We shall discuss this possibility shortly.

In the present system, addition is represented by $a + b = \left|\overline{\left|\overline{a}\right.\;\left|\overline{b}\right.}\right.$ and the rule $\left|\overline{\left|\overline{x}\right.}\right. = x$. The first rule applies whenever a and b are crossed rows of marks. The second rule applies for any expression x. Together, the two rules produce the operation of addition by formalizing the idea that if a number is represented by a "container of marks," then the sum of two numbers is represented by removing the containers for the individual numbers and placing the result in a single container. The reader should compare this description with the formalism shown below.

Let $a = \left|\overline{R}\right.$ and $b = \left|\overline{S}\right.$, where R and S are rows of marks. Then

$$a + b = \left|\overline{\left|\overline{a}\right.\;\left|\overline{b}\right.}\right. = \left|\overline{\left|\overline{\left|\overline{R}\right.}\right.\;\left|\overline{\left|\overline{S}\right.}\right.}\right. = \left|\overline{RS}\right.$$

Of course, RS is a row of marks, the number of which is the sum of the number of marks in R plus the number of marks in S.

As we remarked in the previous section, this representation admits multiplication as well as addition via the use of the transfer

$$ab = a\overline{|R} = a\overline{|\overline{|\ }\ \overline{|\ }\cdots\overline{|\ }} = \overline{|\overline{|a}\ \overline{|a}\cdots\overline{|a}}$$

$$= \overline{|\overline{|\overline{|S}}\ \overline{|\overline{|S}}\cdots\overline{|\overline{|S}}} = \overline{|SS\cdots S}\,.$$

The result of this calculation is that ab contains a row of marks in the form $SS \cdots S$, where S is a's row and there are as many copies of S as there are marks in b's row.

Thus the rule of cancellation, $\overline{|\overline{|x}} = x$, in conjunction with transfer, $x\overline{|\overline{|y}\ \overline{|z}} = \overline{|\overline{|xy}\ \overline{|xz}}$, produces a working model of the arithmetic of the natural numbers. We recognize this arithmetic as a generalization of *Laws of Form*, where cancellation, $\overline{|\overline{|\ }} =$, is allowed, but condensation, $\overline{|\ }\ \overline{|\ } = \overline{|\ }$, has been allowed only under restricted circumstances. For example, we cannot allow condensation to occur in the expression $\overline{|\overline{|\ }\ \overline{|\ }}$ and have the number 2 retain its integrity. But $\overline{|\ }\ \overline{|\ } = \overline{|\ }$ expresses perfectly the fact that $0 \times 0 = 0$.

In order to make a consistent theory of arithmetic that embodies these formalisms, it is necessary to determine how an *arbitrary* expression involving the mark can have an arithmetical value in such a way that the restrictions on condensation are fully articulated. The clue comes from considering the alternative possibility of representing a number by an uncrossed row of marks. In this alternative universe, addition of rows of marks R and S is represented by juxtaposition RS and multiplication is represented by $\overline{|\overline{|R}\ \overline{|S}}$. To see how this works, let $S = \overline{|\ }\ \overline{|\ }\ \cdots\ \overline{|\ }$. Then

$$\overline{|\overline{|R}\ \overline{|S}} = \overline{|\overline{|R}\ \overline{|\overline{|\ }\ \overline{|\ }\cdots\overline{|\ }}} = \overline{|\overline{|\overline{|R}}\ \overline{|\overline{|R}}\cdots\overline{|\overline{|R}}}$$

$$= \overline{|\overline{|R}}\ \overline{|\overline{|R}}\cdots\overline{|\overline{|R}} = RR\cdots R$$

Thus, again using transfer, $\overline{|\overline{|R}\ \overline{|S}}$ reduces to S copies of the row R, and this is indeed the product. In this alternative arithmetic universe the mark $\overline{|\quad}$ denotes 1 rather than 0, and the crossed mark $\overline{|\overline{|\quad}}$ denotes 0 since it is equivalent to the void. A void row has zero marks.

The moral is that *the value of the mark depends on its context*. This value depends on what the mark contains or where in an expression it sits. Let us consider the case of an empty mark. Is it zero or is it one? This can look like an unsolvable conundrum until the ground shifts from looking at the marks themselves to looking at the spaces between them and surrounding them.

The empty mark with value 1 (in the alternative universe) is in an additive space. Juxtaposition represents addition in this universe. The value of the void in an additive space is 0. If you juxtapose nothing to a row of marks their number does not change.

We can now ask: Let a mark stand in an additive space; what is the character of the space within that mark? The answer: The space within the mark is a multiplicative space when the space outside it is additive. Crossing the boundary of the mark does not change the value of the expression within the mark, but it does change the context from additive to multiplicative or from multiplicative to additive.

$$\text{Additive space} = \overline{|\text{multiplicative space}}$$

$$\text{Multiplicative space} = \overline{|\text{additive space}}$$

$$\overline{|\text{Value}} = \text{value}$$

These three equations indicate the principles that we finally arrive at. Note how they work in the interpretation of the equation

$$\overline{|\overline{|\quad}} =$$

If the outer space is additive, then the void on the right-hand side has the value 0. By the principles stated above, this means that the inner mark on the left-hand side has value 0 in the multiplicative space that is enclosed by the outer mark. Indeed, the value of an empty mark in a multiplicative space is 0.

If the outer space is multiplicative, then the void on the right-hand side has the value 1. By the principles stated above, this means that the inner mark on the left-hand side has value 1 in the additive space that is enclosed by the outer mark. Indeed, the value of an empty mark in an additive space is 1.

We see that we can now specify a universe for arithmetic by specifying the type of the outermost part of the indicational space. Our standard universe has a multiplicative outer space. The alternative universe has an additive outer space. In working with expressions we change universes at each crossing but the arithmetical values that are transmitted across a crossing remain the same.

These considerations give a method for finding the arithmetical value of any given expression in a specified universe. To see this, let us specify that we are working in the standard universe. Then spaces of even depth are multiplicative and spaces of odd depth are additive. (The opposite is the case in the alternative universe.) The deepest spaces in an expression are voids of either additive or multiplicative type, depending on the parity of their depth. (Recall that the depth of a space in an expression is the number of inward crossings needed to reach it in the expression.) Since additive or multiplicative voids have the value 0 or 1, respectively, each deepest space has a value. This value is then transmitted to the spaces of one less depth, and by combining values according to the types of the spaces, these values transmit upward in the expression to give a value for that expression that is a well-defined function of the structure of the expression.

For example, in the standard (multiplicative) universe, the expression $\overline{\mid \overline{\mid\;\;\;} \; \overline{\mid\;\;\;} \; \overline{\mid\;\;\;} \; \overline{\mid \overline{\mid\;\;\;}}\;}$ has three multiplicative deepest voids of value 1 and one deepest additive void of value 0 as shown, $\overline{\mid \overline{\mid 1\;\;} \; \overline{\mid 1\;\;} \; \overline{\mid 1\;\;} \; \overline{\mid \overline{\mid 0\;\;}}\;}$. These transmit three 1's and a zero to the additive space underneath the mark, giving the sum of three 1's beneath that mark, $\overline{\mid 1 + 1 + 1 + 0} = \overline{\mid 3\;\;} = 3$. Thus the value of the original expression is 3.

In this way, any expression in the standard (or alternative) universe has a well-defined arithmetical value. We can then use this value as the definition for equivalence in our arithmetic of forms. We shall say that expressions A and B are equal ($A = B$) if they express the same value in their universe (standard or alternative). Note that expressions that are equal in one universe may be distinct in the other universe.

With this notion of equality, we can at once prove the rule of cancellation ($\overline{\mid\overline{\mid A\;}} = A$ for any expression A in either the standard or the alternative universe), for note that both sides are in the same universe and have the value A in that universe. This means that cancellation applies anywhere in any expression. On the other hand, transfer ($C|\overline{|\overline{A}\ |\overline{B}} = |\overline{|\overline{CA}\,|\overline{CB}}$) applies exactly when C is in a multiplicative space. In this case expressions A and B are also in multiplicative spaces, and transfer is exactly the fact that multiplication distributes over addition in ordinary arithmetic.

In the next section we give a formal version of this discussion of the principles of arithmetic, and we prove that two expressions in this arithmetic have the same value if and only if one can be obtained from the other by a sequence of applications of

1. $|\overline{|\overline{A}} = A$ (applicable in all contexts)
2. $|\overline{}\ |\overline{} = |\overline{}$ (applicable in multiplicative contexts)
3. $C|\overline{|\overline{A}\ |\overline{B}} = |\overline{|\overline{CA}\ |\overline{CB}}$ (applicable in multiplicative contexts)
4. $AB = BA$ (implicit in all contexts)

Note that in a given expression all contexts are determined by the choice of context for the shallowest space. For example, if the shallowest space in E is multiplicative, then all spaces of even depth in E are multiplicative and all spaces of odd depth in E are additive.

This completes the account of the relative consistency with ordinary arithmetic of this version of formal arithmetic (an extension of *Laws of Form*). There remain many interesting questions to pursue. We shall indicate some of these in the discussion at the end of this section and the sections to follow.

Remark on Ordinary Arithmetic

To construct this formal arithmetic, which we have come to see as fundamental to the structure of ordinary arithmetic, we have seen fit to use the properties and values of ordinary arithmetic as a yardstick with which to measure the structure and consistency of the formal arithmetic. This may seem circular (it is not) or externally referential (that it is). External reference could have been avoided by including a discussion of ordinary arithmetic, taking it to the desired level of our discussion. We can represent the number 3 by three strokes: $|||$. We can point out how to add and multiply rows of strokes and how to compare them by pairing off strokes from one row with strokes from another row. We can then show all the needed rules of arithmetic. We can state our belief that this system of arithmetic is consistent and that all the numbers |, ||, |||,... are distinct from one another. Then our algorithm for the evaluation of an expression can be regarded as a method for converting a given expression into a row of strokes. This algorithm requires the ability to search the indicational space and to assign parity for the determination of context. This is how we could program a computer to evaluate expressions in this arithmetic.

