ADVANCES IN APPLIED MATHEMATICS **14**, 267–306 (1993)

Knots, Tangles, and Electrical Networks

JAY R. GOLDMAN*

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

AND

LOUIS H. KAUFFMAN†

Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, Illinois 60680

The signed graphs of tangles or of tunnel links (special links in $\{R^3$-two parallel lines$\}$) are two terminal signed networks. The latter contain the two terminal passive electrical networks. The conductance across two terminals of a network is defined, generalizing the classical electrical notion. For a signed graph, the conductance is an ambient isotopy invariant of the corresponding tangle or tunnel link. Series, parallel, and star triangle methods from electrical networks yield techniques for computing conductance, as well as giving the first natural interpretation of the graphical Reidemeister moves. The conductance is sensitive to detecting mirror images and linking. The continued fraction of a rational tangle is a conductance. Algebraic tangles correspond to two terminal series parallel networks. For tangles, the conductance can be computed from a special evaluation of quotients of Conway polynomials and there is a similar evaluation using the original Jones polynomial.

CONTENTS

*Partially supported by NSA Grant No. MDA 904 67-H-2016.

†Partially supported by NSF Grant No. DMS-8822602 and the Program for Mathematics and Molecular Biology, University of California at Berkeley, Berkeley, California.

0196-8858/93 $9.00

1. Introduction

In this paper, we show how a generalization of the conductance of a classical electrical network gives rise to topological invariants of knots and tangles in three-dimensional space. The invariants that we define are chirality sensitive and are related, in the case of tangles, to the Alexander–Conway polynomial at a special value. The most general class of invariants defined here are the conductance invariants for special *tunnel links*. A tunnel link is an embedding of a disjoint collection of closed curves into the complement (in Euclidean three-space) of two disjoint straight lines or tubes.

We were led to these invariants by an analogy between the Reidemeister moves in the theory of knots and certain transformations of electrical networks. A knot or link projection is encoded by a (planar) signed graph (a graph with a $+1$ or a -1 assigned to each edge). The signed graphs are a subset of the signed networks (graphs with a nonzero real number assigned to each edge). These networks can be viewed as generalizations of passive resistive electrical networks, where the number on an edge is a generalized conductance. The series, parallel, and so-called star-triangle (or delta-wye) transformations of an electrical network, leave the conductance across *two terminals* unchanged. These notions generalize to invariance of a generalized conductance across two terminals of a signed network under corresponding transformations. When specialized to planar signed graphs, these transformations are the graph-theoretic translations of the Reidemeister moves for link projections (see Fig. 2.6 for a glimpse of this correspondence). Hence the invariance of conductance across two terminals of our graph yields an ambient isotopy invariant for the corresponding special tunnel link or tangle. Note that this electrical setting yields the first "natural" interpretation of the graph-theoretical version of the Reidemeister moves.

The paper is organized as follows. Section 2 gives background on knots, tangles, and graphs, and the translation of Reidemeister moves to graphical Reidemeister moves on signed graphs. The notion of a tunnel link is introduced and we discuss the relation with tangles. Section 3 reviews the background in classical electricity which motivates the connection between graphical Reidemeister moves and conductance-preserving transformations on networks. In Section 4, we define conductance across two terminals of a signed network in terms of weighted spanning trees. This coincides with the classical notion in the special case of electrical networks. We prove the invariance of conductance under generalized series, parallel, and star-delta transformations which then gives us the topological invariant for the corresponding tangles and special tunnel links. In Section 5, we show how the conductance invariant behaves under mirror images and we provide numerous examples. We also treat the relation between continued fractions, conductance, and Conway's rational tangles. Section 6 shows how, for tangles, the conductance invariant is related to the Alexander–Conway polynomial. The proof uses a combination of state models for the Conway polynomial and the Jones polynomial. Finally, a short appendix traces the flow of ideas from classical electrical calculations to spanning trees of networks.

2. Knots, Tangles, and Graphs

The purpose of this section is to recall basic notions from knot and tangle theory, introduce the new notion of a tunnel link and to show how the theory is reformulated in terms of signed planar graphs. This will enable us to pursue the analogy between knot theory and electrical networks via the common language of the graphs.

A *knot* is an embedding of a circle in Euclidean three-space $\mathbf{R}^3$. (The embedding is assumed to be smooth or, equivalently, piecewise linear [6].) Thus a knot is a simple closed curve in three-dimensional space. The standard simplest embedding (Fig. 2.1) will be denoted by the capitol letter U and is called the unknot. Thus in this terminology the unknot is a knot! Two knots are said to be *equivalent* or *ambient isotopic* if there is a continuous deformation through embeddings from one to the other. A knot is said to be *knotted* if it is not ambient isotopic to the unknot. The simplest nontrivial (knotted) knot is the trefoil as shown in Fig. 2.1. As Fig. 2.1 illustrates, there are two trefoil knots, a right-handed version K and its left-handed mirror image K^*. K is not equivalent to K^* and both are knotted [16].

A *link* is an embedding of one or more circles (a knot is a link). Equivalence of links follows the same form as for knots. In Fig. 2.2 we illustrate an *unlink* of two components, the *Hopf link*, consisting of two unknotted components linked once with each other, and the *Borommean rings*, consisting of three linked rings such that each pair of rings in unlinked.

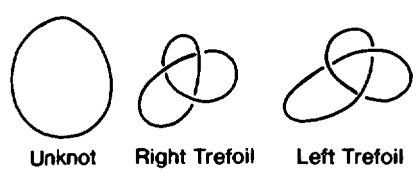

Figure 2.1

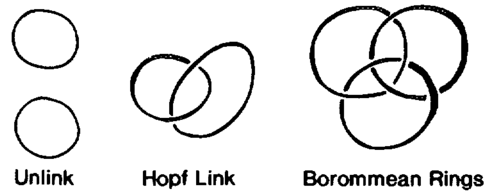

FIGURE 2.2

As these figures show, it is possible to represent knots and links by diagrammatic figures (*projections*). These projections are obtained by a parallel projection of the knot to a plane so that the strands cross transversely (Figs. 2.1, 2.2). The projections can also be thought of as planar graphs where each vertex (crossing) is 4-valent and where the diagram also shows which strand crosses over the other at a crossing.

