Secrets of Calculus I

by LK

These notes will show you how to take derivatives without having to use limits. We introduce you to the $\boxed{\text{zeroid}}$ $\ast$.

The zeroid is a new number unlike all the other real numbers. It has the following properties:

1. The zeroid is greater than zero.

   $0 < \ast$

2. The zeroid is less than any ordinary real number $r > 0$.

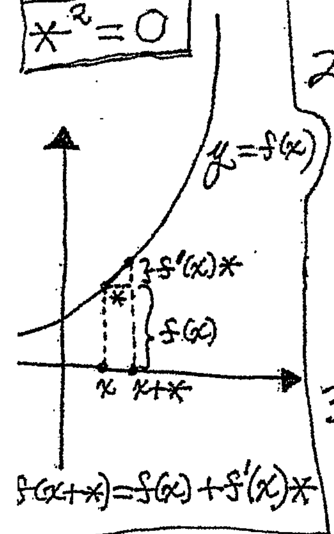
   If r is an ordinary real number, and $0 < r$, then $0 < \ast < r$.

3. The square of the zeroid is equal to zero.

   $\boxed{\ast^2 = 0}$.

Zeroid Calculus

$\boxed{\ast^2 = 0}$



The zeroid is so small that when you sqare it, it vanishes!

Let $f'(x) = df/dx$ be the derivative of the function $f(x)$. The Secret Formula

$$\boxed{f(x + \ast) = f(x) + f'(x)\ast}$$

$$\boxed{f(x+\ast) = f(x) + f'(x)\ast}$$

Secret Zeroid Formula

<u>Example</u>: $(x+\ast)^2 = x^2 + 2x\ast + \underline{\underline{\ast^2}}$

$$= x^2 + (2x)\ast$$

Hence if $f(x) = x^2$

then $f'(x) = 2x$.

<u>Example</u>: $(x+\ast)^3 = (x+\ast)^2(x+\ast)$

$$= (x^2 + 2x\ast)(x+\ast)$$

$$= x^3 + 2x^2\ast + x^2\ast + 2\underline{\underline{x\ast^2}}$$

$$= x^3 + (3x^2)\ast$$

Hence if $f(x) = x^3$, then $f'(x) = 3x^2$.

<u>Exercise</u>. Use the zeroid to show that $\frac{d}{dx}(x^4) = 4x^3$.

<u>Exercise</u>. We say that for ordinary real numbers a, b, c, d : $a + b\ast = c + d\ast \iff a = c$ and $b = d$.

a) When does $\left(\frac{1}{a+b\ast}\right)$ make sense? (You can't divide by a zeroid.

b) What can you say about numbers like $7\ast$?

Secrets of Calculus II

LK

$\hat{\mathbb{R}} = \{a + b* \mid a, b \text{ are real numbers}\}$, $\boxed{*^2 = 0}$

$0 < * < r$ for any positive real number r.

$*$ is the "zeroid". Differentiation is defined by the formula: $f(x+*) = f(x) + f'(x)*$.

More generally, $\boxed{f(x + t*) = f(x) + f'(x)t*}$ where t is any real number. (Note $(t*)^2 = t^2 *^2 = 0$)

Thus $f(x+t*)$ looks like a "microscopic" tangent line approximation to $y = f(x)$ at the point $(x, f(x))$.

Recall that $e \approx (1 + \frac{1}{N})^N$ for large N. Thus $\boxed{e^{\frac{1}{N}} \approx 1 + \frac{1}{N}}$. For $*$, we take $\boxed{e^* = 1 + *}$.

Since $\sin(\theta) \approx \theta$ for small θ, we take $\boxed{\sin(*) = *}$.

Since $\sin^2(\theta) + \cos^2(\theta) = 1$ we have $*^2 + \cos^2(*) = 1$

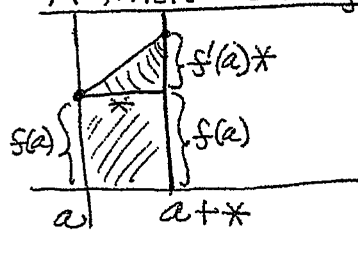
So we take $\boxed{\cos(*) = 1}$. Now a few derivatives:

- $e^{x+*} = e^x e^* = e^x(1+*) = e^x + e^x * \Rightarrow (e^x)' = e^x$.
- $\sin(x+*) = \sin(x)\cos(*) + \cos(x)\sin(*) = \sin(x) + \cos(x)*$
  $\Rightarrow \sin'(x) = \cos(x)$.
- $\cos(x+*) = \cos(x)\cos(*) - \sin(x)\sin(*) = \cos(x) - \sin(x)*$
  $\Rightarrow \cos'(x) = -\sin(x)$.

Rules of Calculus:

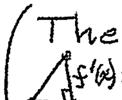
- $(f+g)(x+*) = f(x+*) + g(x+*) = f(x) + g(x) + (f'(x) + g'(x))*$
  $\Rightarrow (f+g)' = f' + g'$.
- $(fg)(x+*) = f(x+*)g(x+*) = (f(x) + f'(x)*)(g(x) + g'(x)*)$
  $= f(x)g(x) + (f'(x)g(x) + f(x)g'(x))*$ (N.B. $*^2 = 0$)
  $\Rightarrow (fg)' = f'g + fg'$.
- $(f \circ g)(x+*) = f(g(x+*)) = f(g(x) + g'(x)*)$
  $= f(g(x)) + f'(g(x))g'(x)*$
  $\Rightarrow (f(g(x)))' = f'(g(x))g'(x)$. (the chain rule).

A Micro-Integral



$$\int_a^{a+*} f(x)\,dx = f(a)* + \underbrace{\tfrac{1}{2}*(f'(a)*)}_{=0 \text{, since } *^2=0}$$

$$\boxed{\int_a^{a+*} f(x)\,dx = f(a)*}$$

(The micro-triangle 

has area zero.)