The whole purpose of this construction of formal arithmetic has been to show that ordinary arithmetic is really quite extraordinary—that it grows out of a mixture of our abilities to remember and repeat and our abilities to form patterns and contexts. The particular value of this version of formal arithmetic is the exhibition of the direct connection with *Laws of Form*. This is a realm where there is a duality between the operations AB and $\overline{\lvert \overline{\lvert A}\;\overline{\lvert B}}$. In our arithmetic this duality is expressed in the reversal of operations between the standard and alternative universes. This duality is best expressed by defining the mapping $\mathbf{L}: \mathbf{S} \to \mathbf{S}$ where $\mathbf{S}$ denotes the standard universe.

$$\mathbf{L}(X) = \overline{\lvert X}$$

(We have shifted X from a multiplicative space to an additive space.)

Proposition: $\mathbf{L}(X \times Y) = \mathbf{L}(X) + \mathbf{L}(Y)$. Here the operation $\times$ denotes multiplication in $\mathbf{S}$, and the operation $+$ denotes addition in $\mathbf{S}$. L is a formal precursor to the logarithm.

Proof: $\mathbf{L}(X \times Y) = \mathbf{L}(XY) = \overline{\lvert XY} = \overline{\lvert \overline{\lvert\overline{\lvert X}}\;\;\overline{\lvert\overline{\lvert Y}}} = \overline{\lvert X} + \overline{\lvert Y}$
$= \mathbf{L}(X) + \mathbf{L}(Y)$.//

The significance of the formal arithmetic resides in the fact that in it there is no longer any distinction between the elements of the arithmetic and the operations of that arithmetic. Everything rests on the values of the void.

Two-Boundary Notation

It is useful to designate the character of the space inside or outside a given mark. Let $+$ denote an additive space and $*$ denote a multiplicative space. We can adopt a special notation for two types of boundary. Let $(A) = \langle A + \rangle *$ and $[A] = \langle A * \rangle +$. Round brackets denote a boundary whose inside is additive and square brackets indicate a boundary whose inside is multiplicative. Thus,

$$() = \langle + \rangle * = 0 \quad \text{in multiplicative context}$$

$$[\,] = \langle * \rangle + = 1 \quad \text{in additive context}$$

With this notation we have the rules $[(A)] = A$, $([A]) = A$, and $A([B][C]) = ([AB][AC])$ for *arithmetical expressions*. An arithmetical expression is an expression in which there are no contradictory indications of type in a given space. Thus all marks at a given depth parity must be of the same type (square or round). Thus $[\,][\,][\,]$, $[()()()]$, $[([\,])()]$ are arithmetical expressions, and $[\,]()$ is not an arithmetical expression.

Note that when we use this two-boundary notation, an arithmetical expression declares its allegiance to either the standard or the alternative universe. Thus $[\,][\,][\,]$ is 3 in the alternative universe and $[()()()]$ is 3 in the standard universe. In order to multiply or add two expressions they must be in the same universe. Note that $[\,][\,] \neq [\,]$, but $()() = ()$. The quotient map to the calculus of indications is simply the map that forgets the distinction between round and square brackets.

The rules for using this two-boundary notation all follow from its interpretation in spatial context. In particular, we understand that $[\,] = 1$, $() = 0$ but that this does not imply that $[\,]()$ has a value because this expression is not arithmetical. Conversely, every arithmetical expression has a unique value by the results of this section and their formalization in the next section.

Remark: A two-boundary notation for arithmetic similar to the one given above is used in James (1993). The value of the approach given in this paper to such notations is that by developing the notation as a shorthand for the full spatial context of arithmetical forms we prove its relative consistency with ordinary arithmetic.

Exponentiation

We now define a^b for a and b in the formal arithmetic. Here it is most convenient to use the two-boundary notation. *By definition, a^b is defined only if a and b are in opposite universes.*

Then we define $a^{([\,][\,]\cdots[\,])} = [(a)(a)\cdots(a)]$ when $a = [\,]\cdots[\,]$. In other words, we insert a into a pair of round brackets for every pair of square brackets in the reduced form of the exponent. Here a is in the additive universe. By inserting into switched brackets in this manner we obtain the product of a with itself b times.

It is interesting to note that this definition of exponentiation gives it the appearance of a transfer of a into b. In Spencer-Brown's original manuscript on arithmetic, exponentiation was denoted by such a transfer. By making the definition in this form, we keep the spirit of this idea without its logical difficulties.

Note also the special case in which there are no square brackets in b. Then $a^{([\,][\,]\cdots[\,])} = [(a)(a)\cdots(a)]$ becomes $a^{()} = [\,]$. Thus $a^0 = 1$ for any a, and this includes the case $0^0 = 1$.

FORMAL ARITHMETIC

Void

The void is unmarked.

Arithmetic

In arithmetic there are two voids: the additive void and the multiplicative void.

Value

The value of the additive void is 0. The value of the multiplicative void is 1.

Context

All divisions of a space are endowed with a context. A context is either multiplicative or additive.

Operation

In an additive space juxtaposition creates addition. In a multiplicative space juxtaposition creates multiplication.

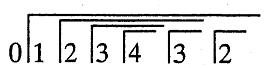
Depth

An expression divides the indicational space into separated spaces. A connected component of this severance is said to be a *division* of the expression. The depth of a division is the number of inward crossings from the unbounded region that are needed to reach the space of the division.

The unbounded region has depth zero and is referred to as the shallowest space.

In the case of a single mark, there are two divisions—one of depth zero and one of depth one: $\mathit{0}\langle \mathit{1}\rangle$ (depth is italicized).

Depths are indicated for all divisions in the following example:


$$= \mathit{0}\langle \mathit{1}\langle \mathit{2}\langle \mathit{3}\langle \mathit{4}\rangle\rangle\langle \mathit{3}\rangle\rangle\langle \mathit{2}\rangle\rangle$$

Note that this example has one division of depth 0, one division of depth 1, two divisions of depth 2, two divisions of depth 3, and one division of depth 4. The number of divisions of a given depth in an expression is the number of spaces distinguished by the expression at that depth level.

Parity for the Standard Universe

Let the parity of the depth of a division designate its context. Let a division of depth zero have multiplicative context. Let a division of depth one have additive context. Depth of even parity connotes multiplicative context. Depth of odd parity connotes additive context.

Expressed Value

Let the value indicated by a token (a token is a cross in the context of an expression) be equal to the value indicated on the inside of the token.

Since the additive void has value zero, the value indicated by a single mark in a multiplicative space, $= \langle\ \rangle$, is zero.

Since the multiplicative void has value one, the value indicated by a crossed mark in a multiplicative space, $= \langle\langle\ \rangle\rangle$, is one.

Conversely, the value indicated by a single mark in an additive space is one; the value indicated by a crossed mark in an additive space is zero.

Crossing

Crossing a boundary changes the *context* of the content. Crossing the boundary does not change the value of the content.

Example: ⟨⟨ ⟩⟨ ⟩⟨ ⟩⟩ contains three divisions of depth two. These are each voids in a multiplicative context. Each void has value 1.

⟨⟨1⟩⟨1⟩⟨1⟩⟩

Therefore the value indicated on the inside of the expression

⟨⟨ ⟩⟨ ⟩⟨ ⟩⟩

is $1 + 1 + 1 = 3$. Therefore the value of the expression is equal to 3 (in a multiplicative context).

The inside of the expression ⟨ ⟩ is an additive void, hence the expression has value zero. ⟨⟨ ⟩⟨ ⟩⟩⟨ ⟩ has value equal to 2 times 0. Hence ⟨⟨ ⟩⟨ ⟩⟩⟨ ⟩ has value zero.

Evaluation

Each deepest space is an additive or multiplicative void. Label each deepest space according to the value of this void. Use these evaluations of deepest spaces to obtain, by multiplication or addition, the value of each next shallowest space.

Continue until an evaluation of the space of depth zero is obtained. This is the value of the expression.

Value

Each expression indicates a unique arithmetical value. Two expressions shall be connected with a sign of equality, =, if they express identical values.

Example: Evaluate ⟨⟨ ⟩⟨ ⟩⟨ ⟩⟩⟨⟨ ⟩⟨ ⟩⟩. The deepest spaces are all multiplicative voids.

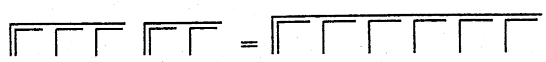
⟨⟨1⟩⟨1⟩⟨1⟩⟩⟨⟨1⟩⟨1⟩⟩

Depth one is additive.

⟨3⟩⟨2⟩

Depth zero is multiplicative.

6



⟨⟨ ⟩⟨ ⟩⟨ ⟩⟩⟨⟨ ⟩⟨ ⟩⟩ = ⟨⟨ ⟩⟨ ⟩⟨ ⟩⟨ ⟩⟨ ⟩⟨ ⟩⟩

Example: ⟨⟨⟨ ⟩⟩⟨ ⟩⟨ ⟩⟨ ⟩⟨ ⟩⟩. The deepest level is of depth 3, hence additive. The value of the additive void is zero. The other deepest spaces are of even parity and hence multiplicative.

⟨⟨⟨0⟩⟩⟨1⟩⟨1⟩⟨1⟩⟨1⟩⟩

Crossing a boundary does not change the value of a content.

⟨0 + 1 + 1 + 1 + 1⟩

⟨4⟩

4

⟨⟨⟨ ⟩⟩⟨ ⟩⟨ ⟩⟨ ⟩⟨ ⟩⟩ = ⟨⟨ ⟩⟨ ⟩⟨ ⟩⟨ ⟩⟩

Transfer

$\langle\langle x\rangle\langle y\rangle\rangle z = \langle\langle xz\rangle\langle yz\rangle\rangle$ when the space pervading z is multiplicative.

Proof: The shallowest space is multiplicative. Therefore if x, y, z are the values of the respective parts of the expression, then the expression reads, upon evaluation,

$$(x + y) * z = x * z + y * z$$

Since this equation is true in ordinary arithmetic, the proposition is proved. Q.E.D.