Not only can one represent knots and links by diagrams, but the equivalence relation defined by ambient isotopy can be generated by a set of transformations of these diagrams known as the *Reidemeister moves*. Reidemeister [21] proved that two knots or links are equivalent (ambient isotopic) if and only if any projection of one can be transformed into any projection of the other by a sequence of the three Reidemeister moves (shown in Fig. 2.3) coupled with ordinary planar equivalence of diagrams given by homeomorphisms (i.e., continuous deformations) of the plane. We indicate the latter as a type-zero move in Fig. 2.3 (zero is indicated in

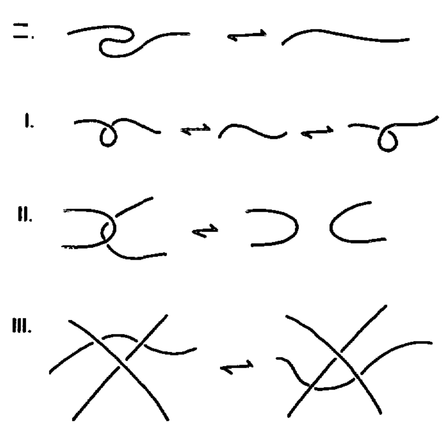

FIG. 2.3. Reidemeister moves.

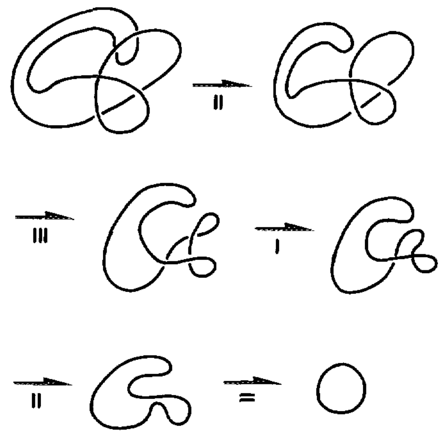

FIGURE 2.4

the Roman style by two parallel lines), showing the straightening of a winding arc. Each move, as shown in Fig. 2.3, is to be performed locally in a disc-shaped region without disturbing the rest of the diagram and without moving the endpoints of the arcs of the local diagrams that indicate the moves. In Fig. 2.4 we illustrate the use of the Reidemeister moves in unknotting a sample diagram.

We now translate this diagrammatic theory into another diagrammatic theory that associates to every knot or link diagram an arbitrary *signed planar graph* (a planar graph with a $+1$ or -1 assigned to each edge) as follows. Recall that a planar graph is a graph, together with a given embedding in the plane, where we usually do not distinguish between two such graphs that are equivalent under a continuous deformation of the plane.

Let K be a connected knot or link *projection* (i.e., it is connected as a 4-valent planar graph). It is easy to see (via the Jordan curve theorem [15] or by simple graph theory [22, Theorem 2-3]) that one can color the regions of K with two colors so that regions sharing an edge of the diagram receive different colors. Call such a coloring a *shading* of the diagram. Letting the colors be black and white, we refer to the black regions as the *shaded regions* and the white regions as the *unshaded* ones. To standardize the shading, *let the unbounded region be unshaded*. We can now refer to the shading of a connected diagram (Fig. 2.5).

Now there is a connected planar graph **G(K)** associated to the shading of K. The nodes (or vertices) of $G(K)$ correspond to the shaded regions of K. If two shaded regions have a crossing in common, the corresponding

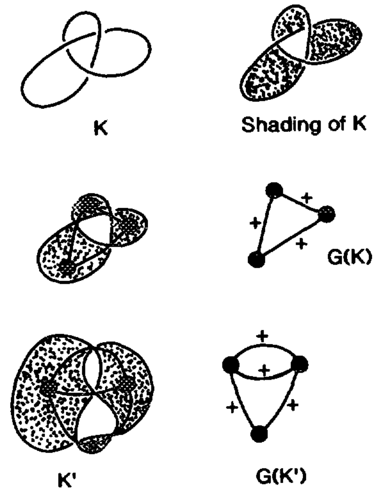

FIG. 2.5. Knots and their graphs.

nodes in $G(K)$ are connected by an edge (one edge for each shared crossing). Each edge of $G(K)$ is assigned a number, $+1$ or -1, according to the relationship of the corresponding crossing in K with its two shaded regions. If by turning the over crossing strand at a crossing counterclockwise, this line sweeps over the shaded region, then this crossing and the corresponding edge in $G(K)$ is assigned a $+1$; otherwise a -1. In Fig. 2.5, the construction of the signed graph is shown for the right-hand trefoil and for the figure eight knot. We shall refer to $G(K)$ as the *(signed) graph of the knot or link diagram* K. Note, by the construction, that since K is connected, $G(K)$ is connected.

Conversely, given a connected planar signed graph H, there is a link diagram $L(H)$ such that $G(L(H)) = H$ (see Fig. 2.5 and [31, 17]). Note that H connected implies that K is connected. Moreover, $L(G(K)) = K$ and thus $K \leftrightarrow G(K)$ is a bijection between connected link projections and connected signed planar graphs. (More precisely, the bijection is between equivalence classes of each under continuous deformation of the plane.) This correspondence is called the *medial* construction; K and H are called the *medials of each other*. The correspondence can be extended to nonconnected link diagrams and graphs by applying the constructions separately to each component of the link diagram or graph. This extension, which we also call the medial construction, is a bijection only if we ignore the relative positions of the components in the plane.

