CLASSIFYING FINITE SUBGROUPS OF SO_3

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ABSTRACT. The goal of this paper is to prove that all finite subgroups of SO_3 are isomorphic to either a cyclic group, a dihedral group, or the rotational symmetry group of a regular solid. We begin by building up a set of definitions pertaining to groups, particularly permutation groups and matrix groups. Armed with these definitions and with the orbit-stabilizer and counting theorems, we will tackle the proof of the classification.

CONTENTS

INTRODUCTION

This paper will prove that all finite subgroups of SO_3 are isomorphic to either a cyclic group, a dihedral group, or the rotational symmetry group of a regular solid. In order to get to this result we will need a number of definitions, as well as some ancillary results. We will start by defining the most basic concepts of group theory, including groups, subgroups, and isomorphisms. With these definitions in hand we will move on to definitions of conjugacy and cosets which will prove useful later on. All of the definitions are drawn from [1] and [3]. Then we will discuss permutation groups such as dihedral groups, which are one of the types of groups that feature in our main result. The next section, on the platonic solids, also pertains directly to that result: the section discusses the isomorphisms between symmteric groups and the rotational symmetry groups of the solids. A section on matrix groups follows, where we introduce special orthogonal groups, alongside other matrix groups. We next discuss group actions, orbits, and stabilizers. These concepts and definitions allow us to prove the orbit-stabilizer and counting theorems, which we will use extensively in the proof of the main result. Finally, the last section goes through

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two proofs: first, that of a less powerful classification of finite subgroups of O_2, and then that of our main result, the classification of finite subgroups of SO_3. Exposition on types of groups and the ancillary theorems is drawn from [1], and the proofs in the last section are also based on [1], with some explication adapted from [2].

1. Basic Group Theory

In order to classify anything, we have to know just what that thing is. To classify a finite subgroup, we must define what a subgroup is, and, before that, a group. These definitions constitute the essential mathematical framework in which our proof will reside. Therefore, we will begin by defining some central concepts of group theory.

Definition 1.1. A **group** is a set G together with a multiplication on G which satisfies the following:

(1) The multiplication is associative, i.e. for all x, y, z in G, $(xy)z = x(yz)$.
(2) There is an element e in G, called the identity, such that for every x in G, $xe = x = ex$.
(3) For each element x in G, there exists an element x^{-1} in G, called the inverse of x, such that $x^{-1}x = e = xx^{-1}$.

Definition 1.2. The **order** of a finite group is the number of elements in the group. An infinite group has infinite order. If x is an element of a group, then the **order of** x is the smallest positive number n such that $x^n = e$, if such an n exists. If no such n exists, then x is said to have infinite order.

Once we know what a group is and how many elements its set contains, we need a way to figure out when two groups have the same structure. This leads us to the following definition:

Definition 1.3. Two groups G and G' are **isomorphic** if there is a bijection ϕ from G to G' such that $\phi(xy) = \phi(x)\phi(y)$ for all x, y in G. We call the map ϕ an **isomorphism**.

Note that the following are true of isomorphisms:

(1) An isomorphism $\phi : G \longrightarrow G'$ is a bijection; therefore, G and G' must be of the same order.
(2) ϕ maps the identity in G to the identity in G', and inverses to inverses.
(3) If $\phi : G \longrightarrow G'$ is an isomorphism, and H is a subgroup of G, then $\phi(H)$ is a subgroup of G'.
(4) An isomorphism preserves the order of each element.

Given a group G, we can define a subgroup of G. Subgroups are more than just subsets of the set of G, since subgroups preserve the group's product.

Definition 1.4. A **subgroup** of a group G is a subset of G which forms a group under the product of G.

By the definition of a subgroup, it is clear that every subgroup must contain, at the very minimum, the identity element. Therefore, there is no "empty subgroup." The next definition shows us one way in which a subgroup can be constructed: using a generating element.

Definition 1.5. The subgroup **generated by a subset** X **of** G is the smallest subgroup of G that contains X. If the subgroup fills out all of G, then we say X is a set of **generators** for G. If there is an element x in G that generates all of G, then we call G a **cyclic group**.

2. Conjugacy and Cosets

This next section deals with some ways in which group elements can interact. They may seem a bit tangential, but we will need these concepts to understand theorems which we will use in our classification proof.

Definition 2.1. Let x, y be elements of a group G. We say that x is **conjugate to** y if $gxg^{-1} = y$ for some g in G. The collection of elements conjugate to an element x is called the **conjugacy class of** x.