Generalized Transfer

$\langle\langle a\rangle\langle b\rangle\langle c\rangle\rangle d = \langle\langle ad\rangle\langle bd\rangle\langle cd\rangle\rangle$ when the space pervading d is multiplicative.

Proof:

$$\langle\langle a\rangle\langle b\rangle\langle c\rangle\rangle d$$

$$= \langle\langle\langle\langle a\rangle\langle b\rangle\rangle\rangle\langle c\rangle\rangle d$$

$$= \langle\langle\langle\langle a\rangle\langle b\rangle\rangle\rangle\langle c\rangle\rangle d$$

$$= \langle\langle\langle\langle a\rangle\langle b\rangle\rangle d\rangle\langle cd\rangle\rangle$$

$$= \langle\langle\langle\langle ad\rangle\langle bd\rangle\rangle\rangle\langle cd\rangle\rangle$$

$$= \langle \qquad \langle ad\rangle\langle bd\rangle \qquad \langle cd\rangle\rangle$$

$$= \langle\langle ad\rangle\langle bd\rangle\langle cd\rangle\rangle \quad \text{Q.E.D.}$$

Example. The Transfer of Zero:

$$\langle\langle\,\rangle\langle\,\rangle\cdots\langle\,\rangle\rangle\langle\,\rangle = \langle\langle\langle\,\rangle\rangle\langle\langle\,\rangle\rangle\cdots\langle\langle\,\rangle\rangle\rangle = \langle \qquad \rangle = \langle\,\rangle.$$

Thus $n \times 0 = 0$ for any n.

The Algebraic Expression of Value

Theorem: Two expressions in the formal arithmetic express the same value if and only if one expression can be obtained from the other by a sequence of replacements as indicated by the initials given below:

(i) $\langle\langle A\rangle\rangle = A$ (applicable in all contexts)
(ii) $\langle\,\rangle\langle\,\rangle = \langle\,\rangle$ (applicable in multiplicative space)
(iii) $\langle\langle A\rangle\langle B\rangle\rangle C = \langle\langle AC\rangle\langle BC\rangle\rangle$ (applicable only when the space containing C is multiplicative)

Proof: The theorem will be proved by showing that any expression may be reduced to one of the forms $\langle\,\rangle$, $\langle\langle\,\rangle\rangle$, $\langle\langle\,\rangle\langle\,\rangle\rangle$, $\langle\langle\,\rangle\langle\,\rangle\langle\,\rangle\rangle, \ldots$. These forms all have distinct arithmetical values (0, 1, 2, 3, . . .). Furthermore, we have shown that the replacements given above do not change the value of an expression. Thus it suffices to prove that any expression can be reduced to one of the standard forms. Let E be an expression, and consider a deepest space in E. This space is empty and is surrounded by a mark $\langle\ \rangle$. This mark is either adjacent to another mark as in $\langle\ \rangle\langle\ \rangle$, or it is enclosed in a mark as in $\langle\langle\ \rangle\rangle$, or the mark stands alone in a shallowest space. In the last case the expression is already reduced. In the next to last case, the expression can be simplified by one application of (i) above. In the first case the two adjacent marks $\langle\ \rangle\langle\ \rangle$ may stand in a division of even depth, in which case the simplification $\langle\ \rangle\langle\ \rangle \rightarrow \langle\ \rangle$ is available by (ii). Finally, if the two adjacent marks stand in a division of odd depth, then they stand in a form $\langle\langle\langle\ \rangle\langle\ \rangle B\rangle A\rangle$ where A is in a multiplicative space (even depth) and A is either empty or A has the form $A = \langle\langle\ \rangle\langle\ \rangle \cdots \langle\ \rangle\rangle$ and B has the form $B = \langle\ \rangle\langle\ \rangle \cdots \langle\ \rangle$ (including the empty expression). (These remarks follow from the assumption that the two adjacent marks have maximal depth.) Under these circumstances a simplification (in the sense of depth reduction) is obtained after an application of (iii) followed by (i) as in $\langle\langle\langle\ \rangle\langle\ \rangle\rangle\langle\langle\ \rangle\langle\ \rangle\langle\ \rangle\rangle\rangle = \text{(iii)}\langle\langle\langle\langle\langle\ \rangle\langle\ \rangle\langle\ \rangle\rangle\rangle\langle\langle\langle\ \rangle\langle\ \rangle\langle\ \rangle\rangle\rangle\rangle\rangle =$ $\text{(i)}\langle\langle\langle\ \rangle\langle\ \rangle\langle\ \rangle\langle\ \rangle\langle\ \rangle\rangle\rangle = \text{(i)}\langle\ \rangle\langle\ \rangle\langle\ \rangle\langle\ \rangle\langle\ \rangle$, This completes the proof of the theorem.//

Comment

The system that we have elucidated is based on a contextual interpretation of evaluation. With the convention that the shallowest space is multiplicative, we have

$$0 = \langle\ \rangle, \quad 1 = \langle\langle\ \rangle\rangle, \quad 2 = \langle\langle\ \rangle\langle\ \rangle\rangle, \quad 3 = \langle\langle\ \rangle\langle\ \rangle\langle\ \rangle\rangle, \quad \text{and so on}$$

Since context is determined by the parity of depth, it is automatically true that $\langle\langle A\rangle\rangle = A$ for any A.

Multiplication goes via transfer:

$$3 * a = \langle\langle\ \rangle\langle\ \rangle\langle\ \rangle\rangle * \langle a\rangle = \langle\langle\langle a\rangle\rangle\langle\langle a\rangle\rangle\langle\langle a\rangle\rangle\rangle = \langle aaa\rangle$$

To define addition in this context we must shift the space. This is accomplished by the strategem $a + b = \langle\langle a\rangle\langle b\rangle\rangle$. By our conventions

$\langle a \rangle$ and $\langle b \rangle$ are sharing an additive space. Therefore they must be added together. The resulting value is their sum.

Nota bene

$\langle\langle\,\rangle\langle\,\rangle\rangle + \langle\langle\,\rangle\langle\,\rangle\rangle$

$$= \langle\langle\langle\langle\,\rangle\langle\,\rangle\rangle\rangle\langle\langle\langle\,\rangle\langle\,\rangle\rangle\rangle\rangle$$

$$= \langle\langle\,\rangle\langle\,\rangle\ \langle\,\rangle\langle\,\rangle\rangle$$

$$= \langle\langle\,\rangle\langle\,\rangle\langle\,\rangle\langle\,\rangle\rangle$$

With this proof that $2 + 2 = 4$ we conclude our presentation of formal arithmetic.

CODA

We have written natural number arithmetic in the context of an indicational calculus. The benefits of such an endeavor are not yet fully apparent. This is partially because we are so familiar with arithmetic that foundational investigations of this sort take a while to sink in. The important conceptual turn that made our construction possible was the realization that the numbers one and zero held equal standing as the values of a void (an empty space) in multiplicative and additive contexts. From this vantage, and through an interpretation of the multiplicative and additive contexts at different levels of parity in the form, we obtain an infinity of values for these expressions that are in one-to-one correspondence with the well-known arithmetic of the natural numbers.

Since the original context of *Laws of Form* (Spencer-Brown, 1969) led directly to Boolean algebra, our study shows that ordinary arithmetic can be regarded as a direct relative of Boolean algebra in which the structure is more complex due to extra distinctions. Our arithmetic has the two-valued calculus of indications as a natural quotient structure. In this sense ordinary arithmetic deserves to be seen as a multiple-valued logic or as a form of reason in which each integer is an imaginary value! (Compare Kauffman, 1987.)

In fact, the relationship of arithmetic to recursion and imaginary value (in the sense of Kauffman, 1987) is highlighted here by the theorem of the preceding section. This theorem states that the arithmetic of forms is generated by the rules $\langle\langle\,\rangle\rangle$ = "nothing" (in any depth parity), $\langle\ \rangle\langle\ \rangle = \langle\ \rangle$ (in even depth parity), and $\langle\langle A\rangle\langle B\rangle\rangle C = \langle\langle AC\rangle\langle BC\rangle\rangle$ (in even depth parity). Taken abstractly, it is not obvious at first that these rules will generate ordinary arithmetic. In this order of presentation, the first rule is the law of crossing, the second is a restricted form of the law of calling, and the third rule can be viewed as the essence of multiplication. It makes each expression at even depth into an operator ready to accept a multiple copy of another expression into itself.

Addition and multiplication are inherent in the structure that generates the natural numbers from expressions in indicational space. Numbers are not collections that can be combined via the properties of addition and multiplication. Rather, numbers are the *residues* of the equivalences of forms under the processes of crossing, calling, juxtaposition, and recursion, the true precursors to addition and multiplication. The numbers occur inextricably mixed with their own operative powers. All these remarks are at once obvious to anyone with experience in the theory of numbers, where the most basic theorems involve the relations between additivity and multiplicativity. Here, we have taken a step back into the foundations and have seen that numbers and their operations are inseparable at the most fundamental level. The next stage involves going forward into the basics of number theory, using these forms as a guide.

It may be helpful, in this coda, to take a peek into this forward direction. Consider the

Puzzle: How many ways are there to distribute N identical balls in M bins?

Consider the case where $N = 7$ and $M = 3$: seven balls in three bins. One such arrangement could be $[\langle\ \rangle\langle\ \rangle\langle\ \rangle][\langle\ \rangle\langle\ \rangle][\langle\ \rangle\langle\ \rangle]$ where the bins are the outer paired brackets and the balls are the inner paired brackets, $\langle\ \rangle$. Another look at this pattern reveals that by transmuting each pair of opposing brackets into a pair of matched brackets, *we transmute the seven balls in three bins to nine balls in one bin, with two specially selected balls* (the transmuted bracket pairs).

$[\langle\ \rangle\langle\ \rangle\langle\ \rangle][\langle\ \rangle\langle\ \rangle][\langle\ \rangle\langle\ \rangle]$

$[\langle\ \rangle\langle\ \rangle\langle\ \rangle[\ ]\langle\ \rangle\langle\ \rangle[\ ]\langle\ \rangle\langle\ \rangle]$

The walls between the bins can be transmuted into balls! The distribution of seven balls in three bins is equivalent to the number of ways to choose two distinct balls from nine distinct balls, hence 36 ways in all.