One can transfer the Reidemeister moves (of Fig. 2.3) to a corresponding set of moves on signed graphs [31]. These *graphical Reidemeister moves*

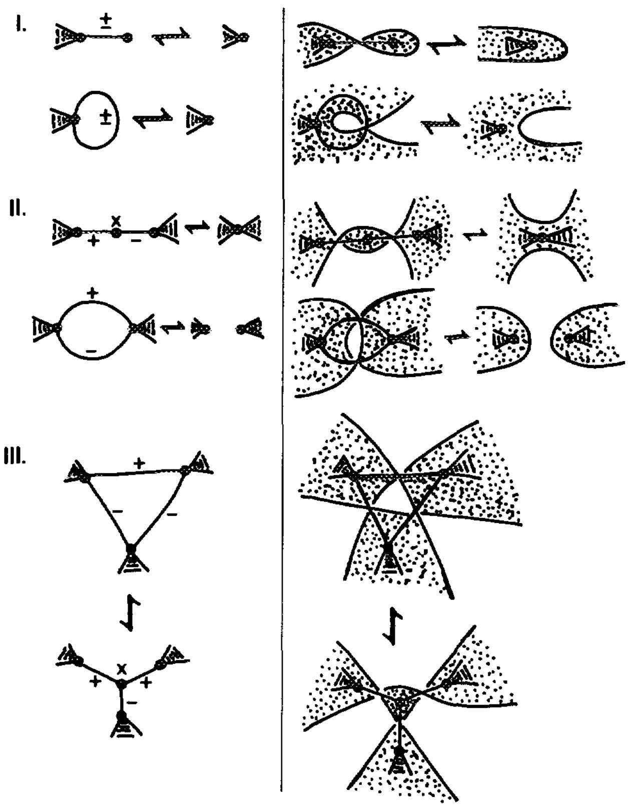

FIG. 2.6. Graphical Reidemeister moves.

are shown in Fig. 2.6, where, as for link diagrams, they are local moves, restricted to the signed configurations enclosed by the black triangles (which represent connections to the rest of the graph) and not affecting the other part of the graph. In Fig. 2.6, we also indicate informally how the graphical moves come about from the link diagram moves. The vertices incident to the triangles can have any valency, but the nodes marked with an x are *only* incident to the edges shown in the figures. Note that there are two graphical Reidemeister moves corresponding to the first Reidemeister move on link diagrams and two more corresponding to the second. This multiplicity is a result of the two possible local shadings. Move zero corresponds to allowing two-dimensional isotopies of the planar graph. The different possibilities for over and undercrossings in move three require exactly two of the three number on the triangle to have the same sign and the same for the star.

FIGURE 2.7

We now turn to the notions of tunnel knots and links, tangles, and their associated graphs—the topological concepts that are our main objects of study.

A *tunnel link* is a link that is embedded in Euclidean three-space with two infinite straight tunnels removed from it. More precisely, let D_1 and D_2 be two disjoint closed disks in the plane $\mathbf{R}^2$. Regard $\mathbf{R}^3$ as the Cartesian product $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}^1$. Then let the *tunnel space* $\tau(\mathbf{R}^3)$ be defined by $\tau(R^3) = \mathbf{R}^3 - [(D_1 \times \mathbf{R}^1) \cup (D_2 \times \mathbf{R}^1)]$. The theory of knots and links in the tunnel space is equivalent to the theory of link diagrams (up to Reidemeister moves) in the punctured plane $\mathbf{R}^2 - (D_1 \cup D_2)$. In other words, tunnel links are represented by diagrams with two special disks such that no Reidemeister moves can pass through those disks. For example, the three "knots" in Fig. 2.7 are distinct from each other in this theory, where the disks D_1, D_2 are indicated by the black disks in these diagrams. The relevance of tunnel links will become apparent shortly.

A (two input–two output) *tangle* is represented by a link diagram, inside a planar rectangle, cut at two points of the diagram. Two adjacent lines of one cut are regarded as the top of the tangle and emanate from the top of the rectangle, and the other two lines from the bottom. Hence, inside the rectangle, no lines with endpoints occur. Examples of tangles are shown in Fig. 2.8.

The two simplest tangles, shown in Fig. 2.8, are "∞" and "0." In "∞" each top line is connected directly to its corresponding bottom line. In "0" the two top lines are connected directly to each other, as are the two bottom ones. The third tangle shown in Fig. 2.8 is the *Borommean tangle* B. Note the two general tangle constructions shown in Fig. 2.8. The *numerator* of a tangle, denoted **n(T)**, is the link obtained from T by joining the top strands to each other and the bottom strands to each other, on the outside of the tangle box. The *denominator*, **d(T)**, is the link obtained by joining top strands to bottom strands by parallel arcs—as shown in Fig. 2.8. The figure illustrates that the numerator, $n(B)$, of the Borommean tangle is the Borommean rings (see Fig. 2.2) and the denominator, $d(B)$, is the Whitehead link (see [6]).

Two tangles are *equivalent* if there is an ambient isotopy between them that leaves the endpoints of the top and bottom lines fixed and is

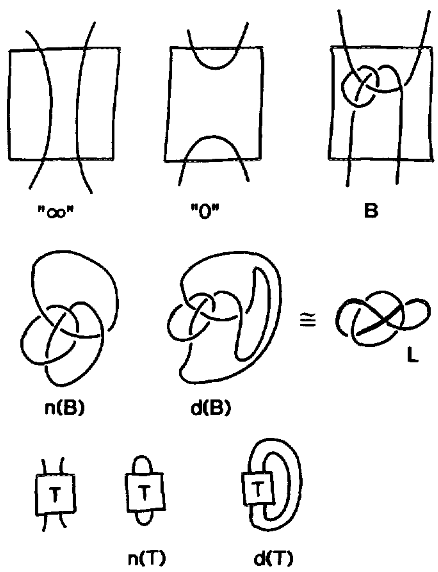

FIG. 2.8. Tangles: Numerators and denominators.

restricted to the space inside the tangle box. For more details on tangles, see [11, 26, 16, 6, 7].

