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Virtual Logic

Cantor's Paradise and the Parable of Frozen Time

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I. Introduction

Georg Cantor (Cantor, 1941; Dauben, 1990) is well-known to mathematicians as the inventor/discoverer of the arithmetic and ordering of mathematical infinity. Cantor discovered the theory of transfinite numbers, and an infinite hierarchy of ever-larger infinities. To the uninitiated this Cantorian notion of larger and larger infinities must seem prolix and astonishing, given that it is difficult enough to imagine infinity, much less an infinite structure of infinities.

Who has not wondered about the vastness of interstellar space, or the possibility of worlds within worlds forever, as we descend into the microworld? It is part of our heritage as observers of ourselves and our universe that we are interested in infinity. Infinity in the sense of unending process and unbounded space is our intuition of life, action and sense of being.

In this column we discuss infinity, first from a Cantorian point of view. We prove *Cantor's Key Theorem* about the power set of a set. Given a set **X**, the power set **P(X)** is the set of all subsets of **X**. Cantor's Theorem states that **P(X)** is always larger than **X**, even when **X** itself is infinite.

What can it mean for one infinity to be *larger* than another? Along with proving his very surprising theorem, Cantor found a way to compare infinities. It is a way that generalizes how we compare finite sets. We will discuss this generalization of size in the first section of the column below.

In the world of Cantor, if **X** is an infinite set, then $\mathbf{X < P(X) < P(P(X)) < ...}$ in an unending ascent into higher infinities.

Set theory in the classical mode is a theory about forms and structures that are eternally unchanging. When we speak of the set of integers or the set of all sets they are assumed to exist in eternity.

When set theory is propelled into the world of Time, a different and equally beautiful structure emerges. This temporal structure is the domain of performative language. Performative language is language whose very utterance commits the speaker to a process and a contract for action. When we speak of all sets or all distinctions, then the very act of speaking creates new sets and new distinctions. The only eternity here is in the moment, and time is an integral part of the structure of our acts and our mathematics.

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We are led into a speculation about the nature of time. With Cantor's Infinite, the infinity is frozen time. Temporal process is an intimation of infinity. Infinite sets are inherently self-referential, containing copies of themselves within themselves. (Infinite sets are like those fractal pictures where a person reads a book whose cover shows a person reading a book whose cover shows a person reading a book ...) Time and self-reference (an act that plays itself out in time) work together in a balance and movement that is the life of awareness.

In this essay, we introduce the set *Aleph* of all sets whose members are sets. Aleph is the simplest of all the concepts of sets. Aleph is the concept of sets whose members are sets – all the way down to nothing at all. Aleph is the concept of making sets from previously created sets, with the creation based on emptiness. But surprise! Aleph is its own power set. After all, any element of Aleph is a set whose members are sets in Aleph. And so any member of Aleph is a subset of Aleph. Similarly, as soon as you make a subset of Aleph, that set becomes a member of Aleph. This is the performative aspect of Aleph's creation. So

Aleph = P(Aleph).

Therefore P(Aleph) is not greater than Aleph.
Aleph is a counterexample to Cantor's Theorem.
How can this be?
Some have said that Aleph is not a set.
We say that Aleph is too *great* to be a set that exists only in eternity.

The reality of Aleph is a continual process of creation. No sooner have you written down a fragment of Aleph, then this fragment can be collected to form new members of Aleph. No matter how great Aleph becomes, she is always greater than that.

Aleph is the eye of the storm where thought can think itself and the world begins. When thought thinks itself and awareness becomes aware of awareness, then action occurs in the world and action being energy-through-time is physical. Pure thought and physicality are one in a creation that is a temporal eternity.

II. Cantor's Paradise

How did Cantor find this infinite hierarchy of infinities? We are all familiar with the infinity of the counting numbers **1,2,3,4,5,6,....** Mathematicians call these the *natural numbers* and write **N = {1,2,3,4,...}** to denote the entire set of natural numbers. Certain subsets of the natural numbers are key objects for the general concept of counting. For example, let **n** denote the set of the natural numbers (and zero) up to **n-1**. Thus

0 = { } (the empty subset)
1 = { 0 }
2 = { 0, 1}
3 = { 0, 1, 2 }

and so on. **n** is a set-theoretic yardstick for sets with n elements.

A set **S** is said to be finite with cardinality **n** if **S** can be put into one-to-one correspondence with the special subset **n** of the natural numbers. For example **{ A, B, C }** has cardinality **3** since it can be matched one-to-one with **3 = { 0, 1, 2 }**. One-to-one correspondence captures the essence of counting. We have all learned to grasp this concept, and we easily move from two arms to two fingers to two parents and understand that these are all instances of the number two.