The First Fundamental Theorem of Calculus

$$\int_a^{x+*} f(t)\,dt = \int_a^{x} f(t)\,dt + \int_x^{x+*} f(t)\,dt$$

$$= \int_a^{x} f(t)\,dt + f(x)* \qquad \text{(by the micro-integral)}$$

$$\Longrightarrow \boxed{\frac{d}{dx}\int_a^{x} f(t)\,dt = f(x)}.$$

The Second Fundamental Theorem of Calculus

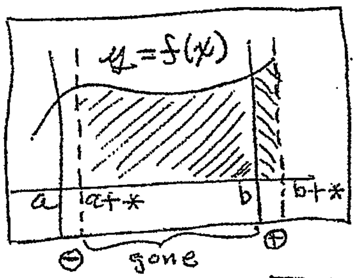
$$\int_a^b f'(x)\,dx = \int_a^b \big(f(x+*) - f(x)\big)\,dx \quad //*$$

$$\boxed{\begin{array}{l} f(x+*) = f(x) + f'(x)* \\ \big(f(x+*) - f(x)\big)//* = f'(x) \end{array}}$$

Here $//*$ means "remove the $*$ factor". For example $\boxed{3* \; //* = 3}$. You can't fully divide by $*$, but this is OK.

$$\text{So } \int_a^b f'(x)\,dx = \int_a^b f(x+*)\,dx - \int_a^b f(x)\,dx \quad //*$$

$$= \int_{a+*}^{b+*} f(x)\,dx - \int_a^b f(x)\,dx \quad //*$$



$$= \int_b^{b+*} f(x)\,dx - \int_a^{a+*} f(x)\,dx \quad //*$$

$$= f(b)* - f(a)* \quad //*$$

$$= f(b) - f(a).$$

$$\Longrightarrow \boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}.$$

Constancy

Suppose $f'(x) = 0$ for all x in $[a,b]$.

Then $0 = \int_a^b f'(x)\,dx = f(b) - f(a)$.

Thus $\boxed{f'(x) = 0 \text{ on } [a,b] \Rightarrow f(x) = \text{constant on } [a,b]}$.

Another Way

Euler tells us that

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Since $e^{i*} = 1 + i*$,

we have

$$1 + i* = \cos(*) + i\sin(*).$$

Therefore

$$\boxed{\begin{aligned} 1 &= \cos(*) \\ * &= \sin(*) \end{aligned}}.$$

Infinitesimal Calculus, Differential Forms and Stokes Theorem ①

L Kauffman, Fall 1999

I. Infinitesimals, Derivatives, Integrals

We are going to use a version of calculus that involves a new number concept that shall be referred to as a square zero infinitesimal. A square zero infinitesimal $\delta \neq 0$ has the following properties: (0) $x\delta = \delta x$ for all real x.

(1) δ can be compared to 0 and either $\delta < 0$ or $\delta > 0$.
If $a \in \mathbb{R}$ (i.e. a is a real number), then
$$\begin{aligned} \delta > 0,\ a > 0 &\Rightarrow a\delta > 0, \\ \delta > 0,\ a < 0 &\Rightarrow a\delta < 0, \\ \delta < 0,\ a > 0 &\Rightarrow a\delta < 0, \\ \delta < 0,\ a < 0 &\Rightarrow a\delta > 0, \\ \delta \text{ any},\ a = 0 &\Rightarrow a\delta = 0. \end{aligned}$$
If $a, b \in \mathbb{R}$ then
$$\begin{aligned} (a+b)\delta &= a\delta + b\delta \\ (ab)\delta &= a(b\delta) \\ 1\delta &= \delta. \end{aligned}$$

(2) δ can be compared with other real numbers. If $a \in \mathbb{R}$, $a > 0$ and $\delta > 0$ then $0 < \delta < a$.
(δ is smaller than any ordinary real number.)
If $a \in \mathbb{R}$, $a < 0$ and $\delta < 0$ then $a < \delta < 0$.

(3) If a, b, c, d are real numbers then
$$a + b\delta = c + d\delta$$
$$\Leftrightarrow a = c \text{ and } b = d.$$

(4) $\delta^2 = 0$. (δ is "so small" that its square is indistinguishable from zero.)

In understanding how to work with δ you should think of it as a very small number, but remember that it is actually quite distinct from all the usual reals. For example, δ is neither rational nor irrational.

We will use δ to do calculus as follows:

(A) Let $f(x)$ be a real-valued function of the variable x, and suppose that we know how to define $f(x+\delta)$

② for any x (real) and any square-zero infinitesi[mal] δ. Then $f'(x) = df/dx$ is <u>defined</u> by the following equation:

$$\boxed{f(x+\delta) - f(x) = f'(x)\delta.}$$

(If the difference $f(x+\delta) - f(x)$ is not of the form [Function of (x)]δ, then we will say that <u>f is not differentiable at x.</u>) Here are some examples:

(1) $(x+\delta)^2 - x^2 = x^2 + 2x\delta + \delta^2 - x^2$
$= 2x\delta$

$\Rightarrow \quad d(x^2)/dx = 2x.$

(2) $f(x) = ax^2 + bx + c \quad (a, b, c \text{ real}).$

$$f(x+\delta) - f(x) = a(x+\delta)^2 + b(x+\delta) + c - ax^2 - bx - c$$
$$= a(x^2 + 2x\delta + \delta^2) - ax^2 + bx + b\delta - bx + c - c$$
$$= 2ax\delta + b\delta$$
$$= (2ax+b)\delta$$
$$\Rightarrow \frac{d(ax^2+bx+c)}{dx} = 2ax+b.$$

(3) <u>Exercise</u>: $d(x^n)/dx = nx^{n-1}$.