Tangles are related to tunnel links as follows. If T is a tangle, let $\bar{n}(T)$ be the tunnel link obtained by placing disks in the ("top and bottom") regions of the numerator $n(T)$ (see Fig. 2.9a for an example). A tunnel link K gives rise to a tangle if its disks are in regions adjacent to the unbounded region of the diagram (or to a common region if the diagrams

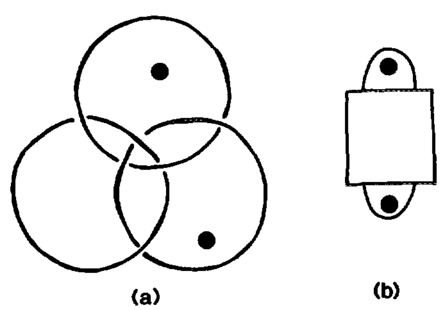

FIG. 2.9. Borommean tunnel link.

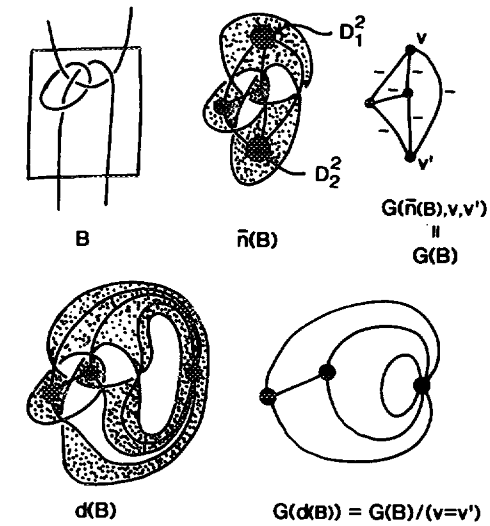

FIGURE 2.10

are drawn on the sphere). If this is the case the diagram will have the form of Fig. 2.9b, and hence $K = \bar{n}(T)$, for a tangle T. In this sense, tunnel links are generalizations of tangles.

A tunnel link K is said to be *special* if its disks lie in the shaded regions of K. This is the case for tunnel links of the form $\bar{n}(T)$. Note that the disks of a special tunnel link remain in shaded regions throughout isotopies generated by Reidemeister moves. *From now on, when we say tunnel link, we mean a special tunnel link.*

The *graph of a tunnel link* **K** has two distinguished nodes corresponding to the shaded regions occupied by the disks D_1 and D_2. Let these nodes by v and v', respectively. The graph of the special tunnel link (K, D_1, D_2) will also be called the *tunnel graph of* **K**, denoted by $(\mathbf{G(K)}, \mathbf{v}, \mathbf{v'})$ (Fig. 2.10). A signed planar graph with designated nodes v and v' will be called a *two-terminal (signed) graph with terminals* **v** *and* **v'**. By the medial construction, every such two-terminal graph is a tunnel graph for some tunnel link.

In the case of a tangle T, where $\bar{n}(T)$ defines a special tunnel link (disks in the top and bottom regions), the two terminal graph $(G(\bar{n}(T)), v, v')$, also denoted by $(\mathbf{G(T)}, \mathbf{v}, \mathbf{v'})$, is the *tangle graph of* T (Fig. 2.10). We also use $G(T)$ for short, when no confusion can arise. Again, by the medial construction, a two-terminal graph is a tangle graph if and only if its terminals are both incident to the unbounded region in the plane. Note that $G(d(T))$, the graph of the link $d(T)$, is obtained from the tangle graph $(G(T), v, v')$ by identifying the terminals v and v' (Fig. 2.10). Thus, *as graphs*, $G(n(T)) = G(T)$, while $G(d(T)) = G(T)/(v = v')$.

In order to be faithful to the restrictions on Reidemeister moves for tangle equivalence (moves only in the tangle box), we must add a corresponding restriction on the graphical moves allowed on the two terminal graphs corresponding to tangles. These restrictions are as follows:

1. If a node labeled x in a signed graph in Fig. 2.6 is a terminal, then the corresponding Reidemeister move is not allowed.

2. In the graph of a tangle, extend lines from each terminal to infinity, as shown in Fig. 2.11. No isotopy of the plane (graphical move zero) can carry parts of the graph across these lines.

Restriction 1 is general for the two terminal graphs corresponding to any tunnel link. Restriction 2 is necessary for tangles as shown in Fig. 2.11; otherwise, we could move a locally knotted piece on one strand over to another strand, which does not correspond to an ambient isotopy of tangles.

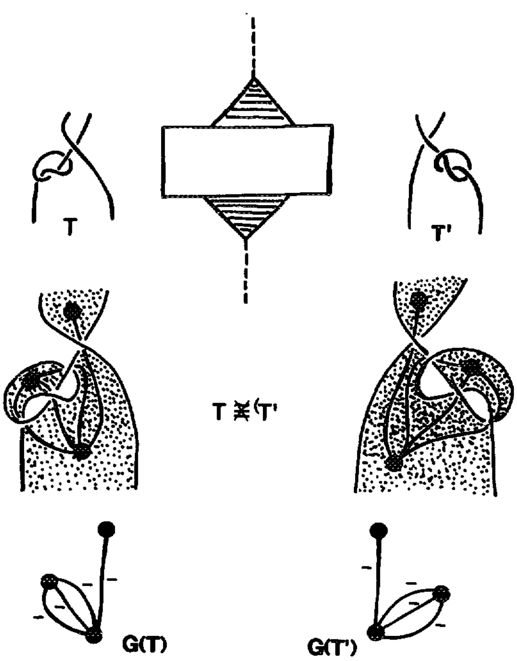

FIGURE 2.11

3. Classical Electricity

A classical (passive) electrical network can be modeled by a graph (with multiple edges allowed) such that each edge e is assigned a positive real number $r(e)$, its *resistance*. The basic theory of electrical networks is governed by Ohm's law and Kirchoff's laws (see [3, 23]).

Ohm's law, $V = ir(e)$, relates the current flowing in an edge e to the resistance $r(e)$ and the potential difference (voltage) measured across the endpoints. Kirchoff's *current law* states that the sum of the currents at a node (taking account of the direction of flow) is zero, while his *voltage law* says that the potential $V(v, v')$, measured across an arbitrary pair of nodes v and v', equals the sum of the changes of potential across the edges of any path connecting v and v'.