In mathematics we find it easier to understand a number as a relationship than to understand a number as a thing. The essence of **3** is not the conventional notation for three. But by making a standard instance of **3** in the set **3 = { 0, 1, 2 }** we provide the possibility of matching to any and all occurrences of three. In so doing we weave a relationship between the general (all possible instances of three) and the particular instance of three that we have chosen for our standard of comparison.

Now you might start to think that this set theoretic discussion of numbers is a wee bit circular! First I said that we would assume familiarity with the numbers **1, 2, 3, ...** and then I asked you to take the set **n = { 0, 1, 2, 3, ... n-1 }** as a yardstick for the number **n**. But this is not a circularity. It is what we really meant by the number **n** as a symbol (like **1** or **2** or **3**). The number **n** as a symbol is a special sign that denotes that place in the sequence of counting where the **n-th** count has happened. Thus when you count to ten, you reach the symbol **10** at the point when you have counted ten times! The symbol **10** denotes the achievement of a 1-1 correspondence between your acts of counting and the elements of the set **10 = { 0,1,2,3,4,5,6,7,8,9 }.**

Each number **n** is a name of an ordered counting sequence that starts at **0** and ends at that number **n-1**. We determine the cardinality of a set of things by matching them with some standard set of things. Our process of "counting in order" facilitates such making of correspondences. Thus, for finite numbers, the concepts of cardinality and order are intertwined. We understand how to make 1-1 correspondence through our experience in counting. Numbers are used for the dual purposes of cardinality and counting.

Something strange happens when we attempt to count infinite collections. First of all, it becomes impossible to just speak

1,2, 3, ...

until we have chanted the names of all the natural numbers. We do have names for them, but we do not have the time to make a complete count. In fact I would wager that very few readers of this column have ever counted even to one thousand, saying each natural number in the sequence **1,2,3, ..., 999, 1000**.

While we cannot physically count infinite collections in order, we can make significant 1-1 correspondences. For example, let **E** denote the even natural numbers so that **E = { 2,4,6,8,10,... }**. Then there is a 1-1 correspondence between **N** and **E** given by associating the number **n** in **N** to the number **2n** in **E**. This is a perfect matching between the elements of **N** and the elements of **E**. Every element in **N** is matched to a unique element in **E** and every element of **E** has a mate in **N**.

$$N = \{ 1, 2, 3, 4, 5, ... \}$$

$$\updownarrow$$

$$E = \{ 2, 4, 6, 8, 10 , ...\}$$

We shall say that any two sets **X** and **Y** have the same size (cardinality) f there is a 1-1 correspondence between the elements of **X** and the elements of **Y**. Cardinality for infinite sets is a natural generalization of size for finite sets. We have just shown that **N** and **E** have the same cardinality and hence the same (generalized) size.

For example, you will agree that {a,b,c} and {1,2,3} are sets with the same number of members, while {1,2,3,4} is a set that is larger than {a,b,c}. You can see this by making a one-to-one correspondence between the members of {a,b,c} and the numbers {1,2,3}. For example, you can match a with 1, b with 2 and c with 3. But by the same token you can attempt to match each of a, b,c with one of the numbers 1,2,3,4 and there will always be one number that is left over. We say that a set Y is *bigger than* another set X if it is possible to match each element of X with a unique element of Y, but when this is done, there will *always* be elements of Y left over. The word *always* is very important here.

For example **E** is a proper subset of **N**, and by matching numbers in **E** to numbers in **N** in this way there are plenty of elements of **N** left over – all the odd numbers! But this, by our Cantorian definition does not mean that **E** is smaller than **N**. **E** is the same size as **N** because it is possible to establish a 1-1 correspondence between the members of **E** and the members of **N**, as we have done above. To show that one set Y is bigger than another set X, we must show the impossibility of such a 1-1 correspondence.

How should we think about this for infinite sets? What we see is that it is possible to have a mapping of an infinite set **X** to an infinite set **Y** that is *injective* (injective means that different elements of **X** are matched with different elements of **Y**) and that this does not preclude the possibility of another mapping that is both injective and *surjective* (every element of **Y** is matched with some element of X) and hence a 1-1 correspondence.

We can map **E** to **N** by the map **i(m) = m** and also by the map **j(m) = m/2.** The first map is injective but not surjective, and it exhibits **E** inside **N** in the usual way. The second map is a perfect 1-1 correspondence between the elements of **E** and the elements of **N**.

This sort of thing just cannot happen for finite sets. *Finite sets are never in 1-1 correspondence with a proper subset of themselves.* This sort of self-embedding is a characteristic property of the infinite.

Cantor discovered these phenomena and he defined size for infinite sets just as we have done in the paragraphs above. He set out to measure the size of infinite sets.