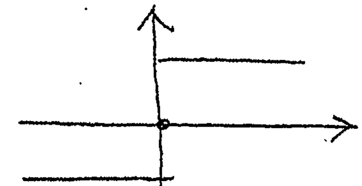
(4) How do we define $e^{x+\delta}$?
Answer, use $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Then
$$e^\delta = 1 + \delta$$
$$e^{x+\delta} = e^x e^\delta = e^x + e^x\delta$$
$$e^{x+\delta} - e^x = e^x\delta$$
$$\Rightarrow \frac{d(e^x)}{dx} = e^x.$$

(How do you know that $e^x e^y = e^{x+y}$?)

5) Check that $\frac{d(e^{ax})}{dx} = a e^{ax}$.

6) Let $f(x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ +1 & x>0. \end{cases}$



If $x<0$ and δ is a square zero infinitesimal, then $x+\delta<0$ so we can define

$$f(x+\delta) = \begin{cases} -1 & x<0,\ \delta<0 \text{ or } \delta>0 \\ +1 & x=0,\ \delta>0 \\ -1 & x=0,\ \delta<0 \\ +1 & x>0,\ \delta<0 \text{ or } \delta>0 \end{cases}$$

This definition makes sense because <u>it is how $f(x+\delta)$ would behave if δ were a very small ordinary real number.</u>

<u>Check</u>: a) $f'(x) = 0$ if $x \neq 0$.

b) $f(0+\delta) - f(0) = f(\delta) = \pm 1$.

$+1$ and -1 are <u>not</u> of the form $F(x)\delta$. Hence <u>$f(x)$ does not have a derivative at the origin.</u>

Note how this corresponds to the usual proof that $f'(0)$ does not exist. In the usual proof we look at

$$f(x+\Delta x) - f(x) \longleftrightarrow$$

$$f(\Delta x) - f(0) = f(\Delta x) = \pm 1.$$

So $\frac{f(\Delta x)}{\Delta x} = \frac{\pm 1}{\Delta x} \longrightarrow \pm\infty$ as $\Delta x \to 0$.

Since we do not take $\pm\infty$ as a value for a derivative, we conclude that the function has no derivative at zero.

7) Does $1/\delta$ have a meaning?

<u>Ans</u>. No. For since $\delta^2 = 0$,

if $\frac{1}{\delta}$ was meaningful, then $\delta = \frac{1}{\delta}(\delta^2) = \frac{1}{\delta}(0) = 0$. But we take $\delta \neq 0$. This is the ~~analog of the statement that~~ $\lim_{\Delta \to 0} \frac{1}{\Delta}$ does not exist.

You can't divide by 0, and you can't divide by δ!

8) $F(x) = f(x) + g(x)$

$$
\begin{aligned}
F(x+\delta) - F(x) &= f(x+\delta) + g(x+\delta) - f(x) - g(x) \\
&= f(x+\delta) - f(x) + g(x+\delta) - g(x) \\
&= f'(x)\delta + g'(x)\delta \\
&= (f'(x) + g'(x))\delta.
\end{aligned}
$$

$\therefore \ (f+g)' = f' + g'$ when the derivatives exist.

9) $F(x) = f(x)g(x)$.

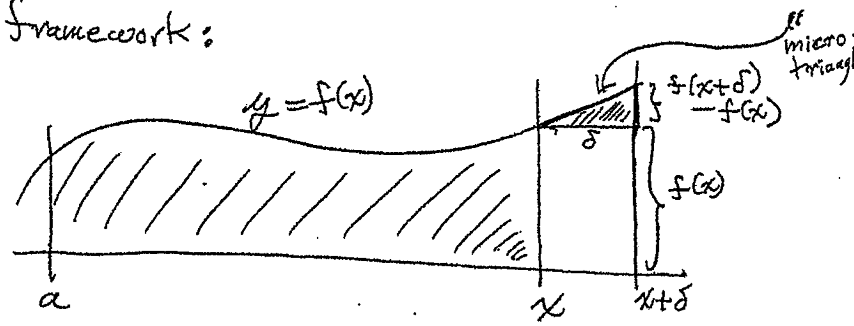
$$
\begin{aligned}
F(x+\delta) - F(x) &= f(x+\delta)\, g(x+\delta) - f(x)g(x) \\
&= (f(x) + f'(x)\delta)(g(x) + g'(x)\delta) - f(x)g(x) \\
&= f(x)g(x) + f'(x)g(x)\delta + f(x)g'(x)\delta \\
&\quad + f'(x)g'(x)\delta^2 - f(x)g(x) \\
&= (f'(x)g(x) + f(x)g'(x))\delta
\end{aligned}
$$

$\therefore \ (fg)' = f'g + fg'$ when f' and g' exist.

10) $F(x) = f(g(x))$.

$$
\begin{aligned}
F(x+\delta) - F(x) &= f(g(x+\delta)) - f(g(x)) \\
&= f(g(x) + g'(x)\delta) - f(g(x)) \\
&= f'(g(x))g'(x)\delta \Rightarrow (f \circ g)' = f'(g)g'.
\end{aligned}
$$

11) Note how the proof of the fundamental theorem of calculus looks in this framework:



$$A(x)=\int_a^x f(t)\,dt \stackrel{\text{def}}{=} \text{area under the curve } y=f(x) \text{ from } x=a \text{ to } x.$$

$$\Rightarrow A(x+\delta)-A(x) = \underset{\substack{\uparrow\\ \text{area of extra rectangle}}}{f(x)\delta} + \underset{\substack{\uparrow\\ \text{area of extra triangle}}}{\tfrac{1}{2}\delta\left[f(x+\delta)-f(x)\right]}$$

$$= f(x)\delta \quad + \underbrace{\tfrac{1}{2}\delta\left[f'(x)\delta\right]}_{=\,0\ \text{(the "micro triangle" has area zero!)}}$$

$$= f(x)\delta$$

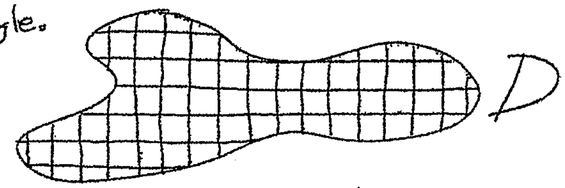
$$\therefore \quad \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

It may seem strange that the micro-triangle has area zero, but note how this compares correctly with the approximations we would use in the usual proof:

$$\frac{A(x+\Delta)-A(x)}{\Delta} \approx \frac{f(x)\Delta}{\Delta} + \frac{\frac{1}{2}\Delta\left[f'(x)\Delta\right]}{\Delta} + \text{higher order terms.}$$

II. Areas, Volumes, Grassman Algebra

In performing multiple integration we want to keep track of infinitesimal areas. Thus when we write $\iint_D f(x,y)\,dx\,dy$ we think of dividing the domain D into small rectangles dx by dy and summing over their areas times the value $f(x,y)$ at a point inside the rectangle.