If a current i is allowed to flow into the network N, only at the node v, and leaves it, only at v', then the *resistance* $r(v, v')$ across v and v' is given by $r(v, v') = V(v, v')/i$. We find it more convenient to deal with the dual concept of the *conductance* $c(v, v')$ (*or* $c(N, v, v')$) across the nodes v and v', which is defined to be $1/r(v, v')$. We also define $c(e)$, the conductance of an edge to be $1/r(e)$.

It is an interesting fact that $c(v, v')$ can be computed strictly in terms of the $c(e)$, for all edges e, independent of any currents or voltages (a formal definition, in a more general context, is given in the next section). Moreover, there are two types of "local" simplifications that can be made to a network, the series and parallel addition of edges, *without changing* $c(v, v')$:

(i) If two edges with conductances r and s are *connected in parallel* (they are incident at two distinct nodes), then $c(v, v')$ does not change when the network is changed by replacing these edges by a single edge, with conductance $t = r + s$, connecting the same endpoints (Fig. 3.1).

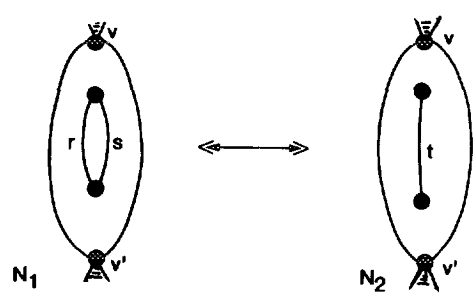

Figure 3.1

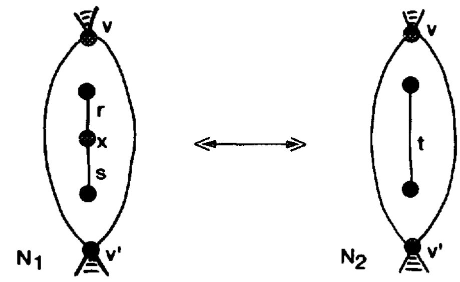

FIGURE 3.2

(ii) If two edges with conductances r and s are *connected in series* (they have exactly one vertex x in common and no other edges are incident to x) and $x \neq v, v'$, then $c(v, v')$ does not change when the network is changed by replacing these edges by a single edge of conductance t, where $1/t = 1/r + 1/s$ or $t = rs/(r+s)$ (Fig. 3.2).

There are two more subtle transformations of networks allowed, the so-called star to triangle and triangle to star substitutions, which leave the conductance invariant. We present it in a more symmetric form, the *star–triangle* relation [3, 19]:

(iii) Let the network N_1 contain the triangle with vertices u, w, y, with conductances a', b', c' on the edges (Fig. 3.3—other edges may also connect u, w, and y). Let the network N_2 be identical to N_1, except that the triangle is replaced by the star with endpoints u, w, y and a new center point x and edge conductances a, b, c (Fig. 3.3—no other edges are incident to x). If the conductances satisfy the relations

$$\begin{aligned} a' &= S/a, & b' &= S/b, & c' &= S/c, \\ a &= D/a', & b &= D/b', & c &= D/c', \end{aligned} \tag{1}$$

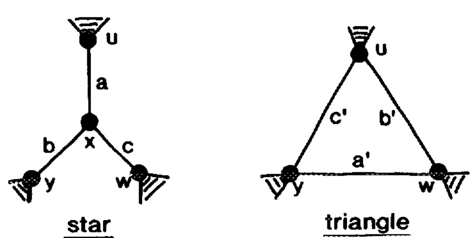

FIGURE 3.3

where $D = a'b' + a'c' + b'c'$ and $S = abc/(a + b + c)$ (of course, by (1), $S = D$), then $c(N_1, v, v') = c(N_2, v, v')$.

Note an absolutely remarkable "coincidence." The operation of replacing a star by a triangle or a triangle by a star corresponds, purely graph-theoretically, to the third graphical Reidemeister move (Fig. 2.6). Moreover, if we let $a = -1$, $b = c = +1$, $a' = +1$, $b' = c' = -1$ in Fig. 3.3, then these values correspond to the signs on graphical Reidemeister move three in Fig. 2.6 and *they also satisfy the relation in* (1). But the relations in (1) refer to *positive* conductances. The observation that it holds for the signed graphs describing the Reidemeister move is amazing and the goal of this paper is to understand it.

There is a generalization of electrical networks, which includes a generalization of the notion of conductance across two nodes. The two terminal signed graphs of tunnel links and tangles are a subclass of these networks. The graphical Reidemeister moves are special cases of series and parallel addition (for move two), star-triangle transformations (for move three), and one other transformation (for move one), and the conductance across two nodes is invariant under these operations. This allows us to define a new ambient isotopy invariant for special tunnel links and tangles. We develop these ideas in the next two sections.

4. Modern Electricity—The Conductance Invariant

Up to now we have discussed (two terminal) signed graphs and classical electrical networks, as well as motivating the need for allowing negative conductances. Now we think of these graphs as embedded in a broader class of "generalized electrical networks" and generalize the notion of conductance across the terminals.

We allow graphs with loops and multiple edges. A (signed) *network* is a graph with a nonzero real number **c(e)** assigned to each edge e, called the *conductance of* **e**. If T is a spanning tree of the network N, the *weight of* **T**, **w(T)**, is the product $\prod c(e)$, taken over all edges e of T. **w(N)**, the *weight of* **N**, is the sum $\Sigma w(T)$, taken over all all spanning trees T of N. If $c(e) > 0$, for all e, then N is an electrical network.

A *two-terminal* network (N, v, v') is a signed network N with two distinguished nodes v and v', called *terminals* (we allow $v = v'$). Let $N + e'$ (really $(N + e', v, v')$ be the net obtained from N by adding a new edge e', connecting v and v', with $c(e') = 1$ (Fig. 4.1). **w′(N + e′)** is the sum $\Sigma w(T)$, taken over all all spanning trees T of $N + e'$ *which contain* e'.