Perhaps at the beginning, after discovering the correspondence between **E** and **N,** he thought that all infinite sets had the same cardinality. But then he made a truly astonishing discovery. He showed that the set of subsets of any set **X** is bigger than **X**.

We shall write **P(X)** (the power set of **X**) for the set of subsets of **X**. Thus for **2 = {0,1}** we have **P(2) = { { }, {0} ,{1} ,{0,1} }**.

How do we prove Cantor's Theorem?
Let **F : X ⟶ P(X)** be any mapping from **X** to **P(X)**. If **X** and **P(X)** are in 1-1 correspondence, then there will exist a mapping **F** that is injective and surjective. We see at once that it is easy to make an injective mapping: just define **F(x) = {x}**, the subset of **X** with the single element **x**. This mapping is certainly not surjective. In fact the empty set **{ }** is not in the image of **F**. We shall show that there does not exist a surjective mapping **F**.

We say that **Y** has greater cardinality than **X** if there is an injective mapping from **X** to **Y**, but there is no surjective mapping from **X** to **Y**. This means that **X** fits inside **Y** but can never encompass **Y**.

Theorem (Cantor). *Let X be any set. Let **P(X)** denote the set of subsets of **X.** Then the cardinality of **P(X)** is greater than the cardinality of X.*

Proof. Given **F: X ⟶ P(X)**, define a subset **C** of **X** by the stipulation

C = { x in X | x is not in F(x) }.

This set is Cantor's basic construction. We shall call **C** *Cantor's Criminal.*

Remember that **x** either is, or is not, a member of **F(x)**. Can **C = F(z)** for some **z** in **X**? We will show that it is not possible for **C** to have the form **F(z)**.

We make the argument concise by adopting a special symbolism for sets and membership. Let "**Ax**" denote the statement "**x is a member of A.**" Then we can define Cantor's set by the equation

Cx = ~F(x)x.

This is a shorthand expression for "***x** is a member of **C** only if it is not the case that **x** is a member of **F(x)**."* With this shorthand before us, we look at what would happen if **C = F(z)** for some **z**. Then

F(z)x = ~F(x)x
for every **x** in **X.**

And so, letting **x =z,** we find

F(z)z = ~F(z)z.

Thus, if **C = F(z)**, we get that **z** is both inside and outside **C!** We have assumed that for all sets and subsets there is no ambiguity of membership in any set. Therefore we conclude that **C** is not of the form **F(z)** and hence that **F** is not surjective. This completes the proof that **P(X)** has greater cardinality **X**. **Q.E.D.**

We have an infinite tower of ever-larger infinite sets:

$$\mathbf{N < P(N) < P(P(N)) < P(P(P(N))) < ...}$$

This is the beginning of the story of Cantor's theory of transfinite numbers.

II. Buzzing Sets and the Great Escape Into Time

Can we take another point of view? If we refuse to be trapped by Cantor's argument, then we shall have to accept the possibility of sets like **F(z)** where there is an intrinsic ambiguity whether **z** is a member of **F(z)**. We might say that **F(z)** has a "buzzing" member **z**:

$$\mathbf{F:X \longrightarrow P(X)}$$
$$\mathbf{Cx = \sim F(x)x}$$
$$\mathbf{C = F(z)}$$
$$\mathbf{F(z)x = \sim F(x)x}$$
$$\mathbf{F(z)z = \sim F(z)z}$$

Buzz!

We shall have to take seriously the idea of a member that is in when it is out and out when it is in.

When z wants in we kick him out.
When z is out, we pull him in.

The condition of **z** is rather like the clapper in a buzzer. When the clapper hits the bell the circuit whose magnet attracted it to the bell is shut off. Then the clapper falls back, closing the circuit and so is attracted once more to the bell.

We have idealized our sets and imagined that they could live in eternity, beyond time and beyond all ambiguity. The price we have paid for this eternal clarity is a tower of highly reticulated infinities and a host of problems about their structure. Cantor's Paradise is a hall of mirrors. Just on the other side of this paradise is the world of time, process, ambiguity and change. Cantor's Paradise gives us a new way to think about the nature of time and process.

But lets go slowly here. Suppose that **F(x) = {x}**. Then **C = { x | x is not in {x}. }** and so **C = { }** is the empty set. Since **F(x)** is non-empty for each **x** in **X**, we see that indeed **C** is not of the form **F(z)**. And there is no buzz. Many mappings **F** are just not strong enough to be candidates for surjections from **X** to **P(X)** and many sets **X** really should be understood as smaller than their power sets.

Here, however is a really big set that challenges Cantor's Theorem. Let Aleph denote the set of *all sets whose members are sets:*

Whenever you look for members of those sets,
They are themselves sets.
And those sets have members that are sets.
It is sets all the way down.