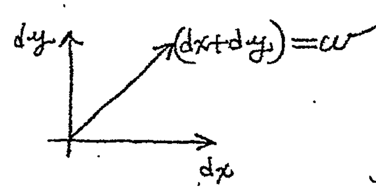


This means that we want products of infinitesimals dx and dy corresponding to different directions to be non-zero.

$$dx\,dy \neq 0.$$

But lets keep $(dx)^2 = 0$ and $(dy)^2 = 0$, so that these are still square-zero infinitesimals. Now what about

$$(dx+dy)^2 \;?$$



$\omega = dx + dy$ represents an ~~inf~~ infinitesimal pointing in a specific direction. We should still have $\omega^2 = 0$ because ω^2 does not represent an area.

So: $0 = (dx+dy)^2 = (dx+dy)(dx+dy)$

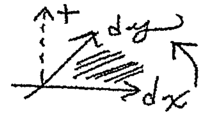
$$= dx\,dx + dx\,dy + dy\,dx + dy\,dy$$
$$= 0 + dx\,dy + dy\,dx + 0$$

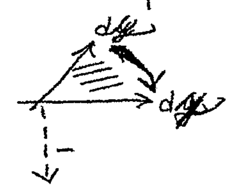
We get $0 = dx\,dy + dy\,dx$ or

$$\boxed{dx\,dy = -dy\,dx}$$

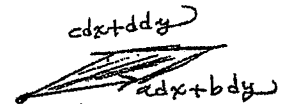
[Note if $dx\,dy = dy\,dx$ then we would get $2dx\,dy = 0 \Rightarrow dx\,dy = 0$ <u>No good!</u>]

Thus we find that infinitesimals pointing in different directions do not commute (just like in vector cross products).





What about $(a\,dx + b\,dy)(c\,dx + d\,dy)$?



Multiply it out:

$(a\,dx + b\,dy)(c\,dx + d\,dy)$

$= ac(dx)^2 + ad\,(dx\,dy) + bc\,(dy\,dx) + bd\,(dy)^2$

$= ad\,(dx\,dy) - bc\,(dx\,dy) \qquad (dx\,dy = -dy\,dx)$

$= (ad - bc)\,dx\,dy.$

Since the area of the parallelogram spanned by (a,b) and (c,d) <u>is</u> <u>$ad-bc$</u>, this formula shows that <u>$(a\,dx + b\,dy)(c\,dx + d\,dy)$ automatically computes the area of the corresponding infinitesimal parallelogram!</u>

One immediate consequence of this is that our non-commutative calculus of infinitesimals automatically gets the right formula for the change of variables in multiple integrals:

e.g. $x = f(r, s)$
$y = g(r, s)$

$$\Rightarrow \quad dx = f_r\,dr + f_s\,ds$$
$$dy = g_r\,dr + g_s\,ds$$

$$\Rightarrow \quad dx\,dy = (f_r g_s - f_s g_r)\,dr\,ds.$$

e.g. $x = r\cos\theta$
$y = r\sin\theta$

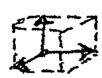
$$\Rightarrow \quad dx\,dy = \big((\cos\theta)\,r\cos\theta - (-r\sin\theta)(\sin\theta)\big)\,dr$$
$$= (r\cos^2\theta + r\sin^2\theta)\,dr\,d\theta$$

$$dx\,dy = r\,dr\,d\theta$$

This algebra of infinitesimals is so nice, it is worth considering all by itself and in arbitrary dimensions. For example, in $\mathbb{R}^3$ we have dx_1, dx_2, dx_3 with $(dx_1)^2 = (dx_2)^2 = (dx_3)^2 = 0$

$$\begin{cases} dx_1dx_2 = -dx_2dx_1 \\ dx_1dx_3 = -dx_3dx_2 \\ dx_2dx_3 = -dx_3dx_2. \end{cases}$$

What about $dx_1dx_2dx_3$? This should also be non-zero, since it represents a tiny volume.

 $dx_1dx_2dx_3$

And by our above non-commuting rules, we have

$$dx_1dx_2dx_3 = -dx_2dx_1dx_3 = +dx_2dx_3dx_1$$
$$= -dx_3dx_2dx_1 = +dx_3dx_1dx_2$$
$$= -dx_1dx_3dx_2.$$

9

The signs on the different ordered products of dx_1, dx_2, dx_3 are the signs of the corresponding permutations of 1, 2 and 3. It will perhaps come as no surprise that

Exercise. $\omega = a\,dx_1 + b\,dx_2 + c\,dx_3$
$\nu = e\,dx_1 + f\,dx_2 + g\,dx_3$
$\lambda = k\,dx_1 + l\,dx_2 + m\,dx_3$

$$\Rightarrow \omega\nu\lambda = \underbrace{\begin{vmatrix} a & b & c \\ e & f & g \\ k & l & m \end{vmatrix}}_{3\times 3 \text{ determinant.}} dx_1 dx_2 dx_3$$