For a fixed element g of the group G the function

$$G \longrightarrow G, x \longmapsto gxg^{-1}$$

is an isomorphism called **conjugation by** g.

We can see that conjugation by an element g of a group G is an isomorphism since it is invertible and preserves a product. Invertibility is obvious, and we have the simple calculation $g(xy)g^{-1} = (gxg^{-1})(gyg^{-1})$ to show that the product is preserved.

Definition 2.2. Let G be a group and H a subgroup of G. The set $gH = \{gh \mid h \in H\}$ for some given element g of G is called the **left coset** of H determined by g. We can define, similarly, a right coset $Hg = \{hg \mid h \in H\}$

3. Permutation Groups

Building on these definitions, we narrow our focus to some of the types of groups that will figure in our classification: dihedral groups, symmetric groups, and alternating groups.

Definition 3.1. A **dihedral group** is a non-commutative group of the symmetries of a polygon, i.e. a group of the rotational and reflectional symmetries of a given polygon. We denote the dihedral group of an n-gon by D_n, and we write its elements as products of rotational symmetries r and reflectional symmetries s. A rotation r is, specifically, a rotation of the plane by $\frac{2\pi}{n}$, and s is any reflection.

It is helpful to consider a simple example of a dihedral group, D_3. This group corresponds to the group of rotational symmetries of a triangular plate. We can rotate this plate about an axis through its center, perpendicular to the plane of the plate, or we can reflect it in a line that passes through a vertex of the plate and the midpoint of the opposite edge. With a little thought, we can see that D_3 has order 6.

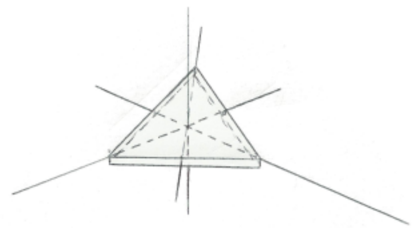

FIGURE 1. Axes of a triangular plate.

The dihedral group of an n-gon has order $2n$: n rotational symmetries and n reflectional symmetries. It is easy to see that an n-gon will have n possible rotations about the axis perpendicular to its center (including the identity). For reflectional symmetries, if n is odd, then each axis of symmetry connects a vertex with the midpoint of its opposite side, giving n axes for the n vertices. If n is even, then there are $n/2$ axes of symmetry connecting pairs of vertices, and $n/2$ axes connecting midpoints of pairs of sides.

Definition 3.2. A **permutation** of some set X is a bijection from X to itself. The collection of all permutations of a set X forms a group S_X under the composition of functions. When X is the set of the first n positive integers, we have a group S_n, which is called the **symmetric group** of degree n.

Since S_n is the collection of permutations of n elements, it is easy to see that there will be $n!$ elements in the set of S_n.

Definition 3.3. A permutation written as $(a_1,\ a_2, \ldots, a_k)$ inside a single pair of brackets is called a **cyclic permutation**. The notation indicates that the permutation sends a_1 to a_2, a_2 to a_3, etc., and finally, a_k to a_1.

Cycles are sometimes referred to in terms of their length: a three-cycle would have three elements; an n-cycle, n elements. A two-cycle, which exchanges only two elements, is called a **transposition**. Every n-cycle can be written as a product of transpositions. While there can be several different ways to write an n-cycle as a product of transpositions, if the given cycle can be written as a product of an even number of transpositions, it cannot be expressed with an odd number, and vice versa.

Symmetric groups and permutations are extremely useful. For example, we can express a dihedral group, say D_3 again, as a subgroup of a symmetric group of degree 3 by labelling the vertices of our triangular plate and seeing where each element of D_3 moves the various vertices. A rotation by $\frac{2\pi}{3}$ about the axis perpendicular to the center of the plate would correspond to a permutation cycle $(1, 2, 3)$.

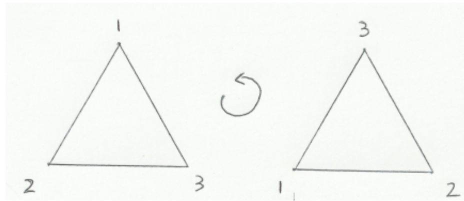

FIGURE 2. Rotation by $\frac{2\pi}{3}$.

Definition 3.4. An element of S_n which can be expressed as the product of an even number of transpositions is called an **even permutation**, and the rest are, naturally, **odd permutations**.