The same reasoning applies to the puzzle of N balls in M bins. Transmute to $M - 1$ balls chosen from $N + M - 1$ balls, and hence the answer to the puzzle is $(N + M - 1)!/(M - 1)!\,N!$

The next stage in formal arithmetic is its interpenetration with the combinatorics of number theory. Another direction is the care and description of transfinite forms and numbers. We have $0 = \langle\,\rangle$, $1 = \langle\langle\,\rangle\rangle$, $2 = \langle\langle\,\rangle\langle\,\rangle\rangle, \ldots$ and so the first transfinite ordinal is certainly $\omega = \langle\langle\,\rangle\langle\,\rangle\langle\,\rangle\langle\,\rangle\ldots\rangle$.

Recursively, $\omega = \langle a \rangle$ where $a = \langle\,\rangle a$ (at depth 1). That is, if $a = \langle\,\rangle\langle\,\rangle\langle\,\rangle\cdots$ (ad infinitum), then $\langle\,\rangle a = \langle\,\rangle\langle\,\rangle\langle\,\rangle\langle\,\rangle\cdots = \langle\,\rangle\langle\,\rangle\langle\,\rangle\cdots = a$, but $a\langle\,\rangle = \langle\,\rangle\langle\,\rangle\langle\,\rangle\cdots\langle\,\rangle$ is not equal to a. This is a primary instance of noncommutativity in the form.

Then we can define $\omega + 1 = \langle a\langle\,\rangle\rangle$, $\omega + 2 = \langle a\langle\,\rangle\langle\,\rangle\rangle, \ldots$ $\omega + \omega = \langle aa\rangle, \ldots, \ldots,$ $\omega^{\omega} = \langle aaa\ldots\rangle = \langle b\rangle$ where $b = ab., \ldots, \ldots$

$$\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}}}} = \tau = \tau^{\tau}$$

It is useful to have a concrete picture of the (countable) transfinite ordinals as they provide a mirror of processes in the finite arithmetic (see Appendix 2) and they themselves are a significant arithmetical domain. The countable ordinals are so constructed so that there are no infinite descending sequences of ordinals (they are well ordered). Here we see the ordinals as an extension of formal arithmetic to expressions with infinitely many marks.

Sets, Ordinals, and the Vanishing of the Comma

It is interesting to note that we could have begun our reconstruction of number by following the formalism of ordinary set theory in the context of using the mark $(\overline{|\quad})$. Then the mark has the syntax of the set theoretic brackets: $\{\,\} = \overline{|\quad}$. In this setting $\overline{|\overline{|A}} = \{\{A\}\}$ is distinct from A, so the law of crossing is not allowed. However, with an appropriate choice of notation, we do get the law of calling. This choice is as follows: Let it be understood that a finite *set* S is an expression in the mark in the form $S = |\overline{E}$ where E is any finite expression in the mark. Since E is equal to a juxtaposition of expressions contained in individual marks, $E = |\overline{E_1}\,|\overline{E_2} \cdots |\overline{E_n} = S_1S_2 \cdots S_n$, we have that E is itself a juxtaposition of the sets $S_1, S_2, \ldots, S_n$. We shall call these sets the *members* of S. Thus we can write sets without putting commas between their members. For example, $\{\{\,\}, \{\{\,\}\}\} = |\overline{|\overline{\;\;}\;|\overline{|\overline{\;\;}}}$.

Note that the empty set, $|\overline{\;\;}$, is distinct from the void. The empty set is a container for the void.

To obtain sets, rather than multisets (a multiset can have many duplicate members as in $\{\{\,\}, \{\,\}\}$), we adopt the rule

Condensation: $ST = S$ whenever S and T are sets with the same members.

Note that condensation implies that $|\overline{EF} = |\overline{FE}$ for any expressions E and F, and that $|\overline{\;\;}\;|\overline{\;\;} = |\overline{\;\;}$. Condensation implies the commutativity of forms that we need for sets. We say that sets S and T are *equal*, $S = T$, if they have the same members. This specifies the construction of finite set theory as a version of the indicational calculus of *Laws of Form* with a restriction on crossing and a special condensation rule that generalizes calling.

In this language, the natural numbers appear as a sequence of sets with distinct members (the von Neumann construction in set theory):

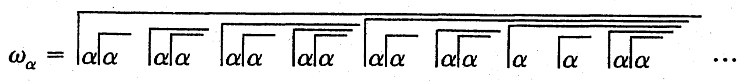
$$|\overline{\;\;} = 0$$

$$|\overline{|\overline{\;\;}} = 1$$

$$|\overline{|\overline{\;\;}\;|\overline{|\overline{\;\;}}} = 2$$

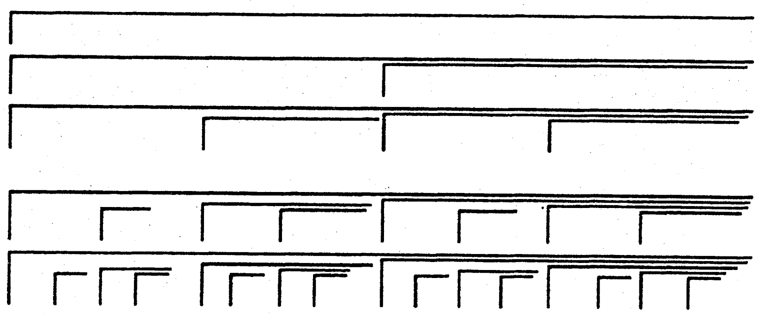
$$|\overline{|\overline{\;\;}\;|\overline{|\overline{\;\;}}\;|\overline{|\overline{\;\;}\;|\overline{|\overline{\;\;}}}} = 3$$

In general, if $N = |\overline{E}$ then $N + 1 = |\overline{E\;|\overline{E}}$. This gives an inductive construction of distinct sets such that $N + 1$ always has one more member than N. The limit of this construction is the first countable ordinal ω. It is interesting to generalize these notations for sets to infinite expressions in the mark that indicate infinite ordinals. For example, let

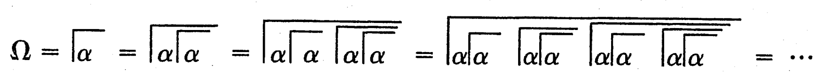


where α is any ordinal or α is void. If α is void, then $\omega_\alpha = \omega$. In general, $\omega_\alpha = \omega_{\alpha\overline{|\alpha}}$ so that $\omega = \omega_1 = \omega_2 = \cdots$. However, $\omega \neq \omega_\omega$.

It is interesting to examine the form that results from taking the reverse limit in this process. By the reverse limit I mean the limit process indicated below:



In this view, the limit takes the form $\Omega = \overline{|\alpha}$ where $\alpha = \alpha\overline{|\alpha}$. This expresses the limit form as a fixed point, and the recursion associated with this fixed point reconstitutes the individual stages of the counting process.



We leave further investigation of these sets and ordinals to the reader.

Imaginary Value

The ordinals are just a small part of the recursive possibilities inherent in infinite forms. Other directions include

$J = \langle J \rangle = \langle\langle\langle\langle\langle \cdots \rangle\rangle\rangle\rangle\rangle$

(See Kauffman and Varela, 1980 and Varela and Gogueu, 1978.) In the Boolean arena this represents the first imaginary value. It is invariant under crossing (which represents negation). In formal arithmetic we do not have $JJ = J$ and so J represents a number whose powers are inscrutable. J is the simplest infinite form with no deepest space. In the set theory discussed above, J is a member of itself. These explorations of the foundations of arithmetic can be extended in these and other directions. We will continue this story in a sequel to this paper.

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APPENDIX A: STRING ARITHMETIC

The purpose of this appendix is to introduce a method of integer arithmetic via string manipulations. The significance of this symbolic arithmetic is its simplicity and the ease with which it can be implemented on a computer. Since most small computers have string-handling capacity, this provides a method for doing arithmetic (and experimental number theory) with integers far beyond the usual range of such a machine.

The equivalence relation on the strings uses local modifications. We expect that this feature will be of use in parallel processing systems (compare Rothstein, 1977).

The arithmetic described in this appendix is distinct from the formal arithmetic given in the body of the paper. It constitutes a separate way to use the parenthetical boundaries to create numbers. Nevertheless the construction given herein is a close relative of *Laws of Form* and of the primary arithmetic. We include it both for its usefulness and for ease of comparison with the main constructions of the paper.

We consider strings involving three symbols: $*$, $\langle$, and $\rangle$. With an appropriate equivalence relation, each (admissible) string reduces to a normal form. The juxtaposition of two strings corresponds to addition. Multiplication involves a reentry process.

The most primitive representation of a natural number will be a row of adjacent stars ($*$). Zero is the empty row, and $*$, $**$, $***$, $****, \ldots$ represent 1, 2, 3, 4, ..., respectively.

This discussion will be restricted to a notation that corresponds to the base 2. We adopt the convention (interpretation) that

$\langle x \rangle$ stands for twice x.

Thus

$$* = *$$

$$** = \langle * \rangle$$

$$*** = \langle * \rangle *$$

$$**** = \langle\langle * \rangle\rangle$$

$$***** = \langle\langle * \rangle\rangle *$$

$$****** = \langle\langle * \rangle * \rangle$$

$$******* = \langle\langle * \rangle * \rangle *$$

$$******** = \langle\langle\langle * \rangle\rangle\rangle$$

The reduced expressions on the right-hand side correspond directly to the binary form of the number. For example,

$$\langle\langle * \rangle\rangle * \to 101$$

$$\langle\langle\langle * \rangle * \rangle\rangle * \rangle \to 11010$$

with the 1's matching the stars and the 0's matching blank spaces between brackets to the right of center.