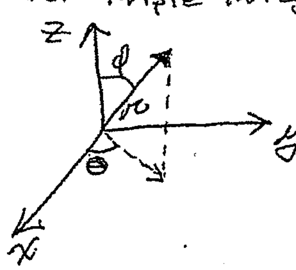
Exercise. Given differentiable functions

$$x = f(r, s, t) \qquad \begin{pmatrix} x \leftrightarrow x_1 \\ y \leftrightarrow x_2 \\ z \leftrightarrow x_3 \end{pmatrix}$$
$$y = g(r, s, t)$$
$$z = h(r, s, t)$$

Show that

$$dx\,dy\,dz = \begin{vmatrix} \partial x/\partial r & \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial r & \partial y/\partial s & \partial y/\partial t \\ \partial z/\partial r & \partial z/\partial s & \partial z/\partial t \end{vmatrix} dr\,ds\,dt$$

This is the change of variables formula for triple integrals.



$$\left.\begin{aligned} x &= r \sin\phi \cos\theta \\ y &= r \sin\phi \sin\theta \\ z &= r \cos\phi \end{aligned}\right.$$

Use the formula to find $dx\,dy\,dz$ in terms of $dr\,d\theta\,d\phi$.

Exercise. Let $\omega = a\,dx_1 + b\,dx_2 + c\,dx_3$

$\nu = g\,dx_1 + h\,dx_2 + k\,dx_3$

Let $W = (a, b, c)$

$T = (g, h, k)$

Let $(W \times T)_i$ denote the i^{th} coordinate of the cross product $W \times T$ $(i = 1, 2, 3)$.

Show that

$$\omega\nu = (W\times T)_1\, dx_2 dx_3 + (W\times T)_2\, dx_3 dx_1 + (W\times T)_3\, dx_1 dx_2$$

Note the geometric interpretation: The sum of the squares of the coefficients of the differentials in $\omega\nu$ is equal to the square of the length of $W \times T$, which in turn equals the square of the parellogram spanned by W & T in $\mathbb{R}^3$. Thus $\omega\nu$ is still directly related to the area of this parallelogram!

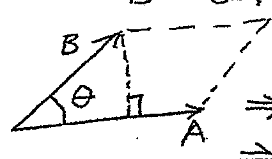
Exercise. (Generalizing to n-space).

$\alpha = a_1 dx_1 + \cdots + a_n dx_n$

$\beta = b_1 dx_1 + \cdots + b_n dx_n$

$A = (a_1, \ldots, a_n) \in \mathbb{R}^n$

$B = (b_1, \ldots, b_n) \in \mathbb{R}^n$



$A \cdot B = |A|\,|B| \cos\theta$

$\Rightarrow$ If area of ▱ $= \mathcal{A}$

$\Rightarrow \mathcal{A} = |A|\,(|B| \sin\theta)$

So $\mathcal{A}^2 = |A|^2 |B|^2 (1 - \cos^2\theta)$

$$\boxed{\mathcal{A}^2 = |A|^2|B|^2 - (A\cdot B)^2}$$

On the other hand,

$$\alpha\beta = \left(\sum_{i=1}^{n} a_i dx_i\right)\left(\sum_{j=1}^{n} b_j\, dx_j\right)$$

$$= \sum_{1\le i,j\le n} a_i b_j\, dx_i dx_j\ .$$

Show: (a) $\alpha\beta = \sum_{1\le i<j\le n} (a_i b_j - a_j b_i) dx_i dx_j$

Note we sum on $i<j$.

(b) If $\mathcal{A} = \sqrt{|A|^2|B|^2 - (A\cdot B)^2}$,

then $\mathcal{A} = \sqrt{\sum_{1\le i<j\le n} (a_i b_j - a_j b_i)^2}\ .$

Thus the area of a paralleogram in $\mathbb{R}^n$ spanned by two vectors A, B is equal to the length of a vector in $\mathbb{R}^{n(n-1)/2}$.

Differentials:

If $f(x_1, \dots, x_n)$ is a function, then we define $df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_n} dx_n$.

An entity $\omega = g_1 dx_1 + g_2 dx_2 + \dots + g_n dx_n$ is called a 1-form. 1-forms are what you integrate along a curve. ($g_1, \dots, g_n$ can all be functions of $x_1, \dots, x_n$.) For example,

Let $\omega = x_1 dx_1 + x_2 dx_2$

and $\alpha(t) = (t^2, t^3)$ be a curve in $\mathbb{R}^2$.

Soln: $x_1 = t^2$, $x_2 = t^3$

$$x_1 dx_1 + x_2 dx_2 = t^2 d(t^2) + t^3 d(t^3)$$
$$= t^2 \cdot 2t\,dt + t^3 \cdot 3t^2 dt$$
$$= (2t^3 + 3t^5)dt.$$

$$\therefore \int_{\substack{\alpha \\ 0 \le t \le 1}} \omega = \int_0^1 (2t^3 + 3t^5)dt = \left(\frac{2t^4}{4} + \frac{3t^6}{6}\right)\Big|_0^1$$
$$= \frac{2}{4} + 3/6 = \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

Along with 1-forms, there are 2-forms, 3-forms, ..., n-forms (in $\mathbb{R}^n$).

Thus in $\mathbb{R}^3$ we have:

Λ^0: 0-forms $\leftrightarrow$ functions $f(x, y, z)$

Λ^1: 1-forms $\leftrightarrow$ $a\,dx_1 + b\,dx_2 + c\,dx_3$

(a, b, c can be functions of x_1, x_2, x_3)

Λ^2: 2-forms $\leftrightarrow$ $a\,dx_1 dx_2 + b\,dx_1 dx_3 + c\,dx_2 dx_3$

Λ^3: 3-forms $\leftrightarrow$ $a\,dx_1 dx_2 dx_3$.

$$\boxed{\text{We let } \Lambda^k \mathbb{R}^n \text{ denote the } k\text{-forms on } \mathbb{R}^n.}$$

Note: 0. 0-forms can be evaluated on points (points have dimension 0).

1. 1-forms can be integrated on curves.

2. 2-forms can be integrated on surfaces.

3. 3-forms can be integrated on volumes.