We now state, without proof, a proposition about an important subgroup of S_n, called the Alternating Group.

Proposition 3.5. *The even permutations in S_n form a subgroup of order $n!/2$ called the **alternating group** A_n of degree n.*

Proof. See [1], p. 29. □

4. The Platonic Solids

The five platonic solids are the only regular convex polyhedra–that is, they are the only convex polyhedra that are made up of congruent regular polygons with the same number of faces meeting at each vertex. Although they may not seem to fit in with the content presented thus far in this paper, they are in fact very much connected to group theory in general, and to symmetry groups in particular. As we aim to show in our final proof, the rotational symmetry groups of the solids are isomorphic to finite subgroups of SO_3. More specifially, their rotational symmetry groups are isomorphic to particular symmetry groups.

The five platonic solids are: the **tetrahedron**, the **cube**, the **octahedron**, the **dodecahedron**, and the **icosahedron**.

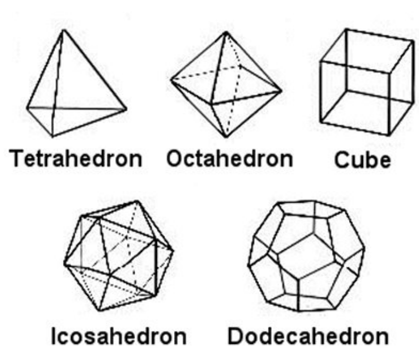

FIGURE 3. The five platonic solids.

The rotational symmetry groups of the solids are isomorphic to symmetry groups as follows:

solid:	isomorphic to:
tetrahedron	A_4
cube	S_4
octahedron	S_4
dodecahedron	A_5
icosahedron	A_5

Note that the rotational symmetry groups of the cube and octahedron, and the icosahedron and dodecahedron, are isomorphic to the same symmetry groups. This is because they are what are called **dual solids**, meaning that one can be constructed from the other by connecting vertices placed at the centers of the faces of its dual. That is, dual solids are pairs of solids with faces and vertices interchanged.

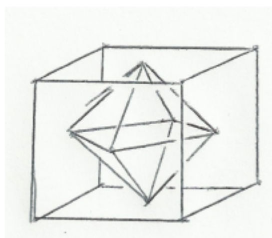

FIGURE 4. An octahedron can be drawn by connecting the centers of the faces of a cube.

The cube and the octahedron are dual solids, and thus they have the same amount of symmetry; the dodecahedron and the icosahedron are also dual. The tetrahedron is self-dual.

5. Matrix Groups

Matrix groups provide a useful way of representing the rotational symmetries we want to classify. In order to use these groups in our proof, we must first define the basic terminology of matrix groups.

The set of all $n \times n$ matrices with real-number entries forms a group under matrix multiplication. Each matrix A in this group determines an invertible linear transformation $f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $f_A(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x}$ in $\mathbb{R}^n$. The product matrix AB determines the composite linear transformation f_{AB}.

Definition 5.1. We call the group of $n \times n$ invertible matrices with real number entries the **General Linear Group**, $GL_n(\mathbb{R})$.

Definition 5.2. An $n \times n$ matrix A over $\mathbb{R}$ is **orthogonal** if A^tA is the identity matrix.

The collection of all $n \times n$ orthogonal matrices is a subgroup of GL_n called the **Orthogonal Group**, O_n. The elements of O_n which have determinant equal to $+1$ form a subgroup of O_n called the **Special Orthogonal Group**, SO_n.

If A is an element of O_n, the corresponding linear transformation f_A preserves distance and orthogonality. We can see this by taking two elements x and y in $\mathbb{R}^n$, applying f_A, and taking a scalar product of $f_A(x)$ and $f_A(y)$.

$$\langle f_A(x), f_A(y)\rangle = \langle A^t Ax, y\rangle = \langle x, y\rangle$$

Since the length of x in $\mathbb{R}^n$ is equal to $\sqrt{x \cdot x}$, taking the scalar product of $f_A(x)$ with itself shows us that f_A preserves the length of x.

A 2×2 orthogonal matrix represents either a rotation of the plane about the origin, or a reflection in a straight line through the origin. Such a matrix is in SO_2 precisely when it represents a rotation. This may sound far-fetched, but we can intuit where the idea comes from by considering a matrix A in O_2. The columns of A are orthogonal, and let's say that they are unit vectors. Now we are working on the unit circle, and our matrix for a rotation counterclockwise through θ is:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Trigonometric identities tell us that this matrix has a determinant of $+1$ for any angle θ.