Note that

$$1 = * \to 1$$

$$2 = \langle * \rangle \to 10$$

$$4 = \langle\langle * \rangle\rangle \to 100$$

$$8 = \langle\langle\langle * \rangle\rangle\rangle \to 1000$$

$$16 = \langle\langle\langle\langle * \rangle\rangle\rangle\rangle \to 10000$$

$$32 = \langle\langle\langle\langle\langle * \rangle\rangle\rangle\rangle\rangle \to 100000$$

$$\ldots$$

Addition is represented by juxtaposition of strings. $X + Y$ is the string XY. For example,

$$2 + 3 = ** + *** = ***** = 5.$$

The distributive law for multiplication, $2 \times A + 2 \times B = 2 \times (A + B)$, becomes (in string language)

$$\langle X \rangle \langle Y \rangle = \langle XY \rangle$$

Thus the underlying instruction for performing the distributive law is the cancellation of opposing brackets.

$$\rangle \langle \; =$$

Thus

$$2 + 2 = \langle * \rangle + \langle * \rangle = \langle * \rangle \langle * \rangle = \langle ** \rangle = \langle \langle * \rangle \rangle = 4$$

We adopt the following rules:

Rules for String Arithmetic

1. $\rangle \langle \rightleftharpoons$ (blank)
2. $** \rightleftharpoons \langle * \rangle$
3. $* W \rightleftharpoons W *$ (W is any string representing a natural number)

Successive application of 1 and 2 is sufficient to bring any string of stars into normal form. For example

$$*******$$

$$\langle * \rangle *****$$

$$\langle * \rangle \langle * \rangle ***$$

$$\langle * \rangle \langle * \rangle \langle * \rangle *$$

$$\langle ** \rangle \langle * \rangle *$$

$$\langle *** \rangle *$$

$$\langle \langle * \rangle * \rangle *$$

Rule 3 is necessary when juxtaposing two strings as in

$\langle\langle * \rangle * \rangle * \langle * \rangle$

$\langle\langle * \rangle * \rangle\langle * \rangle *$ (applying 3)

$\langle\langle * \rangle * * \rangle *$

$\langle\langle * \rangle\langle * \rangle\rangle *$

$\langle\langle * * \rangle\rangle *$

$\langle\langle\langle * \rangle\rangle\rangle *$

It is easy to see that successive application of 1, 2, and 3 is sufficient to perform addition via juxtaposition:

$$
\begin{aligned}
X + Y &= XY \\
7 + 1 &= \langle\langle * \rangle * \rangle * + * \\
&= \langle\langle * \rangle * \rangle * * \\
&= \langle\langle * \rangle * \rangle\langle * \rangle \\
&= \langle\langle * \rangle * * \rangle \\
&= \langle\langle * \rangle\langle * \rangle\rangle \\
&= \langle\langle ** \rangle\rangle \\
&= \langle\langle\langle * \rangle\rangle\rangle = 8
\end{aligned}
$$

The brackets take the roles of place markers *and* "carry operators." Carrying happens automatically when place-marker boundaries collapse through the rule $\rangle\langle =$ (void).

Comment on Rule 3: Any string made up of stars $(*)$ and well-formed parentheses (each left bracket, $\langle$, appropriately paired with a right bracket, $\rangle$, in the usual nested sense) actually represents a unique number. Thus rule 3 can be replaced by an instruction to count left parentheses until a right parenthesis is reached and then to count an equal number of right parentheses. Stars can be passed across such sequences. This counting can be accomplished in the string space by adding the following local rules:

$$*\langle\, = \langle a, \quad a\langle\, = \langle ab, \quad b\langle\, = \langle b, \quad bb\rangle = b\rangle b, \quad ab\rangle = \rangle a, \quad a\rangle = \rangle *$$

For example,

$$*\langle\langle\langle\;\rangle\rangle\rangle = \langle a\langle\langle\;\rangle\rangle\rangle = \langle\langle ab\langle\;\rangle\rangle\rangle = \langle\langle a\langle b\rangle\rangle\rangle = \langle\langle\langle abb\rangle\rangle\rangle$$
$$= \langle\langle\langle ab\rangle b\rangle\rangle = \langle\langle\langle\;\rangle ab\rangle\rangle = \langle\langle\langle\;\rangle\rangle a\rangle = \langle\langle\langle\;\rangle\rangle\rangle * .$$

With these rules in place, the string arithmetic can be handled entirely by local decisions in string handling and hence is susceptible to parallel processing.

Arbitrary Base

It is easy to generalize these ideas to represent numbers to any base. For example, base 3 will have $*$, $**$, and $*** = \langle * \rangle$. The bracket is a direct analog of positional notation, but it also plays an active role via the reduction $\rangle\langle\, =$. This obviates the need for extra carry rules and makes the arithmetic self-contained.

Multiplication and the Cybernetics of Arithmetic

Multiplication of strings X and Y, denoted $X \# Y$, is accomplished by *substituting Y for every occurrence of* $*$ *in* X. For example, $\langle * \rangle * \# Y = \langle Y \rangle Y$ is the form of multiplication by 3.

We now describe the program for multiplication. This is a description of an actual computer algorithm. The reader may also enjoy thinking of this program as a specific instruction whereby the observer of this system of arithmetic is drawn into the system so that her actions become part of the system itself.

The program performs multiplication $X \# Y$ by first placing a dummy symbol, @, for every appearance of $*$ in X and then substituting Y for the leftmost @, reducing the new string as much as possible, then repeating the process. The result $5 \times 2 = 10$ becomes:

$$\langle\langle * \rangle\rangle * \# \langle * \rangle$$

MULTIPLICATION BY REENTRY

$\langle\langle @\rangle\rangle *$

$\langle\langle @\rangle\rangle @$

$\langle\langle\langle *\rangle\rangle\rangle @$

$\langle\langle\langle *\rangle\rangle\rangle\langle *\rangle$

$\langle\langle\langle *\rangle\rangle *\rangle$

finis!

Paradox

This mode of multiplication leads into a recursive paradox. For example, if we try $* \# *@$ then the program substitutes @ for $*$ on the left of #, and then it substitutes $*@$ for this dummy. Then the program looks at the result, sees another dummy @, and again substitutes $*@$, forming $**@ = \langle *\rangle @$. This continues until the machine breaks down or the string space gives out.

Note that in writing $A \# B$ we give the computer two instructions. The first instruction is to replace all occurrences of $*$ in A by the sign @. The second instruction is the substitutional definition $@ = B$. It is no surprise that we shall get an infinite recursion if B contains an instance of @. Indeed, the definition $@ = *@$ is a simple and charming example of this phenomenon. The machine has learned to count!

$* \# *@$

MULTIPLICATION BY REENTRY

$*@$

$**@$

$\langle *\rangle @$

$\langle *\rangle *@$

$\langle *\rangle **@$

⟨∗⟩⟨∗⟩@

⟨∗∗⟩@

⟨⟨∗⟩⟩@

⟨⟨∗⟩⟩∗@

⟨⟨∗⟩⟩∗∗@

⟨⟨∗⟩⟩⟨∗⟩@

⟨⟨∗⟩∗⟩@

⟨⟨∗⟩∗⟩∗@

⟨⟨∗⟩∗⟩∗∗@

⟨⟨∗⟩∗⟩⟨∗⟩@

⟨⟨∗⟩∗∗⟩@

⟨⟨∗⟩⟨∗⟩⟩@

⟨⟨∗∗⟩⟩@

⟨⟨⟨∗⟩⟩⟩@

⟨⟨⟨∗⟩⟩⟩∗@

...

It is remarkable that adding multiplication to additive arithmetic brings the computer so close to the edge of paradox. This is a reflection of the deeper results about incompleteness of formal systems that attempt to describe arithmetic fully and consistently (Gödel, 1962).

Recursive Structure

It is worth noting the form of our definition of multiplication of strings. We regard each string as a function by replacing the ∗ by a variable. Then $F(*) \# G(*) = F(@)$ where $@ = G(*)$. Normally, this process stops with $F(*) \# G(*) = F(G(*))$. However, if $G(*)$ contains an @ sign, the process will continue indefinitely.

In the case of $* \# *@$, we have $@ = *@$, giving rise to the recursion $@ = *@ = **@ = ***@$ and so on, with reductions taking place in the strings so produced. This follows the paradigm that sees a recursive program as a fixed point in an appropriate formalism. It is of interest to see how this issue arises naturally in the string arithmetic.

Relation with the Lambda Calculus

It is interesting to note that this definition of multiplication as insertion can be formalized in terms of the Church-Curry lambda calculus (Barendreght, 1984). That is, we regard an expression in brackets and stars as a function with variables in the place of the stars. For a function of one variable, one writes $\lambda x.f(x)$ to denote the function and composition is indicated by the equation

$$\lambda x.f(x)\ \lambda y.g(y) = \lambda y.f(g(y)).$$

The lambda part of the expression keeps track of the variables and their precedence order in the case of many variables.

The unstoppable recursion resulting from $@ = *@$ then corresponds to the basic fixed-point construction of the lambda calculus:

If $G = \lambda x.F(xx)$ so that $Gx = F(xx)$, then $GG = F(GG)$.

In string arithmetic we can define $Gx = *xx$. Then $GG = *GG$ so that $GG = @$. In fact, any recursive program can be reformulated as a fixed point in analogy to the above formalism.

It is the recursive context of arithmetic that gives rise to so many fascinating problems. Here we see that this context is fundamental and that it is a direct consequence of considering the interface between the operator of the arithmetic and the arithmetic itself. The operator of the arithmetic is herself a mark (a boundary or interface) in a calculus of indications that is broad enough to encompass all of these operations. In a formal sense this larger operational calculus is best modeled by a structure like the lambda calculus. Nevertheless, with an appropriate understanding of context we see that the original arena of indicational space is sufficient to support all these complexities both inside and outside arithmetic.

An Example — The Collatz Problem

The Collatz problem is a well-known unsolved question in iterative arithmetic.

The Collatz Iteration:

1. Choose an odd number N.
2. If N is odd, replace N by $3N + 1$.
3. If N is even, replace N by $N/2$.
4. If $N = 1$ then stop, otherwise return to 2.

For every odd integer N that anyone has ever tried, this process has terminated at 1. It has yet to be proved that this is always the case.