(In n-space with continue with hypervolumes all the way to $\Lambda^n \mathbb{R}^n$.)

We have already
$$d: \Lambda^0 \longrightarrow \Lambda^1.$$
that is, we know how to take the differential of a function $f(x_1, x_2, \dots, x_n)$ by the formula: $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$.

We now extend this definition to
$$d: \Lambda^k \longrightarrow \Lambda^{k+1}$$
as follows: 1) If ω and η are k-forms, then $d(\omega + \eta) = d\omega + d\eta$.

2) If $\omega = f(x_1, \dots, x_n)\, dx_{i_1} \cdots dx_{i_k}$ then $d\omega = (df)\, dx_{i_1} \cdots dx_{i_k}$.

Thus $d\omega = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j\, dx_{i_1} \cdots dx_{i_k}$

and since $dx_j\, dx_{i_1} \cdots dx_{i_k} = 0$ if $j = i_1$ or i_2 or $\dots$ or i_k, we get
$$d\omega = \sum_{j \neq i_1, \dots, i_k} \frac{\partial f}{\partial x_j} dx_j\, dx_{i_1} \cdots dx_{i_k}.$$

- <u>For example</u>, $f(x_1, x_2, x_3)$, $n = 3$:
$$d(f\, dx_1 dx_2) = \frac{\partial f}{\partial x_3} dx_3\, dx_1\, dx_2$$
$$= \frac{\partial f}{\partial x_3} dx_1\, dx_2\, dx_3.$$
$$d(f\, dx_1 dx_3) = \frac{\partial f}{\partial x_2} dx_2\, dx_1\, dx_3$$
$$= -\frac{\partial f}{\partial x_2} dx_1\, dx_2\, dx_3.$$

We usually re-arrange the order of multiplications of the dx_i to fit our conveniences.

• For example, if $n = 3$ and

$$\omega = a\,dx + b\,dy + c\,dz \quad (x = x_1, y = x_2, z =$$

then

$$d\omega = \left(\frac{\partial a}{\partial y}dy + \frac{\partial a}{\partial z}dz\right)dx$$
$$+ \left(\frac{\partial b}{\partial x}dx + \frac{\partial b}{\partial z}dz\right)dy$$
$$+ \left(\frac{\partial c}{\partial x}dx + \frac{\partial c}{\partial y}dy\right)dz$$

$$\boxed{\begin{aligned} d\omega = &\left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}\right)dy\,dz \\ &+ \left(\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x}\right)dz\,dx \\ &+ \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right)dx\,dy \end{aligned}}$$

This formula has an interesting geometric interpretation that we shall now consider. But first,

Exercise. With $\omega = a\,dx + b\,dy + c\,dz$ as above, show that $d^2\omega \stackrel{\text{def}}{=} d(d\omega) = 0$. In general, show that for any k-form ω $d^2\omega = 0$ (assuming that the coefficient functions are continuous and have as many partial derivatives as you like.)

Exercise. Let $\omega \in \Lambda^k$, $\tau \in \Lambda^\ell$ so that $\omega\tau \in \Lambda^{k+\ell}$. Show that $d(\omega\tau) = (d\omega)\tau + (-1)^k\omega(d\tau)$.

Exercise. Let $n=3$, $\tau \in \Lambda^2$.

$$\tau = a\,dy\,dz + b\,dz\,dx + c\,dx\,dy.$$

Show that $d\tau = \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right) dx\,dy\,dz$

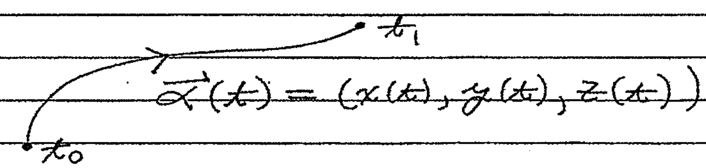
Geometric Interpretations

Let $\vec{W} = (a(x,y,z), b(x,y,z), c(x,y,z))$. We say that $\vec{W}$ is a vector field on $\mathbb{R}^3$. A vector field assigns a (varying) vector to each point in space. This vector can represent a force field that changes from place to place (e.g. an electrical field, a magnetic field, or a field of stresses in a material subjected to external forces, or a gravitational field). In physical interpretations, vector fields may also vary in time, making them functions of four variables x, y, z and t. For now, we just assume that $\vec{W}$ is a function of x, y, and z.

Now consider a 1-form

$$\omega = a\,dx + b\,dy + c\,dz$$

where x, y and z are constrained to a curve $\vec{\alpha}(t)$: $t_0 \le t \le t_1$



Then
$$\int_{\vec{\alpha}\, t_0}^{t_1} \omega = \int_{\vec{\alpha}\, t_0}^{t_1} a\,dx + b\,dy + c\,dz$$

$$= \int_{t_0}^{t_1} a x' dt + b y' dt + c z' dt$$

$(x' = \frac{dx}{dt}$ etc.$)$

$$= \int_{t_0}^{t_1} (a, b, c) \cdot (x', y', z')\, dt$$

$$= \int_{t_0}^{t_1} (\vec{W} \cdot \vec{\alpha}')\, dt$$

This is called the work done by a particle moving in the field $\vec{W}$, along the curve $\vec{\alpha}$ from $\vec{\alpha}(t_0)$ to $\vec{\alpha}(t_1)$. (This comes from the basic formulas in physics: Work = Force × Distance.)