If we keep looking at the unit circle and take the orthogonal columns of a matrix B to be, once again, unit vectors, we find that a reflection matrix across a line through the origin at angle $\theta/2$ is:

$$B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

and this matrix obviously has a determinant of -1.

Similarly, each matrix in SO_3 represents a rotation of $\mathbb{R}^3$ about an axis which passes through the origin.

6. Actions, Orbits, and Stabilizers

Groups become even more interesting when we consider the ways in which they "act" on other mathematical objects. A group action, a way in which a group interacts with a set, can help us to understand both the group and the set it acts on in new ways. In this section, we deal with group actions and some concepts that follow. The definitions and the theorems in this section are integral to our classification proof, as they include the orbit-stabilizer and counting theorems, which we will use extensively.

Definition 6.1. An **action** of a group G on a set X is a homomorphism from G to S_X. That is, we can define a homomorphism $\phi : G \longrightarrow S_X$ where for each g in G, $\phi(g)$ gives us a permutation of the points of our set X. Written another way, an action of a group G on a set X is a map from $G \times X$ to X satisfying the following conditions:

(1) For all $g_1, g_2 \in G, x \in X, (g_1 \cdot g_2)(x) = g_1(g_2(x))$
(2) For all $x \in X, 1 \cdot x = x$.

Definition 6.2. Given an action G on a set X and a point x in X, the set of all images $g(x)$ for all g in G is called the **orbit** of x, and is written $G(x)$. That is, $G(x) = \{g(x) \mid g \in G\}$.

Our dihedral group of a triangular plate provides a very simple example of these concepts. The group D_3 consists of rotations and reflections which, as we noted previously, can be represented by permutations of the numbers $1, 2, 3$ by numbering the vertices of the triangular plate. Thus, the group D_3 acts on the set of the numbered vertices $\{1, 2, 3\}$. Since each vertex can switch places with either of the other two, the action has only one orbit.

Definition 6.3. For a point x in our set X, the elements of G which leave x fixed form a subgroup of G called the **stabilizer of** x, G_x. That is, for a fixed x in X, $G_x = \{g \in G \mid g(x) = x\}$.

The stabilizer of any vertex of our triangular plate will contain two elements: the identity, and rotation by π about the axis through the given vertex and the midpoint of the opposite side.

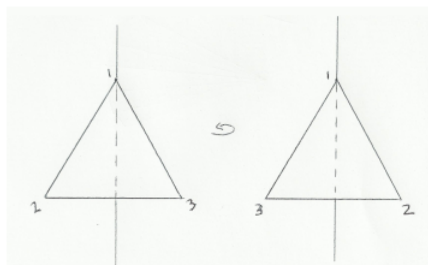

FIGURE 5. A rotation stabilizing 1.

The rotation shown in the figure above would be in the stabilizer of the vertex labelled 1.

We state the following without proof, as it will be used briefly in the proof of the counting theorem.

Proposition 6.4. *Points in the same orbit have conjugate stabilizers.*

Proof. See [1], p. 94. □

Our example of D_3 can help to clarify this proposition. Consider that all three vertices of the triangle are in one orbit. If we look at the reflection in the picture above, and call that reflection s, we can conjugate s by a rotation r which we define, in the left-hand triangle, as the permutation $(1, 2, 3)$. It is easy to see that s fixes 1, and rsr^{-1} fixes 2.

7. Important Theorems

These theorems are justly called important; they feature prominently in the classification proof, as well as being interesting results in their own right. They show us the connections between a group action, its orbits, and the stabilizers of the elements of the set that is acted upon.

Theorem 7.1. ***The Orbit-Stabilizer Theorem:*** *Let X be a set, and G a group acting on X. For each x in X, the correspondence $g(x) \to gG_x$ is a bijection between $G(x)$ and the set of left cosets of G_x in G.*

Proof. We want to show that the correspondence $g(x) \to gG_x$ is bijective. It is clear that this correspondence is surjective: For any x in X, every gG_x obviously corresponds to one $g(x)$ as g runs through G. In order to show that it is injective, let us consider g and g' in G. If $gG_x = g'G_x$, then we can say that $gh_1 = g'h_2$ for some h_1, h_2 in G_x. Then, setting $h_2h_1^{-1} = h$, we get that $g = g'h$ for h in G_x. But this means that

$$g(x) = g'h(x) = g'(h(x)) = g'(x)$$

since h is in G_x (i.e. $h(x) = x$). Thus we see that if $gG_x = g'G_x$, then $g(x) = g'(x)$, proving that the correspondence is also injective, and therefore bijective. □

