Chapter 6

CONWAY'S CHEQUERBOARD ARMY

Games are among the most interesting creations of the human mind, and the analysis of their structure is full of adventure and surprises.

James R. Newman

John Horton Conway is very hard to encapsulate. He is universally acknowledged as a world-class mathematician, a claim strongly substantiated by his occupation of the John von Neumann Chair of Mathematics at Princeton University. His vast ability and remarkable originality have caused him to contribute significantly to group theory, knot theory, number theory, coding theory and game theory (among other things); he is also the inventor of surreal numbers, which seem to be the ultimate extension of the number system and, most famous of all in popular mathematics, he invented the cellular automata game of Life. In chapter 14 we will look at him putting fractions to mysterious use, but here we will be concerned with another cellular game, typically simple, and typically deep.

The Problem

Imagine an infinite, two-dimensional chequerboard divided in half by an infinite barrier, as in figure 6.1. Above the barrier the

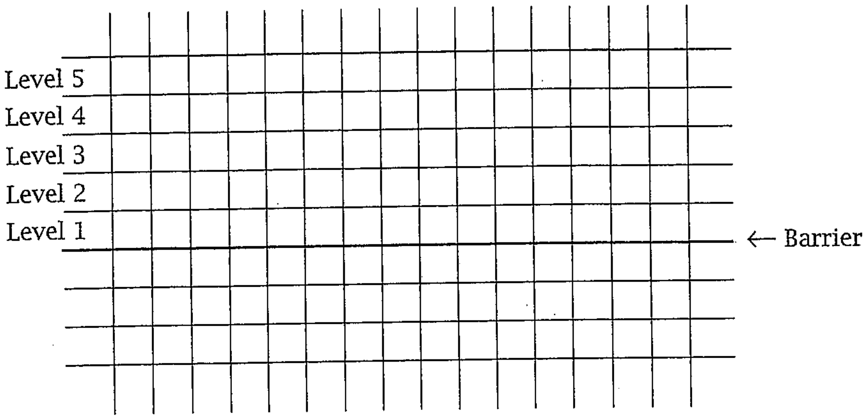

Figure 6.1. The playing area.

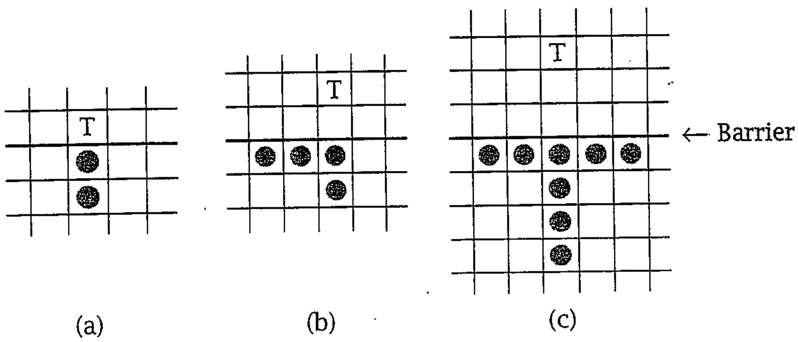

Figure 6.2. Reaching (a) level 1, (b) level 2, (c) level 3.

horizontal levels are numbered as shown. Chequers are placed on the squares below the barrier and can move horizontally or vertically below it or above it by jumping over and removing an adjacent piece. The puzzle Conway associated with this simple situation is to find starting configurations entirely below the barrier which will allow a single chequer to reach a particular target level above the barrier. It's very instructive to experiment with the pieces and, having done so, figure 6.2 shows the minimal configurations required to reach levels 1 to 3; in each case the target square T is reached by a single chequer.

The minimal number of chequers needed to reach levels 1, 2 and 3 is then 2, 4 and 8, respectively. The answer for level 4 is more complicated and, in what might be thought of as our first

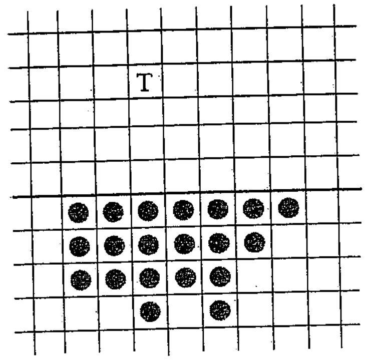

Figure 6.3. Reaching level 4.

Table 6.1. The level/chequer-count comparison.

Level	Minimum no. of chequers to reach level
1	2
2	4
3	8
4	20
5	There isn't one

surprise, figure 6.3 discloses that it is not 16 but a full 20 pieces that are needed to reach the target square T.

The second surprise, and the one which will occupy us for the rest of the chapter, is that level 5 is impossible to reach, no matter how many chequers are placed in whatever configuration below the barrier.

Table 6.1 summarizes the situation. The result is indeed surprising, but then so is Conway's ingenious method of proof, which, apart from anything else, brings in the Golden Ratio.

The Solution

To start with, fix any target square T on level 5 and, relative to it, associate with every square a nonnegative integer power of the variable x, that power being the 'chequerboard distance'

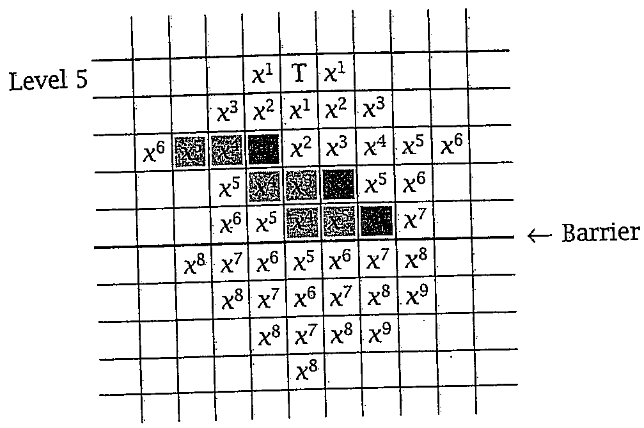

Figure 6.4. The labelling of the squares.

or 'taxicab distance' of the square from T. Such a distance is measured as the number of squares, measured horizontally and vertically from T, which gives rise to figure 6.4.