Note that we have shown that, restricting $\omega = a\,dx + b\,dy + c\,dz$ to the curve $\vec{\alpha}(t)$ we have

$$\omega|_{\vec{\alpha}}(t) = (\vec{W} \circ \vec{\alpha}) \cdot \vec{\alpha}'(t)\, dt$$

rewriting the 1-form as a dot product of the vector field with the "infinitesimal tangent vector" $\vec{\alpha}'(t)\,dt$ to the curve. Integration of a 1-form along a curve is always a computation of the work done by moving a particle (in the field $\vec{W}$) along the curve. //

We define two basic quantities associated with a vector field $\vec{W} = (a, b, c)$. These are the curl $\nabla \times \vec{W}$, a new vector field, and the divergence $\nabla \cdot \vec{W} = div(\vec{W})$, a scalar (i.e. a scalar field, that is, a function defined on all points in space). Sometimes we write $\nabla \times \vec{W} = curl(\vec{W})$. Here are the definitions:

(1) $$\boxed{\nabla \cdot (a,b,c) = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}}$$

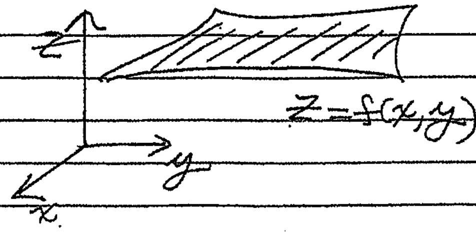
Thus, if $d\vec{S} = (dy\,dz, dz\,dx, dx\,dy)$, then
$$d(\vec{W} \cdot d\vec{S}) = div(\vec{W})\,dx\,dy\,dz$$
(by our previous exercise)

(2) $$\nabla \times (a,b,c) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a & b & c \end{vmatrix}$$
$$= \left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}\right)\vec{i} + \left(\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x}\right)\vec{j} + \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right)\vec{k}$$

$$\boxed{\nabla \times (a,b,c) = \left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}, \frac{\partial a}{\partial z} - \frac{\partial c}{\partial x}, \frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right)}$$

Refer back to the formula for $d\omega$ ($\omega = a\,dx + b\,dy + c\,dz$) and you will see that $d\omega = (\nabla \times \vec{W}) \cdot d\vec{S}$ where $\vec{W} = (a,b,c)$ and $d\vec{S} = (dy\,dz, dz\,dx, dx\,dy)$.

Interpretation of $d\vec{S} = (dy\,dz, dz\,dx, dx\,dy)$:



Suppose x, y and z are constrained to the surface $z = f(x,y)$.

Then $d\vec{S} = (dy\,(f_x dx + f_y dy), (f_x dx + f_y dy)\,dx, dx\,dy)$

$$d\vec{S} = (-f_x\,dx\,dy, -f_y\,dx\,dy, dx\,dy)$$

$$= \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) dx\,dy$$

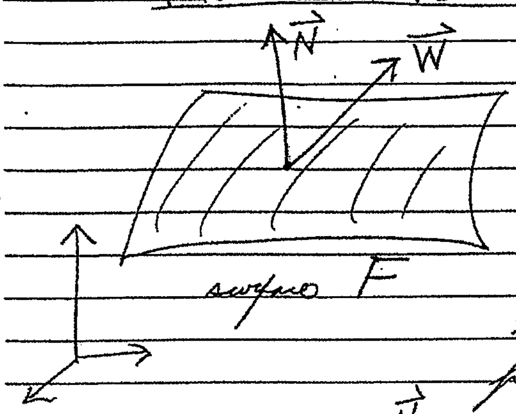
$d\vec{S} = \vec{N}\,dx\,dy$, where $\vec{N} = \nabla(z - f(x,y))$ is a normal vector to the surface.

Thus $d(a\,dx + b\,dy + c\,dz)$

$$= [\nabla \times (a,b,c)] \cdot \vec{N}\,dx\,dy$$

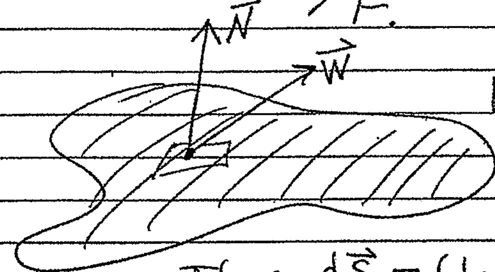
If we were to integrate the dot product of the curl of a vector field with the normal vector to a surface, we would be integrating the differential of the 1-form $a\,dx + b\,dy + c\,dz$ associated with that field. We will use this fact to simplify this integration.

Flux of a Vector Field



$$\iint_F \vec W \cdot \vec N \, dx\, dy$$

is called the flux of the vector field $\vec W$ across the surface F.



For a general surface F we write $d\vec S$ for the area differential.

Thus $d\vec S = (dy\,dz, dz\,dx, dx\,dy)$

and $\boxed{\vec W \cdot d\vec S = \vec W \cdot \vec N \, dS}$

If $\omega = a\,dx + b\,dy + c\,dz$

$\vec W = (a, b, c)$

then
$$\iint_F d\omega = \iint_F (\nabla \times \vec W)\cdot \vec N \, dS$$

Thus $\iint_F d\omega$ equals the flux of the curl of $\vec W$ across F.

Exercise: $\omega \in \Lambda^K$. Show that
$$d^2\omega = d(d\omega) = 0.$$

Exercise. $\nu = a\,dy\,dz + b\,dz\,dx + c\,dx\,dy$,
$$\Rightarrow d\nu = (\nabla \cdot \vec{W})\,dx\,dy\,dz$$
when $\vec{W} = (a, b, c)$.

Exercise. If $\vec{W}$ is a vector field, then $\operatorname{div}(\operatorname{curl}(\vec{W})) = 0$.

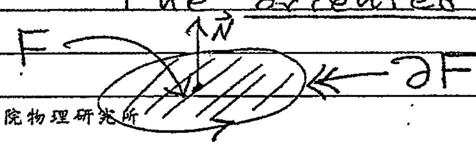
Stokes Theorem (The general case)

$\omega \in \Lambda^K$, a K-form, can be integrated over a K-dimensional space.

$$\left.\begin{array}{l} K=0 \ \text{— points} \\ K=1 \ \text{— curve} \\ K=2 \ \text{— surface} \\ K=3 \ \text{— solid} \end{array}\right\}$$

etc.