Corollary 7.2. *If G is finite, the size of each orbit is a divisor of the order of G.*

Proof. By the Orbit-Stabilizer theorem, we know that the size of the orbit of x is equal to $|G|/|G_x|$, so we have:

$$|G(x)||G_x| = |G|$$

□

Theorem 7.3. ***The Counting Theorem:*** *Let G be a finite group acting on a set X, and write $X^g = \{x \mid gx = x\}$; that is, the set of x in X that are left fixed by a given element g of G. The number of distinct orbits is:*

$$\frac{1}{|G|} \sum_{g \in G} |X^g|.$$

In other words, the number of distinct orbits is the average number of points left fixed by an element of G.

Proof. We have the group action $G \times X \longrightarrow X$ which maps $(g, x) \longmapsto g(x)$, and we will start by counting the pairs (g, x) where $g(x) = x$. By our definition of X^g, that number is the sum:

$$\sum_{g \in G} |X^g|. \tag{7.4}$$

This sum counts pairs by summing the number of elements x of X that are left fixed by g in G as g runs through G. Since we define the stabilizer G_x as the set of all elements g in G that fix a given x, we can rewrite equation 7.4 as

$$\sum_{x \in X} |G_x| \tag{7.5}$$

where we count the same pairs by summing all stabilizing elements g in G over X. Let us call the number of distinct orbits of our action k, and these orbits $X_1, X_2, \ldots X_k$. Since these distinct orbits together make up our set X, we can rewrite equation 7.5 as

$$\sum_{i=1}^{k} \sum_{x \in X_i} |G_x| \tag{7.6}$$

We know from proposition 6.4 that points in the same orbit have conjugate stabilizers, and thus we know that the stabilizers of points in the same orbit are the same size. So, given any element x_1 in X_i, we have:

$$\sum_{x \in X_i} |G_x| = |X_i||G_{x_1}| = |G(x_1)||G_{x_1}| = |G|$$

by the Orbit-Stabilizer theorem. We can then see that combining all of these equations gives

$$\sum_{x\in X}|G_x| = k|G| = \sum_{g\in G}|X^g|$$

And so, dividing through by $|G|$ gives us our desired formula:

$$k = \frac{1}{|G|}\sum_{g\in G}|X^g| \tag{7.7}$$

□

8. Classifying Finite Subgroups of SO_3

The special orthogonal group SO_3 can be identified with the group of rotations of $\mathbb{R}^3$ about a fixed origin. If we take an object and orient it appropriately at the origin, its rotational symmetry group will be isomorphic to a subgroup of SO_3. We will show that there are only three possibilities for these subgroups, provided they are finite: cyclic groups, dihedral groups, and rotational symmetry groups of regular solids. All three of these possibilities do, in fact, occur.

First, we will prove a related but less complex result about finite subgroups of O_2.

Theorem 8.1. *A finite subgroup of O_2 is either cyclic or dihedral.*

Proof. Let G be a nontrivial subgroup of O_2. We have two cases to consider: G can be a subgroup of SO_2 as well as O_2, or it can have some elements in $O_2 - SO_2$.

First, suppose that G is a subgroup of SO_2. As we stated earlier with our definitions of orthogonal and special orthogonal groups, a 2×2 orthogonal matrix is in SO_2 precisely when it represents a rotation of the plane about the origin. Thus, we know that in this case every element of G represents a rotation of the plane.

Take A_θ to be the matrix in G which represents a counterclockwise rotation through θ about the origin, with $0 \le \theta < 2\pi$. Choose A_ϕ in G so that ϕ is positive and as small as possible. Now, we can use the division theorem to write $\theta = k\phi + \psi$ for some positive integer k and with $0 \le \psi < \phi$. This gives us:

$$A_\theta = A_{k\phi+\psi} = (A_\phi)^k A_\psi$$

which can be rearranged to

$$A_\psi = (A_\phi)^{-k} A_\theta.$$

Since A_θ and A_ϕ are both in G, this means that A_ψ will also be in G. We chose ϕ to be the smallest positive rotation in G, but ψ is less than ϕ, so ψ must be zero in order to avoid contradiction. Therefore, A_ϕ generates G, meaning that in this case G is cyclic.