We define **c(N, v, v′)**, the *conductance across the terminals* **v** *and* **v′** in (N, v, v'). For now, let N be understood and just write $c(v, v')$. $c(v, v')$

FIGURE 4.1

will take values in the real numbers with the symbols $0/0$ and ∞ adjoined, with the conventions $r/0 = \infty$, for r a nonzero real number, and $r/\infty = 0$, for any real r.

DEFINITION. (1) If N is connected, then

$$c(v, v') = w(N)/w'(N + e') \tag{1}$$

(if $v = v'$, we let $w'(N + e') = 0$—there are no spanning trees containing the loop e'—and if $N = v$, i.e., there are no edges, we take $w(N) = 1$—there is only the empty spanning tree of weight 1—and, in this case, $c = 1/0 = \infty$).

(2) If N has two connected components, one containing v and the other containing v', then $c(v, v')$ is also given by Eq. (1), where $w(N) = 0$ (N has no spanning trees). In this case, $c(v, v')$ is either 0 or $0/0$.

(3) In all other cases, $c(v, v') = 0/0$.

Remarks. (i) If all edges have positive conductance, then our definition does yield the classical notion of conductance across two terminals in an electrical network (see [5, Theorem 3.4] and our appendix). We also define $r(N, v, v')$, the *resistance* between v and v' to be $1/c(N, v, v')$, which also reduces to classical resistance when all $c(e)$ are positive.

(ii) If e is an edge of the network N, let $N - e$ denote the network obtained from N by deleting e, and let N/e denote the network obtained from N by *contracting along the edge e* (deleting e and identifying its endpoints—Fig. 4.2).

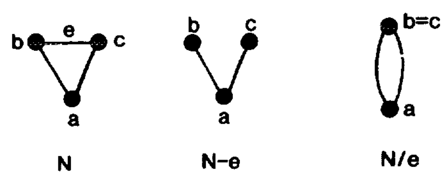

FIGURE 4.2

Now Eq. (1) can also be written as

$$c(v, v') = w(N' - e')/w(N'/e'), \tag{1'}$$

where $N' = N + e'$. Since $N = N' - e'$, the numerators in (1) and (1') are equal. If e' is not a loop, then $T \leftrightarrow T/e'$ is a bijection between those trees T of N which contain e' and all spanning trees of N/e'. Since $w(T) = c(e')w(T/e') = w(T/e')$, we have $w'(N') = w(N'/e')$, and the denominators of (1) and (1') are also equal. When e' is a loop or in other cases, the equality of the denominators is handled by the definition of $w(N'/e)$.

In the case of a tangle graph (G, v, v') of a tangle K, the somewhat more symmetric form (1') is important because $(G + e') - e'$ and $(G + e')/e'$ are the graphs of the numerator and denominator of K, respectively.

(iii) Both tangle and tunnel graphs are planar. However, conductance is a function of the underlying abstract graph and independent of any special planar embedding.

The following theorems show that the operations discussed in the last section, together with the hanging edge result in Theorem 1 which are routinely used to calculate resistance and conductance in electrical networks, are true for our more general networks. Special cases of these operations correspond to the invariance of conductance under the graphical Reidemeister moves and therefore the conductance of the graph of a tunnel knot or a tangle is an invariant of the corresponding tunnel knot or tangle.

Unless otherwise stated, *tree* will mean *spanning tree*. Recall that an edge e with endpoints a and b is a *loop* if $a = b$ and a *link* if $a \neq b$. When no confusion can arise, *we use the same symbol to denote an edge and its conductance*.

An edge e of the network N is a *hanging edge* if it is a loop or if it is a link connected to the rest of the graph at exactly one endpoint (Figs. 4.3a, b). Hanging edges play no role in computing conductance, i.e.,

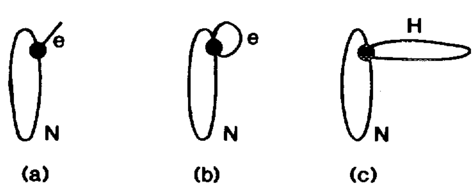

FIGURE 4.3

THEOREM 1. *If (N, v, v'), $v \neq v'$, has hanging edge e, then $c(N, v, v') = c(N - e, v, v')$.*

Proof. If e is a loop, it is not in any tree of N. Therefore, the trees of N and $N - e$ are identical, as are the trees of $N + e'$ and $N - e + e'$ containing e'. So the corresponding weights are the same and we are done.

If e is a link, then every tree of N (resp. $N + e'$) contains e, and the correspondence $T \leftrightarrow T - e$ defines a bijection between the trees of N and the trees of $N - e$ (resp. between the trees of $N + e'$ containing e' and the trees of $N - e + e'$ containing e'). Since $w(T) = ew(T - e)$, in all cases, e cancels from the numerator and denominator of the formula for $c(N, v, v')$ and $c(N, v, v') = c(N - e, v, v')$.

When the conductance of the hanging edge is ± 1, Theorem 1 implies that the conductance of a tangle graph or a tunnel graph across the terminals is invariant under Reidemeister move I. More generally, it is easy to prove that we can delete a "hanging" subnetwork H from a network N without changing the conductance (Fig. 4.3c), if $w(H) \neq 0, 0/0, \infty$. Figure 4.10 at the end of this section illustrates the problem when $w(H) = 0$.

THEOREM 2. *Let (N_1, v, v') be a two-terminal network which contains two links r and s connected in series, with their common node x not incident to any other edges, and assume $x \neq v, v'$.*

(i) If $r + s \neq 0$, let N_2 be the network obtained from N_1 by replacing the edges r and s by the single edge t (Fig. 3.2). If $1/t = 1/r + 1/s$ or $t = (rs)/(r + s)$, then

$$w(N_1) = (r + s)w(N_2), \qquad w'(N_1 + e') = (r + s)w'(N_2 + e'), \tag{2}$$

and

$$c(N_1, v, v') = c(N_2, v, v'). \tag{3}$$

(ii) If $r + s = 0$, let N_2 be the network obtained from N_1 by contacting the edges r and s to a point (Fig. 4.4). Then

$$w(N_1) = (-r^2)w(N_2), \qquad w'(N_1 + e') = (-r^2)w'(N_2 + e'), \tag{4}$$

and

$$c(N_1, v, v') = c(N_2, v, v'). \tag{5}$$

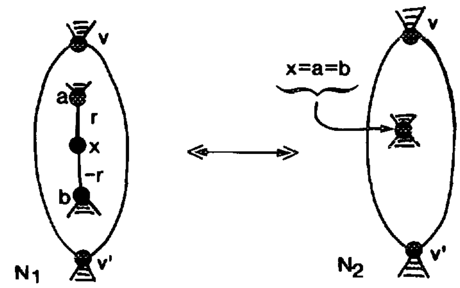

FIGURE 4.4

Proof. We need only prove (2) and (4), since taking quotients yields (3) and (5).