Let X^{K+1} be a $(K+1)$ dimensional space (we leave out the precise definition of space at this point but you can read point, curve, surface or solid...) Let ∂X^{K+1} denote the oriented boundary of X^{K+1}.

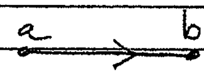


$$\boxed{\begin{array}{l}\text{Generalized Stokes Theorem} \\ \displaystyle\int_{\partial X^{k+1}} \omega = \int_{X^{k+1}} d\omega \\ \text{for } \omega \in \Lambda^k .\end{array}}$$

Lets examine some cases of this result.

$k=0,\ n=1:\quad f \in \Lambda^0 \quad df = f'(x)\,dx$

$$\int_a^b df = \int_{[a,b]} df = \int_a^b f'(x)\,dx$$
$$= f(b) - f(a).$$

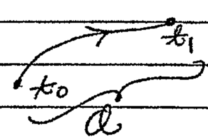


$$\partial[a,b] = [b] - [a]$$

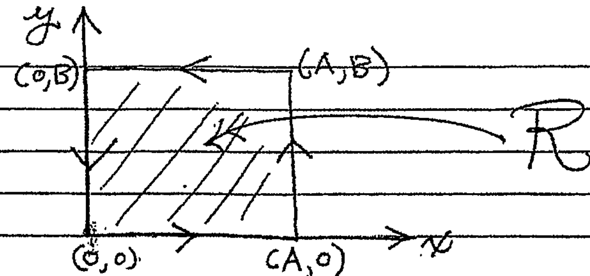
$$\int_{\partial[a,b]} f = \int_{[b]} f - \int_{[a]} f = f(b) - f(a) \quad \checkmark$$

$k=0$, n arbitrary, $\alpha(t) = (x_1(t), \ldots, x_n(t))$

$$\omega = f_1\,dx_1 + \cdots + f_n\,dx_n$$



$$\int_\alpha d\omega = \int_\alpha f_1 x_1'\,dt + \cdots + f_n x_n'\,dt$$
$$= \int_\alpha \frac{df}{dt}\,dt = f(\alpha(t_1)) - f(\alpha(t_0))$$
$$= \int_{\partial\alpha} f \quad \checkmark$$



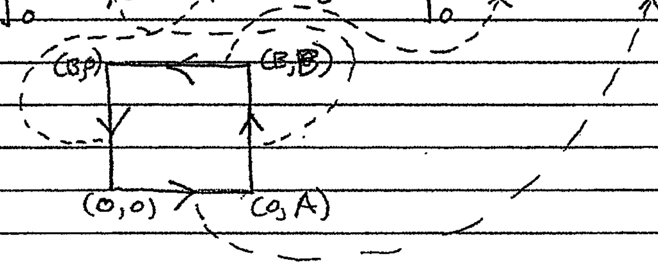
$$\omega = a\,dx + b\,dy$$

$$d\omega = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx\,dy$$

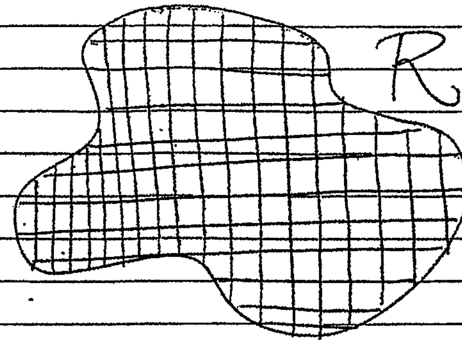
$$\iint_R d\omega = \int_0^B\int_0^A \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx\,dy$$

$$= \int_0^B\int_0^A \frac{\partial b}{\partial x}\,dx\,dy - \int_0^B\int_0^A \frac{\partial a}{\partial y}\,dx\,dy$$

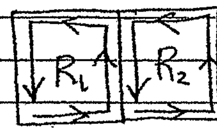
$$= \int_0^B \big(b(A,y) - b(0,y)\big)\,dy - \int_0^A \big(a(x,B) - a(x,0)\big)\,dx$$



$$= \int_{\partial R} a\,dx + b\,dy = \int_{\partial R} \omega \qquad //$$



A more complex region R can be regarded as covered by tiny rectangles.



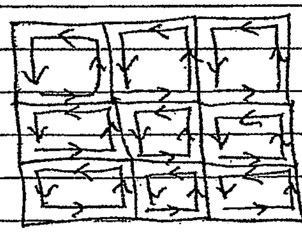
Use limits to get $\boxed{\iint_R d\omega = \int_{\partial R} \omega}$

$$\iint_{R_1 \cup R_2} d\omega = \iint_{R_1} d\omega + \iint_{R_2} d\omega$$

$$= \int_{\partial R_1} \omega + \int_{\partial R_2} \omega$$

$$= \int_{\partial (R_1 \cup R_2)} \omega$$

via the cancellation of the integrals along the common borders.



$$\boxed{\partial\left(\bigcup_i R_i\right) = \sum_i \partial R_i}$$

$$\omega = a\,dx + b\,dy + c\,dz$$

$$\vec{W} = (a, b, c)$$

$$d\omega = (\nabla \times \vec{W}) \cdot \vec{N}\,dS$$

$$\boxed{\iint_F (\nabla \times \vec{W}) \cdot \vec{N}\,dS = \int_{\partial F} \vec{W} \cdot \vec{T}\,ds}$$

where $\vec{T}$ is the tangent vector to the curve along ∂F.

The flux of the curl of $\vec{W}$ across F is equal to the work done along ∂F by $\vec{W}$.

This Theorem about surfaces and boundaries is what is usually called Stokes Theorem.

<u>Exercise</u>. Show that the generalized Stokes Theorem implies the <u>Divergence Theorem</u>. $\vec{W}$ a vector field. S' a surface, $S' =$ boundary of solid B.

$$\boxed{\iiint_B \operatorname{div}(\vec{W})\,dx\,dy\,dz = \iint_{S'} \vec{W} \cdot \vec{N}\,dS}$$