Next we address the other case, where G is not completely contained in SO_2. Set $H = G\cap SO_2$. Then H is a subgroup of G, and by the first case H will be cyclic since it is entirely contained in SO_2. Choose a generator A for H, and an element B which is in G but not in H. B is therefore in O_2 and not SO_2, and represents a reflection. Thus, we have that $B^2 = I$. If $A = I$, then G contains only I and B and is a cyclic group of order 2. Otherwise, A has order $n \ge 2$, and the elements of G are:

$$I, A, \ldots, A^{n-1}, B, AB, \ldots, A^{n-1}B$$

with $B^2 = I$, $A^n = I$, and $BA = A^{-1}B$. We know that these are all the elements of G because the set includes all combinations of A and B where neither is raised to a power greater than its order. If we take A to correspond to rotational symmetry r and B to correspond to reflectional symmetry s, we have an isomorphism between G and the dihedral group of an n-gon D_n. □

Now we can move on to the big theorem: classifying all of the finite subgroups of SO_3.

Theorem 8.2. *A finite subgroup of SO_3 is isomorphic either to a cyclic group, a dihedral group, or the rotational symmetry group of one of the regular solids.*

Proof. We will proceed in stages. First, we will show that a finite subgroup G of SO_3 acts on the set of poles fixed by the rotations in G. We will then use the counting theorem to show that the action must have either 2 or 3 distinct orbits. From the two-orbit case we will get cyclic groups, and from the three-orbit case we will get four subcases leading to dihedral groups and the rotational symmetry groups of the regular solids.

Let G be a finite subgroup of SO_3. Each element in G (except for the identity) represents a rotation in $\mathbb{R}_3$ about an axis through the origin. We notice that each axis defined by an element of G will intersect the unit sphere at two points. For an element g in G, call those two points the poles of g in G. Stated another way, the poles are the only points on the unit sphere that are left-fixed by a given element.

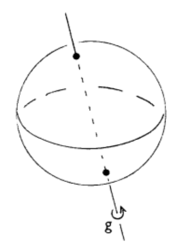

FIGURE 6. The two poles of an element g. [1] p. 105.

It would be nice if G defined a group action on some set–this would allow us to gather more information about these poles and about the group G. Luckily, there is such a set: let X be the set of poles of elements in $G - \{e\}$. We will show that G acts on X.

Take an element x in X and two elements g, h in G. Let x be a pole of the rotation h. Then we have:

$$(ghg^{-1})g(x) = g(h(x)) = g(x)$$

This means that $g(x)$ is left-fixed by ghg^{-1}, and so by definition $g(x)$ is an element of X. Thus, we have found an action $G \times X \longrightarrow X$, showing that G acts on X.

To proceed with the proof, we will apply the counting theoreom to this action to restrict the possibilites for the sizes of X and the orbits of G.

First, we need to find some basic limitations on the points in X. Using the counting theorem, the orbit-stabilizer theorem and some alegbraic manipulation we can discover constraints on X which allow us to reduce our classification problem to a small number of cases.

Let N be the number of distinct orbits of our action. Choose one pole from each orbit, and call these poles $x_1, \dots, x_N$. Every element g in $G - \{e\}$ fixes two poles, and the identity fixes them all. The counting theorem tells us that the number of distinct orbits of an action is equal to the average number of points fixed by an element of G. We can therefore write:

$$N = \frac{1}{|G|} \left(2(|G| - 1) + |X|\right) \tag{8.3}$$

In order to understand what this equation tells us about our action, we have to do some algebraic manipulation. First, we know that by definition, $|X| = \sum_{i=1}^{N} |G(x_i)|$; that is, the size of the set is the sum of the sizes of all the distinct orbits of the action on the set. Plugging this into equation 8.3 and rearranging some terms gives us

$$2\left(1 - \frac{1}{|G|}\right) = N - \frac{1}{|G|} \sum_{i=1}^{N} |G(x_i)| \tag{8.4}$$

Using the corollary to the orbit-stabilizer theorem, we can replace our sum of orbits with a sum of stabilizers, yielding the following equation:

$$2\left(1 - \frac{1}{|G|}\right) = N - \sum_{i=1}^{N} \frac{1}{|G_{xi}|} \tag{8.5}$$

And finally, we bring N inside the sum:

$$2\left(1 - \frac{1}{|G|}\right) = \sum_{i=1}^{N} (1 - \frac{1}{|G_{xi}|}) \tag{8.6}$$