In terms of string arithmetic, $3N + 1$ is accomplished as follows:

$$N = \langle W \rangle * \text{ (an odd number)}$$

$$\begin{aligned} 3N + 1 &= \langle N \rangle N * \\ &= \langle \langle W \rangle * \rangle \langle W \rangle * * \\ &= \langle \langle W * \rangle W \rangle \end{aligned}$$

Thus after division by 2 (once) the recursion becomes

$$\langle W \rangle * \rightarrow \langle W * \rangle W$$

The program for the Collatz problem, using string arithmetic, performs this transformation and then removes outer brackets to divide by 2. Such a program can significantly extend the capacity of a small computer, and it will be useful on larger systems with parallel architecture for investigating the upper reaches of the Collatz problem.

Ad Astra per Mysterium

A more mystical reason for writing the Collatz in string arithmetic is the hope that there is a subtle pattern right in the notation of string arithmetic that will show the secrets of the iteration. With this aim in mind, it is quite compelling to watch the symbols streaming past on the computer screen, knowing that every single step of the process is being performed before one's eyes. String arithmetic is written on top of the highest language available on a given computer. It has the effect of turning the computer inside out, so that the full workings of arithmetic are there for all to see, a formal system propelled electronically into time as well as space.

APPENDIX B: ORDINALS, CONWAY NUMBERS AND THE GOODSTEIN SEQUENCE

The purpose of this appendix is to say a few words about a very general construction of numbers due to Conway (1976), to discuss ordinals in this context, and to show how ordinals can be used as "imaginary values" to prove very real properties of recursions on natural numbers.

In the body of the paper we have restricted ourselves to finite ordinals except for an indication in the coda that our constructions can be extended to infinite ordinals.

Conway builds all numbers from void by a process of creation. Each number is uniquely determined by two sets of previously created numbers, called the left set and the right set. No member of the left set is greater than or equal to any member of the right set. If L and R denote a given pair of left and right sets, then the new number generated by them is denoted by the brackets $\{L \mid R\}$. If L and R are both empty, then it is true that every member of L is less than every member of R. Thus $\{\ |\ \}$ is a number. It is zero; $0 = \{\ |\ \}$.

Order

$\langle L \mid R\rangle \geq \langle L' \mid R'\rangle$ exactly when no member of R is $\leq \langle L' \mid R'\rangle$ and $\langle L \mid R\rangle$ is $\leq$ no member of L'. $X = Y$ means $X \leq Y$ and $Y \leq X$.

Operation

$$X + Y = \{x^L + Y, X + y^L \mid x^R + Y, X + y^R\}.$$

Here x^L denotes any element of the left set of X, and x^R denotes any element of the right set of X. The left and right sets of $X + Y$ consist of all the sums indicated within the brackets.

$$XY = \{x^LY + Xy^L - x^Ly^L, x^RY + Xy^R - x^Ry^R \mid$$
$$x^LY + Xy^R - x^Ly^R, x^RY + Xy^L - x^Ry^R\}$$

These definitions, handled inductively, are sufficient to create the ordered field of Conway numbers. These include all real numbers, all ordinals, and an extension of the real numbers that includes infinitesimals of all orders and infinite numbers of all orders.

Here are the first few numbers. $\{\mid\} = 0$. Having created 0, we see that it is true that every member of the empty set is less than every member of the set consisting of zero. Thus $\{\ \mid 0\}$ is a number, as is $\{0 \mid\ \}$. By the definition above, $\{\ \mid 0\} < \{\ \mid 0\}$. In fact, $-1 = \{\ \mid 0\}$ and $+1 = \{0 \mid\ \}$, $2 = \{1 \mid\ \}$, $3 = \{2 \mid\ \}, \dots, \omega = \langle\{0, 1, 2, \dots\} \mid \rangle$, $\omega + 1 = \langle \omega \mid \ \rangle, \dots, 2\omega = \langle\{\omega,\ \omega + 1,\ \omega = 2, \dots\} \mid \rangle, \dots$ while $-2 = \{\ \mid -1\}$, $-3 = \{\ \mid -2\}, \dots\ -\omega = \{\ \mid -1, -2, -3, \dots\}$ and so on. Conway numbers continue on out into ordinals of all sizes, but that is not all!

Fractions occur early on. Consider $\{0 \mid 1\}$. We have $0 = \{\mid\} < \{0 \mid 1\} < \{0 \mid\ \} = 1$. In fact, $\{0 \mid 1\} = 1/2$, $\{0 \mid 1/2\} = 1/4, \dots, \{0 \mid 1/2^n\} = \{0 \mid 1/2^{n+1}\}, \dots, \{0 \mid 1, 1/2, 1/4, \dots, 1/2^n, \dots\} = 1/\omega$. The number $1/\omega$ is the first infinitesimal in the system. This system of numbers bootstraps itself into existence. Read Conway (1976) for a full accounting of this.

Goodstein Sequence

A natural number M is said to be written in base N (N a natural number greater than 1) when it is presented in the form $M = a_0 + a_1N + a_2N^2 + a_3N^3 + \cdots + a_kN^k$ where $0 \le a_i \le N - 1$. M is said to be in *complete base* N if it is presented in base N and all the exponents of powers of N are also presented in base N, and their exponents as well, as far as this process can go. We say this inductively by saying that all the exponents of the powers of N are in complete base N and that any nonnegative integer less than N is in complete base N. For example,

$$M = 1 + 2 + 2^{1+2+2^{2^{(2^{1+2}+2)}}}$$

is in complete base 2.

Given M written in complete base N, we can form $\{R\}(M)$, a new number in complete base R, by replacing all occurrences of N in M with R. For example, with M above we have

$$\{3\}M = 1 + 3 + 3^{1+3+3^{1+3+33^{3^{(3^{1+3}+3)}}}}$$

A *Goodstein sequence* $[N]$ is obtained by starting with a number M in complete base N. Let $G = M(N)$ where the N denotes that M is written in complete base N. Let $G' = (\{N+1\}M) - 1$ written in complete base $N+1$. In other words, G' is obtained from G by shifting G to complete base $N+1$, subtracting 1, and rewriting the result in complete base $N+1$. The Goodstein sequence obtained from a given G is the sequence $G, G', G'', G''', \ldots$. At each state the next number is obtained by shifting the value of the base upward by one, subtracting one, and rewriting in the new base completely.

Often the values of the numbers in a Goodstein sequence go up very rapidly. For example, if we start with $G = 2 + 2^{2+2^2}$, then $G' = 1 + 3^{3+3^3}$, $G'' = 4^{4+4^4}$,

$$G''' = 5^{5+5^5} - 1 = 1 + 4 \times 5 + 4 \times 5^2 + \cdots + 4 \times 5^{4+5^5}$$

Sometimes a Goodstein sequence is decreasing. For example, suppose that $G = 4$ in base 5. Then $G' = 3$ in base 6, $G'' = 2$ in base 7, $G''' = 1$ in base 8, and $G'''' = 0$ in base 9. Remarkably, the following theorem is true (see Loebl & Nesetril, 1990).

Theorem: Every Goodstein sequence terminates in 0 after a finite number of steps.

Proof: Given M in complete base N, let $\{\omega\}M$ denote the ordinal obtained by replacing every occurrence of N in M by the ordinal ω.

For example, if $G = 1 + 5 + 5^{1+5^3}$ in complete base 5, then $\{\omega\}G = 1 + \omega + \omega^{1+\omega^3}$.

Let $G, G', G'', \ldots$ be a Goodstein sequence. Each term is written in a complete base and so we can form $\{\omega\}G^{(k)}$ for each term $G^{(k)}$ in the sequence. It is easy to see that the sequence of ordinals $\{\omega\}G$, $\{\omega\}G'$, $\{\omega\}G''$, $\ldots$ is strictly decreasing. Since there are no infinitely descending sequences of countable ordinals, this sequence must terminate. It follows at once that the original Goodstein sequence hits zero after a finite number of iterations.//

It has been shown that there is no proof of this theorem if the methods of reasoning are restricted to Peano arithmetic (Loebl & Nesetril, 1990). The ordinals form a metalanguage in which a beautifully simple proof can be cradled. From the point of view of this essay and from the point of view of the Conway numbers, the ordinals are part of the whole expression of the concept of a number. They are part and parcel of the context of the construction of number. By including the ordinals in the system, we can avail ourselves of powerful modes of reasoning about the recursive properties of ordinary finite integers. These are reasons for rethinking the construction of numbers and the form of arithmetic. By making the metalanguage part of the language itself, we begin a path to knowing the nature of number.

McCulloch (1960) wrote a remarkable essay entitled "What is a number that a man may know it, and a man that he may know a number?" His answer to the first part was the Bertrand Russell definition of number: "A number is the class of all classes that are in one-to-one correspondence with a particular class." Thus the number seven is the class of all classes in one-to-one correspondence with the days of the week. This definition is knowable even though we do not apprehend the infinity of all classes that it refers to. The knowledge is a knowledge of actions to find correspondences and relationships. The definition of number is both a concept and a program for action.

Here we have given a different approach to number, and it is to be hoped that this too will shed light on both sides of McCulloch's question. All these systems of number have arisen from different ways of construing the form—different ways of finding how an observer (a mark in the form) can appear in relation to his or her context. Context and the observer of that context arise concurrently. Insight forms in the shifting boundary that we take to be a distinction.

APPENDIX C: DUALITY, ELECTRICITY, AND TANGLES

In formal arithmetic, as described in the body of this paper, addition and multiplication are dual to one another through two spatial contexts. In the multiplicative context multiplication is given by juxtaposition of forms while addition has the formula $A + B = \langle\langle A\rangle\langle B\rangle\rangle$. Dually, in additive space, addition is given by juxtaposition of forms, while multiplication is given by the formula $A * B = \langle\langle A\rangle\langle B\rangle\rangle$.

This duality is a reflection of the well-known symmetry of the operations of *or* ($\cup$) and *and* ($\cap$) in Boolean algebra. That is, we have in Boolean algebra that $a \cap b = \langle\langle a\rangle \cup \langle b\rangle\rangle$ where $\langle x\rangle$ denotes *not* x. This is DeMorgan's law. Our construction of formal arithmetic has followed this pattern, but operations are interpreted quite differently.