(i) First we work with $w(N_1)$ and $w(N_2)$. Since a tree in N_1 must contain the node x, it either contains both r and s or exactly one of them. In N_2, a tree either contains t or it does not.

If a tree T_1 in N_1 contains r, but not s, it has weight of the form rw. We pair T_1 with the tree $T_2 = T_1 - r + s$ containing s, but not r, of weight sw. Hence the two trees $\{T_1, T_2\}$ contribute $(r + s)w$ to $w(N_1)$. Let $\{T_1, T_2\}$ correspond to $T' = T_1 - r$ $(= T_2 - s)$, a tree in N_2 of weight w. Hence $w(T_1) + w(T_2) = (r + s)w(T')$.

A tree T in N_1 containing r and s has weight of the form wrs. Let T correspond to $T' = T - r - s + t$, a tree in N_2 of weight $tw = (rs/r + s)w$. Hence $w(T) = (r + s)w(T')$.

Combining both cases, which include all trees in N_1 and N_2, we see that $w(N_1) = (r + s)w(N_2)$. The same argument, restricted to trees containing e' proves that $w'(N_1 + e') = (r + s)w'(N_2 + e')$ and we are done.

(ii) We have two edges r and $-r$ in series. The proof requires a small modification of the proof in (i). Again we work with $w(N_1)$ and $w(N_2)$.

The sum of the weights of a tree T_1 in N_1 containing r but not $-r$, and the tree $T_2 = T_1 - r + (-r)$ is zero; so they contribute zero to $w(N_1)$.

A tree T in N_1 containing r and $-r$ has weight of the form $-r^2w$. Let T correspond to $T' = T - r - (-r)$, a tree in N_2 of weight w; so $w(T) = -r^2w(T')$. Conversely, a spanning tree T' in N_2 must contain x. Thus we can split the node x and reinsert the two edges r and $-r$ in series (going from N_2 back to N_1) obtaining the tree $T = T' + r + (-r)$. Therefore $T \leftrightarrow T'$ is a bijection between trees in N_1 containing r and $-r$ and all trees in N_2. Since these are the only types of trees contributing nonzero terms to $w(N_1)$ and $w(N_2)$, we have $w(N_1) = (-r^2)w(N_2)$. Similarly, $w'(N_1 + e') = (-r^2)w'(N_2 + e')$ and we are done.

Note that if the numerator or denominator in the conductance are zero, our arguments are still okay; so we have included the cases where the conductance may be infinite or $0/0$.

In electrical circuit terminology, the theorem just says that for two edges connected in series, the reciprocals of their conductances, viz., their resistances, can be added without changing $c(v, v')$. If $r + s = 0$, then $t = \infty$ (a "short circuit"), which is the same as contracting t to a point.

When $r = 1$, $s = -1$, Theorem 2 says that the conductance of a tangle graph or tunnel graph across the terminals is invariant under the series version of Reidemeister move II.

THEOREM 3. *Let (N_1, v, v') be a two-terminal network which contains two links r and s* connected in parallel (*the two edges are incident at two distinct vertices*).

(i) If $r + s \neq 0$, let N_2 be the network obtained from N_1 by replacing the edges r and s by the single edge t. If $t = r + s$ (Fig. 3.1), then

$$w(N_1) = w(N_2), \qquad w'(N_1 + e') = w'(N_2 + e'), \tag{6}$$

and

$$c(N_1, v, v') = c(N_2, v, v'). \tag{7}$$

(ii) If $r + s = 0$, let N_2 be the network obtained from N_1 by deleting the edges r and s (Fig. 4.5). Then

$$w(N_1) = w(N_2), \qquad w'(N_1 + e') = w'(N_2 + e'), \tag{8}$$

and

$$c(N_1, v, v') = c(N_2, v, v'). \tag{9}$$

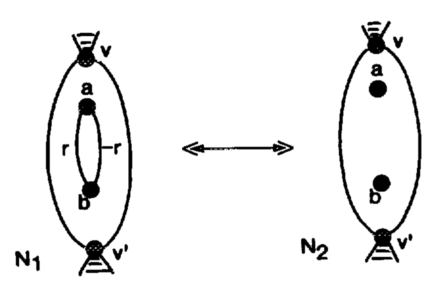

FIGURE 4.5

Proof. Equations (7) and (9) follow from (6) and (8), so we prove the latter. The proofs are very similar to those of Theorem 2 for series connections.

(i) Since r and s are in parallel, every tree in N_1 contains exactly one of the two edges or neither edge. If a tree T_1 in N_1 contains r, but not s, it has weight of the form rw. We pair T_1 with the tree $T_2 = T_1 - r + s$, containing s, but not r, of weight sw. Hence the two trees $\{T_1, T_2\}$ contribute $(r + s)w$ to $w(N_1)$. Let $\{T_1, T_2\}$ correspond to $T_3 = T_1 - r + t$, a tree in N_2 of weight tw. Therefore, $\{T_1, T_2\}$ and T_3 contribute the same weight to $w(N_1)$ and $w(N_2)$, respectively.

A tree T in N_1 which does not contain r or s is also a tree in N_2, not containing t. Hence it contributes the same weight to $w(N_1)$ and $w(N_2)$.