At the beginning of the proof we stated that G was not the trivial subgroup. Therefore, the order of G is at least 2, meaning that the left hand side of equation 8.6 must be between 1 and 2, inclusive. We also know that the stabilizer G_{xi} of a given element x_i contains at least two elements, since each x_i is fixed by both the identity and by the rotation g in G about the axis through the origin which contains x_i. This constrains the right hand side of equation 8.6, giving us

$$\frac{1}{2} \leq 1 - \frac{1}{|G_{xi}|} < 1 \text{ with } 1 \leq i \leq N.$$

These constraints on the two sides of the equation tell us that N must be either 2 or 3.

If $N = 2$, then equation 8.4 gives $2 = |G(x_1)| + |G(x_2)|$. There can only be two poles in this situation, and they determine a single axis. All rotations in $G - \{e\}$ must be about this axis. These rotations take the plane containing the origin and perpendicular to the axis to itself. Thus we see that G is isomorphic to a subgroup of SO_2 and, by theorem 8.1, G must be cyclic.

When $N = 3$, we have multiple cases to deal with. Our first step is to identify these cases. We will begin by renaming our three orbits: for convenience's sake, $G(x_1), G(x_2), G(x_3)$ will be called $G(x), G(y), G(z)$. Equation 8.5 tells us that

$$1 + \frac{2}{|G|} = \frac{1}{|G_x|} + \frac{1}{|G_y|} + \frac{1}{|G_z|} \tag{8.7}$$

where the right hand side of the equation will, obviously, be strictly greater than 1. This narrows the number of cases we have to consider down considerably. Since we know that the cardinality of each of the stabilizers is at least 2, the largest any of the fractions on the right can be is $\frac{1}{2}$. If we start with halves and then reduce the fractions as much as possible, we find that only 4 cases remain where the right hand side of the equation is strictly greater than 1 and the cardinality of each stabilizer is at least 2. The cases are:

(a) $\frac{1}{2}, \frac{1}{2}, \frac{1}{n}$, where $n \geq 2$

(b) $\frac{1}{2}, \frac{1}{3}, \frac{1}{3}$

(c) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

(d) $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$

If we reduce the fractions beyond $\frac{1}{5}$, the right hand side will not be greater than 1; therefore, these are all the possible cases. All that remains for us to do is to treat each of these cases in turn, and to show that all lead to the kinds of subgroups we expect.

Case (a) First consider the subcase where $|G_x| = |G_y| = |G_z| = 2$. By equation 8.7, $|G| = 4$. Every element in G other than the identity has order 2. We can see this by taking an element g in G_x, not the identity, and supposing that g^2 is not the identity. Then g^2 will be equal to some element of another stabilizer, say h in G_y, where h is not the identity. But since g fixes x, simple calculation shows that h would be in G_x, providing a contradiction and confirming that the order of g is 2.

We will call the elements of G (g_1, g_2, g_3, e). Let g_3 generate G_z, and recall that any g in G preserves the distances between points. Selecting the pole called z, we see that x and $g(x)$ are equidistant from z, as are y and $g(y)$. Focusing on x instead, we see that z and $g(z)$, and y and $g(y)$, are similarly equidistant from x. We have, from the corollary to the orbit-stabilizer theorem, three orbits with two points each, and we see that it makes sense to rename the points $g(x), g(y), g(z)$ as $-x, -y - z$, as the conditions of equidistance define the familiar perpendicular axes of 3-dimensional space. Given three nontrivial elements each of order 2, with $g_3 = g_1 g_2$, we see that in this subcase we are dealing with the dihedral group with 4 elements, D_2.

If we have $|G_x| = |G_y| = 2$ and $|G_z| = n \geq 3$, then the order of G will be equal to $2n$ by equation 8.7. The axis through z in X is fixed by g in G_z, so G_z is cyclic of order n. Take g to be the minimal rotation in G_z, the generator of G_z. Then we have $x, g(x), \ldots, g^{n-1}(x)$, all distinct. If we had $g^r(x) = g^s(x)$ with $r > s$ and such that r does not equal s mod n, then $g^{r-s}(x) = x$. But, since z and $-z$ are the only two poles fixed by g^{r-s}, and x cannot equal $-z$ since $|G_x| = 2$, $g^r(x)$ cannot equal $g^s(x)$. Finally, we use the fact that g preserves distance, and we write

$$|x - g(x)| = |g(x) - g^2(x)| = \ldots = |g^{n-1}(x) - x|. \tag{8.8}$$

We can see that $x, g(x), \ldots, g^{n-1}(x)$ must be the vertices of a regular n-gon, and G is the rotational symmetry group of the n-gon, D_n.