With this notation in place, every arrangement of chequer pieces, whether the initial configuration or the configuration at some later stage, can be represented by the polynomial formed by adding each of these powers of x together, for example, the starting positions to reach levels 1 to 4 might be represented by the polynomials x^5+x^6, $x^5+2x^6+x^7$, $x^5+3x^6+3x^7+x^8$ and $x^5+3x^6+5x^7+6x^8+4x^9+x^{10}$, respectively.

We now look at the effect of a move on the representing polynomial by realizing that, for this purpose, the choice of moves reduces to just three essentially different possibilities, which are characterized by the shaded cells in figure 6.4, where counters in the light grey squares are replaced by the counter in the dark grey square in each case. The general forms of these are

$$x^{n+2}+x^{n+1} \text{ is replaced by } x^n,$$
$$x^n+x^{n-1} \text{ is replaced by } x^n,$$
$$x^n+x^{n+1} \text{ is replaced by } x^{n+2}.$$

Any starting configuration will define a polynomial and, with every move that is made, that polynomial will change according to one of the three possibilities detailed above. The variable x

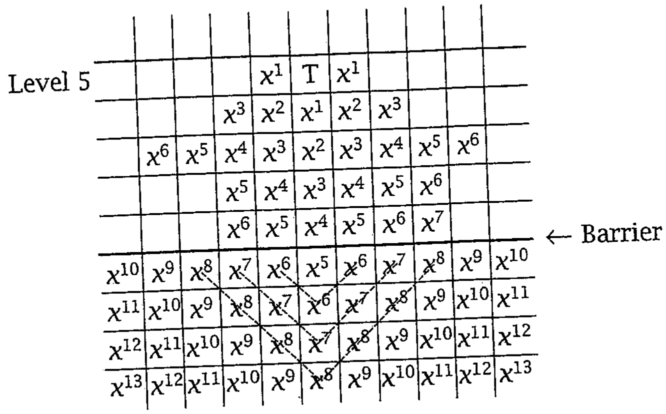

Figure 6.5. The ultimate 'polynomial'.

is arbitrary and we are free to replace it with any value we wish and will look to do so by choosing a value (greater than 0) which will cause the numeric value of the polynomial to decrease in the second and third cases and remain unchanged in the first (this last is for later algebraic convenience) when this number is substituted into it. Since $x > 0$, evidently $x^n + x^{n-1} > x^n$. If $x^n + x^{n+1} > x^{n+2}$, we require that $1 + x > x^2$ and this means that $0 < x < \frac{1}{2}(\sqrt{5} + 1) = \varphi$, which brings about the promised appearance of the Golden Ratio.

To cause the first move to leave the value of the polynomial unchanged we require that $x^{n+1} + x^{n+2} = x^n$, which means $x + x^2 = 1$ and $x = \frac{1}{2}(\sqrt{5} - 1) = 1/\varphi$, and the Golden Ratio appears once more.

So, if we make $x = 1/\varphi$ $(< \varphi)$, we are assured that the requirements are satisfied and further that, for this value of x, $x + x^2 = 1$.

Whatever our starting configuration below the dividing line, there will be a finite number of squares occupied. This means that any starting position evaluated at $x = 1/\varphi$ would be less than that of the 'infinite' polynomial generated by the occupation of every one of the infinite number of squares. We can find an expression for this by adding the terms in 'vertical darts', as illustrated in figure 6.5.

Adding terms in this way results in the expression

$$P = x^5 + 3x^6 + 5x^7 + 7x^8 + \cdots$$
$$= x^5(1 + 3x + 5x^2 + 7x^3 + \cdots).$$

The series in the brackets is a standard one (sometimes known as an arithmetic-geometric series) and is summed in the same way as a standard geometric series

$$S = 1 + 3x + 5x^2 + 7x^3 + \cdots,$$
$$\therefore \quad xS = x + 3x^2 + 5x^3 + 7x^4 + \cdots,$$
$$\therefore \quad S - xS = (1-x)S = 1 + 2x + 2x^2 + 2x^3 + \cdots$$
$$= 1 + 2(x + x^2 + x^3 + \cdots)$$
$$= 1 + \frac{2x}{1-x} = \frac{1+x}{1-x},$$
$$\therefore \quad S = \frac{1+x}{(1-x)^2}.$$

Multiplying by the x^5 term gives the final expression as

$$P = \frac{x^5(1+x)}{(1-x)^2}.$$

Since our chosen value for x satisfies $x + x^2 = x(1+x) = 1$, it must be that $1 + x = 1/x$ and also $1 - x = x^2$. Therefore,

$$P = \frac{x^5(1/x)}{(x^2)^2} = \frac{x^5}{x^5} = 1.$$

This means that the value of any starting position must be strictly less than 1 and since each move reduces or maintains the value of the position, the value of a position can never reach 1. It is impossible, therefore, to reach level 5.

The proof can be seen to fail with the lower levels. For example, with level 4 we finish with the product

$$x^4 S = x^4 \times \frac{1}{x^5} = \frac{1}{x} > 1,$$

leaving room for a reduction of the position to exactly 1.

Nonplussed!

MATHEMATICAL PROOF OF IMPLAUSIBLE IDEAS

Julian Havil

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