In fact, in formal arithmetic the equation $A + B = \langle\langle A\rangle\langle B\rangle\rangle$ in multiplicative space is a referral to addition as juxtaposition in additive space. That is $\langle\langle A\rangle\langle B\rangle\rangle = \langle\langle A\rangle + \langle B\rangle\rangle$ in the context of formal arithmetic. When we take the quotient to get the primary arithmetic (calculus of indications) from formal arithmetic we do so by forgetting the difference between multiplicative and additive spaces. The operations $*$ and $+$ are no longer distinguished, but the operations of juxtaposition, AB, and crossed juxtaposition $\langle\langle A\rangle\langle B\rangle\rangle$ remain and become the patterns of the Boolean algebra. The Boolean marked, $\langle\ \rangle$, and unmarked, $\langle\langle\ \rangle\rangle$, values correspond to the elementary arithmetical values of 0 and 1, but which is which depends upon the choice of context in the arithmetic.

Switching Circuits and Electrical Circuits

There is another way in which Boolean algebra extends to ordinary arithmetic. In order to describe this correspondence it is convenient to use the symbol $+$ for Boolean *or* ($\cup$) and $*$ for Boolean *and* ($\cap$). We shall adopt this convention for the rest of the appendix.

Recall (Shannon, 1938) that switching circuits can be described by Boolean algebra. See Figure 1. An open switch corresponds, let us say,

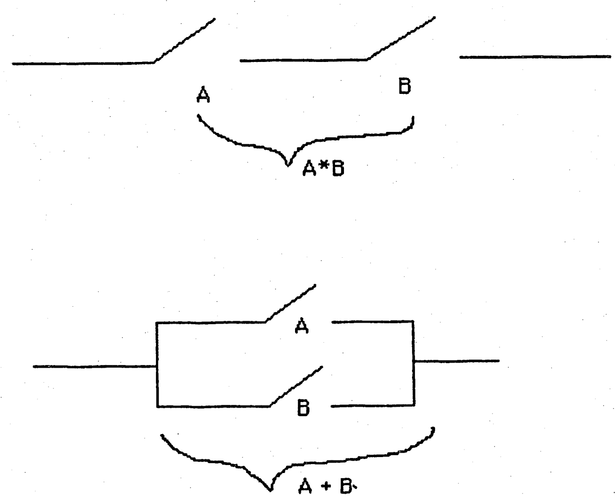

Figure 1.

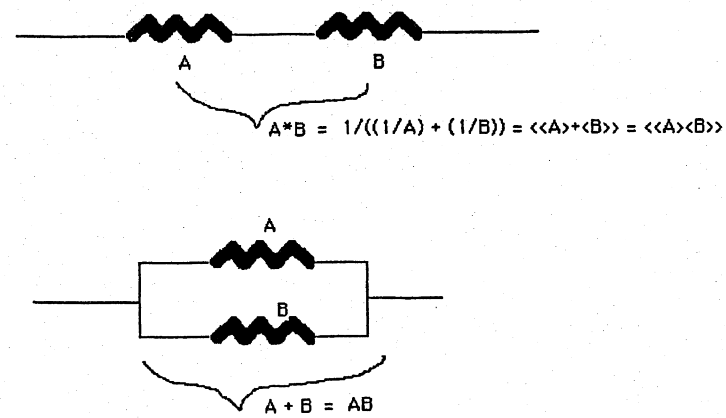

Figure 2.

to the unmarked state and a closed switch corresponds to the marked state. Parallel and series connections of switches then produce the analogs of the operations of $+$ and $*$. However, an open switch and a closed switch are the extremes of possibility for the conductance of an electrical circuit with one input and one output. Conductance (equal to the inverse of resistance) can take any value from ∞ (closed switch with perfect conductance) to 0 (an open circuit). Two elements with conductances A and B have conductance $A + B$ when connected in parallel. The conductance of a series connection of A and B is equal to $A * B = 1/((1/A) + (1/B)) = A \times B/(A + B)$ where $A \times B$ denotes the multiplication of real numbers and $A * B$ is defined by this equation. If we let $\langle A \rangle = 1/A$, then $A * B = \langle\langle A \rangle + \langle B \rangle\rangle$, giving a formula in parallel with the DeMorgan law. See Figure 2.

Let us therefore regard the Boolean values for switching circuits as 0 and ∞. We have

$$\langle 0 \rangle = 1/0 = \infty$$

$$\langle \infty \rangle = 1/\infty = 0$$

$$\infty + \infty = \infty$$

$$\infty + 0 = \infty = \infty + 0$$

By definition, $a * b = \langle\langle a\rangle + \langle b\rangle\rangle$ so that

$$0*0 = \langle\infty + \infty\rangle = \langle\infty\rangle = 0$$

$$0*\infty = \langle\langle 0\rangle + \langle\infty\rangle\rangle = \langle\infty + 0\rangle = \langle\infty\rangle = 0$$

$$\infty*\infty = \langle\langle\infty\rangle + \langle\infty\rangle\rangle = \langle 0 + 0\rangle = \langle 0\rangle = \infty$$

We match this Boolean formalism for circuits with laws of form by taking zero as the unmarked state, $0 = \langle\ \rangle$, and infinity as the unmarked state, $\infty = \langle\langle\ \rangle\rangle$, with $A * B$ designated by the juxtaposition of forms. This makes zero the dominant value.

$$a * b = ab \quad \text{(juxtaposition of forms)}$$

$$a + b = \langle\langle a\rangle\langle b\rangle\rangle.$$

The calculus embeds in a real number calculus via $\langle a\rangle = 1/a$. Then $a * b = 1/((1/a) + (1/b)) = a \times b/(a + b)$.

In this way the (positive) real numbers are a natural extension of the Boolean algebra of zero and infinity. The Boolean $+$ extends directly to the addition of reals. The Boolean $*$ becomes the reciprocal of the product of reciprocals. The system extends directly to negative reals in the same way, and consistently so long as one takes $-\infty = \infty$ (so that $-0 = 0$).

Brackets, and the Principle of Idemposition

Remarkably, there is another route to this "electrical" extension of Boolean algebra. This route is based in fundamental considerations about the boundaries of forms. The basic principle is the

Principle of Idemposition (M. Aintree, private communication, 1979). *Common boundaries cancel.* A distinction is undone when the boundary is seen to join the two sides. A distinction is undone when the boundary is seen as an interface for communication between the two sides. A distinction is undone when edges fuse.

To illustrate this principle, let us consider curves in the plane. Let two curves that share a bit of boundary cancel along the boundary as shown in Figure 3. Two closed curves situated next to one another and sharing a bit of boundary amalgamate to a single curve (Figure 4). Two nested curves sharing all their boundary cancel completely (Figure 5). Thus we see the *forms* of calling and crossing as special cases of idemposition.

The second illustration of the principle of idemposition shows how separate forms with matched boundaries amalgamate to create a single form. For example, see Figure 6. In these illustrations we see arcs with points for boundary joining to form simple closed curves. A typographical instance of this is the understanding that we bring to paired brackets:

$$\langle\, \rangle$$

In the eye of the reader, paired brackets are not separate. They are not separate exactly because the boundaries of the individual kets in the pair have been matched so that the bracket is really a simple closed curve in the plane.

With this in mind, let us write $\delta = \langle\, \rangle$ for a simple pair of paired brackets. We now consider the form $M = \rangle\langle$ of a pair of antibrackets. These brackets are not paired, but note the following calculation:

$$M = \rangle\langle.$$

$$MM = \rangle\langle\, \rangle\langle = \rangle\delta\langle$$

$$MMM = \rangle\langle\, \rangle\langle\rangle\langle = \rangle\delta\delta\langle$$

$$MMMM = \rangle\langle\, \rangle\langle\, \rangle\langle\, \rangle\langle = \rangle\delta\delta\delta\langle$$

and so on.

Although left and right kets do not commute ($\langle\, \rangle \neq \rangle\langle$), it is convenient to allow, in this context $\delta\langle = \langle\delta$ and $\delta\rangle = \rangle\delta$. Then $MM = \rangle\delta\langle = \delta\rangle\langle = \delta M$. This is an abstraction of the quantum mechanical formalism of Dirac (1958). Thus we are led to a new generalization of the mark where $\lceil\ \lceil\ = \delta\lceil\ $.

In the case $\delta = 0$, we have $MM = 0M$ and this can be identified with the value 0. In fact, it is exactly at this point that a significant contact occurs between circuit arithmetic and the "deformation" of logic suggested by $MM = \delta M$ or $\lceil\ \lceil\ = \delta\lceil\ $. In the following, we

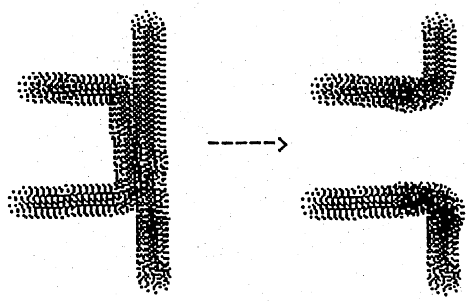

Figure 3.

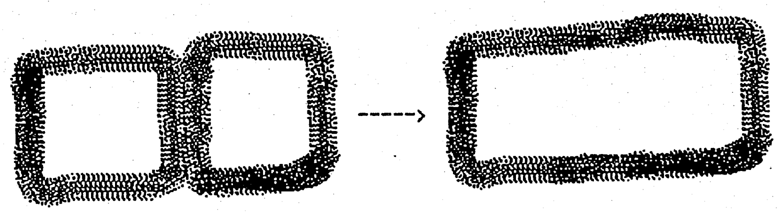

Figure 4.

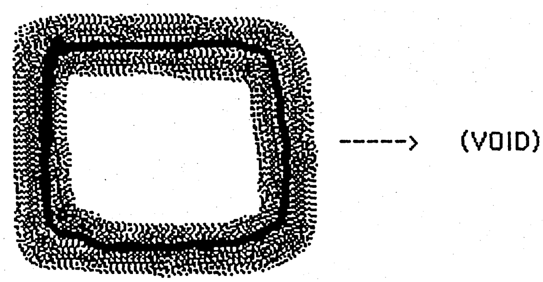

Figure 5.