Case (b) $|G_x| = 2$, $|G_y| = |G_z| = 3$. By equation 8.7, $|G| = 12$. By the corollary to the orbit-stabilizer theorem, the cardinality of the orbit of z is 4. Choose g the generator of G_z, and choose an element u in the orbit of z so that u is not z or $-z$. We can see that $u, g(u), g^2(u)$ are distinct, equidistant from z, and lie at the corners of an equilateral triangle. We can similarly choose to focus on u, or on any one of the other points in the orbit, and we see that for example $z, g(u), g^2(u)$ are equidistant from u and mark the corners of another equilateral triangle. Thus it is obvious that we are dealing with a regular tetrahedron, and that G is the rotational symmetry group of the solid.

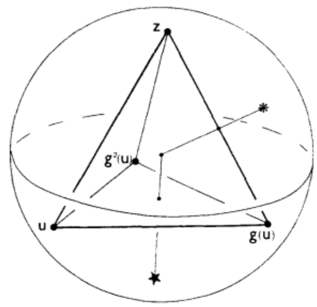

FIGURE 7. The rotational symmetries of a regular tetrahedron. [1] p. 109.

Case (c) $|G_x| = 2$, $|G_y| = 3$, $|G_z| = 4$. By equation 8.7, $|G| = 24$. By the corollary to the orbit-stabilizer theorem, the cardinality of the orbit of z is 6. Choose g the generator of G_z, and choose an element u in the orbit of z so that u is neither z nor $-z$. We can see that $u, g(u), g^2(u), g^3(u)$ are distinct, equidistant from z, and lie at the corners of a square. The last point in the orbit of z is $-z$. Similarly, we know that $-u$ will be in the orbit of u, which is also the orbit of z. $-u$ cannot be z or $-z$, and it cannot be $g(u)$ or $g^3(u)$ because those two points are adjacent to u. Therefore, $-u$ must be $g^2(u)$. We can look at the orbit starting at any one of the points, and we always see four equidistant points forming a square: thus, we see that the points are the vertices of a regular octahedron and G is the rotational symmetry group of the solid. We note in passing that since the octahedron and the cube are dual solids, G is also the rotational symmetry group of a cube.

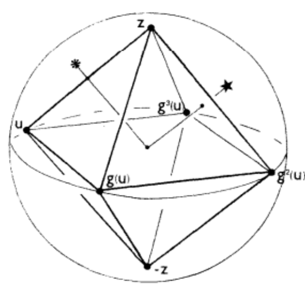

FIGURE 8. The rotational symmetries of an octahedron. [1] p. 109.

Case (d) $|G_x| = 2$, $|G_y| = 3$, $|G_z| = 5$. By equation 8.7, $|G| = 60$. By the corollary to the orbit-stabilizer theorem, the cardinality of the orbit of z is 12. Take g to be the minimal rotation which generates G_z, and choose u, v in the orbit of z such that $0 < |z-u| < |z-v| < 2$. This keeps everything in the unit circle, and allows us to see that we have $u, g(u), g^2(u), g^3(u), g^4(u)$, all distinct and equidistant from z, and similarly $v, g(v), g^2(v), g^3(v), g^4(v)$. These two sets of points lie at the vertices of two regular pentagons, and since the two sets of points are not the same distance away from z we have two separate regular pentagons. The 12th point in the orbit of z must be $-z$. Once again, we can shift our perspective to view our 12 points looking out from any one of the 12, and we will see the same picture: two separate, equidistant regular pentagons, and then the negative point to our viewpoint. Thus we see that our points lie at the vertices of a regular icosahedron, and G is the rotational symmetry group of the solid. As with our octahedron and cube, the icosahedron and the dodecahedron are duals, so G is also the rotational symmetry group of a dodecahedron.

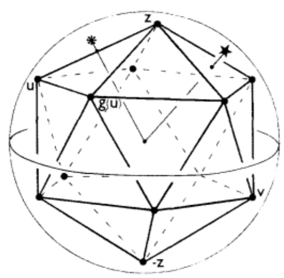

FIGURE 9. The rotational symmetries of an icosahedron. [1] p. 110.

□

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