We first prove as a lemma that for any $B \subset A$, if there is an injection $f : A \to B$, then there is also a bijection $h : A \to B$.

Inductively define a sequence (C_n) of subsets of A by $C_0 = A \setminus B$ and $C_{n+1} = f(C_n)$. Now let $C = \bigcup_{k=0}^{\infty} C_k$, and define $h : A \to B$ by

$$h(z) = \begin{cases} f(z), & z \in C \\ z, & z \notin C \end{cases}.$$

If $z \in C$, then $h(z) = f(z) \in B$. But if $z \notin C$, then $z \in B$, and so $h(z) \in B$. Hence h is well-defined; h is injective by construction. Let $b \in B$. If $b \notin C$, then $h(b) = b$. Otherwise, $b \in C_k = f(C_{k-1})$ for some $k > 0$, and so there is some $a \in C_{k-1}$ such that $h(a) = f(a) = b$. Thus h is bijective; in particular, if $B = A$, then h is simply the identity map on A.

To prove the theorem, suppose $f : S \to T$ and $g : T \to S$ are injective. Then the composition $gf : S \to g(T)$ is also injective. By the lemma, there is a bijection $h' : S \to g(T)$. The injectivity of g implies that $g^{-1} : g(T) \to T$ exists and is bijective. Define $h : S \to T$ by $h(z) = g^{-1}h'(z)$; this map is a bijection, and so S and T have the same cardinality.