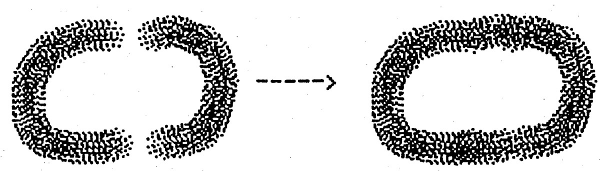

Figure 6.

shall make a direct identification of M as the Boolean zero. It is, however, important to distinguish M from zero in this context. Here $M = \rangle\langle$ is a "square root of zero" when $\langle\,\rangle$ is equal to zero. The following formalism gives a hint of what transpires in this domain.

Let $X = aM + b$ and $Y = cM + d$ where a, b, c, d are numbers commuting with the forms. Multiply and add formally and assume that $MM = 0M$. Then $X * Y = (ad + bc)M + bd$. Thus if we define the fraction of X to be $\mathrm{FRAC}(aM + b) = a/b$, then $\mathrm{FRAC}(X * Y) = \mathrm{FRAC}(X) + \mathrm{FRAC}(Y)$. This is another formal logarithm. The result is that $1/\mathrm{FRAC}(X) = \mathrm{COND}(X)$ (the *conductance* of X) is the evaluation so that $\mathrm{COND}(X + Y) = \mathrm{COND}(X) + \mathrm{COND}(Y)$ and $\mathrm{COND}(X * Y)$ is given as the reciprocal of the sum of the reciprocals of conductances of X and Y. This places the pattern of electrical evaluations right at the source of this deformation of logic.

With $\delta = 1$ we recover the formalism of calling, but if δ is not one, then it is clear that a form of counting has ensued. In fact, this leads directly to an iconic generalization of Boolean algebra. We take "2-strand boxes" as the basic elements. These boxes have two input strands and two output strands just as the form $M = \rangle\langle$ has two left legs and two right legs. We take the strands of the boxes up to topological deformation. Given a box, A, there is a notion of A^{-1} as shown in Figure 7.

Note that $(A^{-1})^{-1} = \mathrm{rot}(A)$ where $\mathrm{rot}(A)$ is the result of turning A upside down (Figure 8).

For the simplest boxes 0 and ∞ we have $0^{-1} = \infty$ and $\infty^{-1} = 0$, $\mathrm{rot}(0) = 0$, and $\mathrm{rot}(\infty) = \infty$, so this inverse is of order two in the beginning (Figure 9).

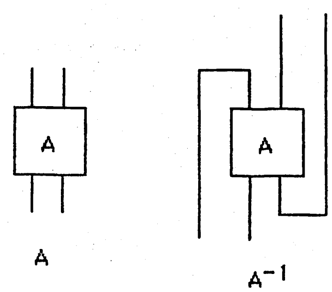

Figure 7.

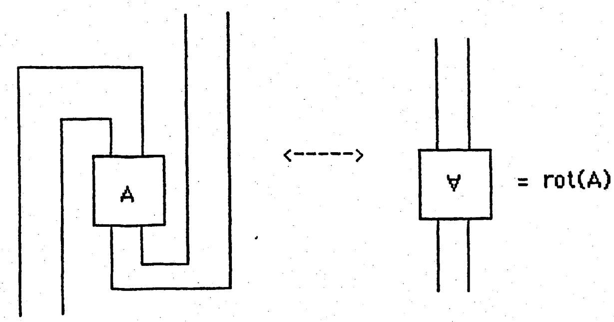

Figure 8.

We define the product of boxes AB by attaching the input strands of B to the output strands of A. We define the sum of boxes, $A + B$, by the analog of parallel circuit connection as shown in Figure 10. Note that we take a simple closed curve in the plane to represent the commuting value δ.

In fact, $\operatorname{rot}(A + B) = ((A^{-1})(B^{-1}))^{-1}$, as shown in Figure 11. This shows that sum and product are categorically dual in exactly the fashion of Boolean algebra.

The boxes generated by our icons for 0 and ∞ yield a generalization of Boolean algebra. The widest generalization occurs when we place topological weaving patterns inside these boxes. These weaving boxes are called *tangles*. The simplest instances of tangles are the left- and right-handed crossings shown in Figure 12. Let these be denoted by I and J. We see that $I * J = \infty$ and that $I + J = 0$. This is exactly consistent with $I = +1$ and $J = -1$ $(I * J = 1/((1/I) + (1/J)))$.

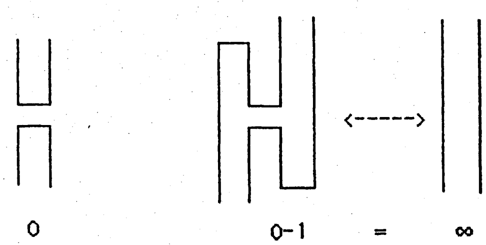

Figure 9.

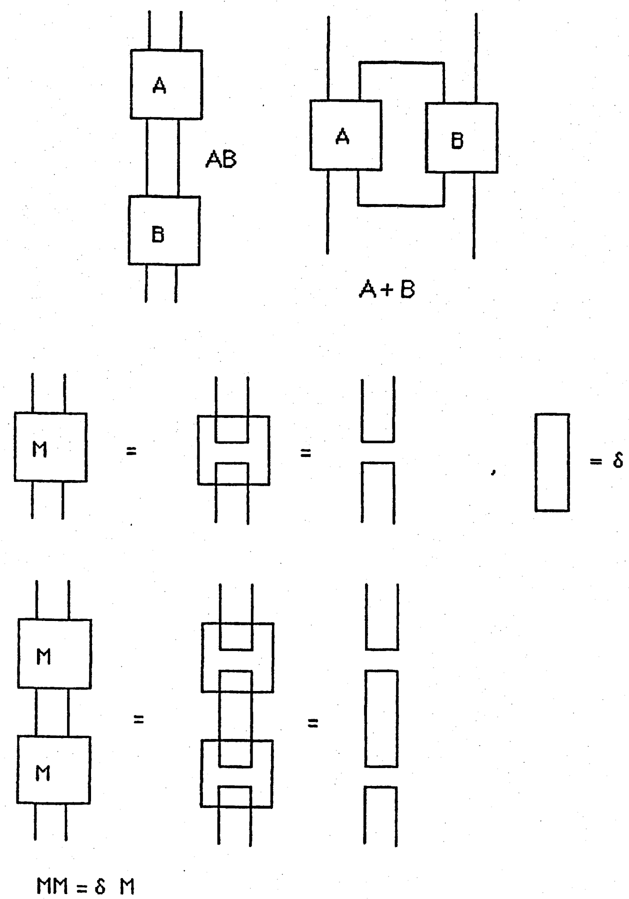

Figure 10.

Counting begins with the introduction of the weave, a kind of abstract form of Quipu!

There is not room here for a full exposition of how the weaves are interrelated with arithmetic, but it should be clear to the reader that the two-stranded braid $I + I + \cdots + I$ represents the integer n in this system, that $J + J + \cdots + J$ represents $-n$, and that fractions occur naturally through the use of tangle products as well as tangle sums. The

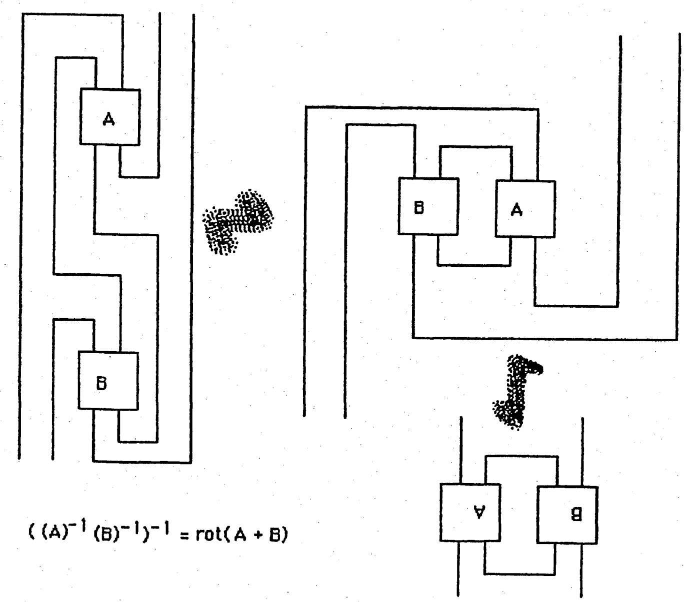

Figure 11.

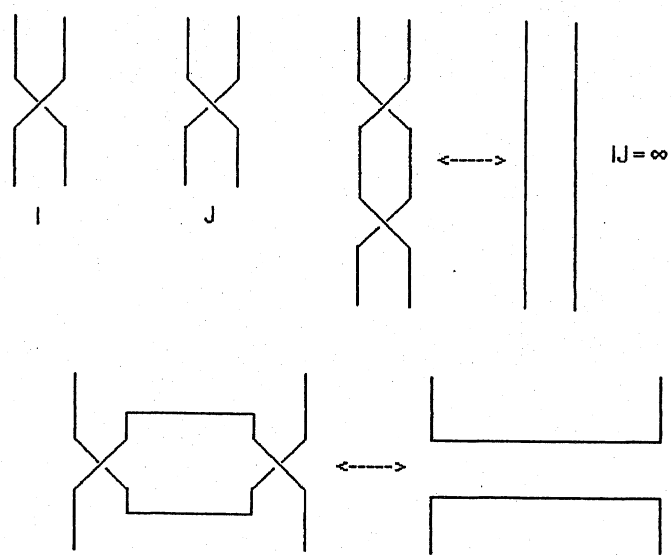

Figure 12.

reader can consult Goldman and Kauffman (1993) and Kauffman (1991, 1994) for more of the story. The main point of this section is that there is a multiplicity of surprising associations of arithmetic and topology with the Boolean realm and that an iconic method that is motivated by topological considerations strikes at the roots of a pattern connecting arithmetic and Boolean algebra.