Does this seem strange? To create Aleph we just begin with the empty set and keep on going, collecting sets made from sets: the empty set, the set whose member is the empty set, the set whose member is the set whose member is the empty set, the set of all the sets listed so far, ...

There are those who whisper, Aleph is too transcendent to be a set. Beware the Absolute. But we shall ignore them for the being of time…

Aleph is the set of all sets of sets and so *every subset of* Aleph *is a member of* Aleph *and every member of* Aleph *is a subset of* Aleph**!**

We map Aleph to its power set P(Aleph) = Aleph by taking the identity map **I(x) = x** for all sets **x** in Aleph. This mapping is certainly injective and surjective. And what about Cantor's set **C**? Recall the definition of **C**.

Cx = ~ I(x)x = ~ xx.
Cx = ~ xx.
C is the set of sets in Aleph that are not members of themselves.

And where is the buzz? Why the buzz is **C** itself! **I(x) = x** for all **x** implies that **C = I(C)**. If we ask whether **C** is a member of **C** we get the cryptic answer:

CC = ~ CC ----Bzzz!

C is in when it is out and out when it is in. And what does this mean? Some have said that the meaning is that **C** is really not a set. I say that **C** is a set acting in time.

In fact, the set C (of all x that are not members of themselves) is the famous Russell set, and the paradox that we have derived from it is the equally famous Russell paradox. See [http://en.wikipedia.org/wiki/Russell's_paradox] for more information. The Russell paradox is inherent in Cantor's work, and was surely known to Cantor in his thinking about the set of all sets. Russell pinpointed the difficulties of this paradox for logicians and philosophers, and he proposed a way to avoid it that is known as the Theory of Types. We shall not enter into a discussion of Russell's Theory here. The Theory of Types survives today in those who separate sets and classes. A set is a member of a class, but a class cannot be a member of a set. In this view, C is a class but not a set. There is no set whose member is C. The entity {C} is neither a set nor a class in the Goedel-Bernays Set Theory. See [http://en.wikipedia.org/wiki/Class_(set_theory)] for more information about these matters.

Aleph is its own power set. This contradicts directly Cantor's Theorem. Surely Aleph can not be larger than itself! But Aleph can *become* larger than itself. Aleph is a set that is growing in time. The buzz of contradiction becomes the sound of growth and recursion.

Aleph itself is so defined that it defies being pinned down. Any approximation to Aleph will have to include itself and become larger. No limit is possible here unless we wish to allow the simultaneity of being in and being out at the same time. Multiple universes loom here and there are those who would have them as well. The Cantorian paradise of eternal forms has emerged into temporal process. Infinite sets are processes. None are bigger or smaller than any others except as we care to compare them in the Cantorian way.

You cannot avoid Aleph. Aleph is the concept of the set of all sets emanating from the void. Aleph is simplicity, process and eternity all rolled up into one. When we reach for the large and simple idea of Aleph, eternity must be replaced by temporal process and we have returned from the immutable realms of pure zero-one classical logic to the imaginary and temporal states of the living world. When you imagine Aleph, you imagine thought. You cannot imagine thought without the presence of thought. Aleph brings itself into being and cannot be denied by logic or by eternity. Thought thinks itself and becomes real.

III. Imaginary Worlds in Time

Little by little they inched their way through the infinitesimal wormhole. It was not at all the way theory had predicted. No tidal forces. No terrible distortions of spatial reality. No apparent shifts of time-perception. Just a loss of innumerable distinctions. It was as if the world had become very stark, almost binary. Gradations disappeared. The familiar geometric boundaries of the sensual world were gone. Replacing them were very large distinctions such as *myself* and *external world, number, finite, infinite* (but specific numbers were long gone). The old way of dealing with criteria for such distinctions had disappeared. How did you know that you had a self when every movement seemed indistinguishable from *yours* or *theirs* or *its.* And yet you knew who you were with a clarity that was astounding. The clarity grew in proportion to the external lack of indication. There was no longer any reference to anything but the form of distinction itself. And then .... nothing. Nothing at all but a clarity so blinding that it canceled all thought, all normal consciousness, all reference. This *claritas* was the central balance, the pause between universes. Time was no more. And yet, after that would return indication, sensation, apparent world, three dimensionality, feeling. And it would be a new world, an entirely new world reached through the transgression of indication. The shifting of frames. The participation in Infinity.

IV. At First

At first there was nothing. But nothing is an unstable state. And so time and something came into being with a First Distinction, an act of creation. And this act was temporal as all acts must be. But this first act of creation happened in no time and no space and there was nothing but the idea of a distinction, a thought in the void. And that idea, that Word, was enough to engender actuality and time and space. The universe was born in eternity.

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