Since we have considered all trees in N_1 and N_2, we have $w(N_1) = w(N_2)$. Similarly, $w'(N_1 + e') = w'(N_2 + e')$.

(ii) When $r + s = 0$, the only modification we need in (i) is to note that $w(T_1) + w(T_2) = rw + (-r)w = 0$ and there is no corresponding T_3. Hence only the trees T not containing r or s may contribute nonzero weights to $w(N_1)$ and $w(N_2)$, and they are the same trees in both networks.

In electrical circuit terminology, the theorem just says that for two edges connected in parallel, their conductances can be added without changing $c(v, v')$. If $r + s = 0$, then $t = 0$ (no current can flow), which is the same as deleting the edge.

When $r = 1$, $s = -1$, Theorem 2 says that the conductance of a tangle graph or a tunnel graph across the terminals is invariant under the parallel version of Reidemeister move II.

The last class of operations we need to generalize from electrical theory are the star–triangle (or wye–delta) transformations, which are also used in a variety of other network theories (switching, flow, etc.) as well as in statistical mechanics (see [1, 20, 2, 18, 24]). A *triangle* (or *delta*) in a network is, of course, a subset of three edges uy, yw, wu, with distinct nodes u, y, w (Fig. 3.3). A *star* (or *wye*) in a network is a subset of three edges xu, xy, xw, on the distinct nodes u, y, w, x, where x, the *center* of the star, has degree three (Fig. 3.3).

Given a star in a network N, with conductances a, b, c and end-nodes u, y and center x, such that $\mathrm{Y} = a + b + c \neq 0$, a *star–triangle* (or *wye–delta*) transformation N consists of replacing the center and edges of the star by the triangle with nodes u, y, w and conductances a', b', c' (Fig. 3.3), given by

$$a' = S/a, \qquad b' = S/b, \qquad c' = S/c, \tag{10}$$

where $\mathbf{S} = abc/Y$. Equivalently,

$$a' = bc/Y, \qquad b' = ac/Y, \qquad c' = ab/Y. \tag{10'}$$

Given a triangle in a network N, with conductances a', b', c' and nodes u, v, w, such that $\mathbf{D} = a'b' + b'c' + c'a' \neq 0$, a *triangle–star* (or *delta–wye*) transformation N consists of replacing the triangle by the center and edges of the star with end-nodes u, y, w, center x, and conductances a', b', c' (Fig. 3.3), where

$$a = D/a', \qquad b = D/b', \qquad c = D/c'. \tag{11}$$

These operations are inverse to one another and Eqs. (10) and (11) have a pleasing symmetry. To prove this and obtain some useful relations, we start with a triangle, but state the results as a purely arithmetic lemma.

LEMMA. *Let a', b', c' be nonzero real numbers, such that $D = a'b' + b'c' + c'a' \neq 0$. Define a, b, c, by Eqs.* (11), *and let $Y = a + b + c$ and $S = abc/Y$. Then*

(i) $D^2 = Ya'b'c'$,
(ii) $Y, S \neq 0$,
(iii) $DY = abc$ *and* $S = D$,
(iv) *Eqs.* (10) *are true.*

Proof. (i)

$$Y = a + b + c = D/a' + D/b' + D/c', \qquad \text{by (11)}$$
$$= D(1/a' + 1/b' + 1/c') = D((a'b' + b'c' + c'a')/a'b'c') = D^2/a'b'c'.$$

(ii) By assumption, $D, a', b', c' \neq 0$, so $Y \neq 0$, by (i), and $S = abc/Y \neq 0$.

(iii) Multiplying the three equations in (11) yields $abc = D^2D/a'b'c'$, which by (i) equals $(Ya'b'c')D/a'b'c' = YD$, and, by the definition of S, $S = D$.

(iv) By Eq. (11), $a' = D/a$, which equals S/a, by (iii). Similarly for b' and c'.

Of course, (iv) yields the invertibility; i.e., beginning with a triangle ($D \neq 0$), replacing it with a star ($S \neq 0$, by (ii)), and replacing this star with a triangle, yields the original triangle. Conversely, we could start with a star (with $S \neq 0$, and a', b', c' defined by (10)) and prove that $D \neq 0$ and Eqs. (11) are satisfied. Thus $S \neq 0 \Leftrightarrow D \neq 0$.

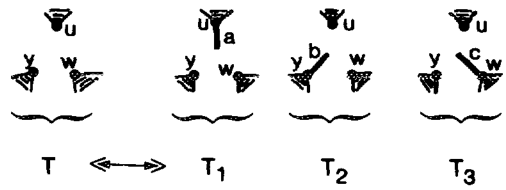

FIGURE 4.6

THEOREM 4. *If the network* (N_2, v, v') *is obtained from* (N_1, v, v') *by a star–triangle or a triangle–star transformation, where the center of the star is not a terminal, then*

$$w(N_2) = Yw(N_1), \qquad w'(N_2 + e') = Yw'(N_1 + e'),$$

and

$$c(N_1, v, v') = c(N_2, v, v').$$

Proof. Assume we go from N_1 to N_2 by a triangle–star as labeled in Figure 3.3. We first consider $w(N_1)$ and $w(N_2)$. A tree in N_1 contains 0, 1, or 2 edges of the triangle, while a tree in N_2 contains 1, 2, or 3 edges of the star (it must contain the center x). Let T be a tree in N_1.

(1) If T does not contain any edges of the triangle, let it correspond to the three trees $\{T_1 = T + a, T_2 = T + b, T_3 = T + c\}$ in N_2 (Fig. 4.6). Then $\Sigma w(T_i) = (a + b + c)w(T) = Yw(T)$.

(2) If T contains exactly one edge of the triangle, say a', let it correspond to the tree $T' = T - a' + b + c$ in N_2, where we replaced a' by the two edges b, c of the star which are incident to it (Fig. 4.7). Then $w(T') = w(T)bc/a'$ which, by the first equation in (10′), equals $Yw(T)$, and similarly if T contains b' or c'.

(3) If T contains two edges of the triangle, say a' and b', then replacing these edges by the pair b', c' or c', a' yields two more spanning

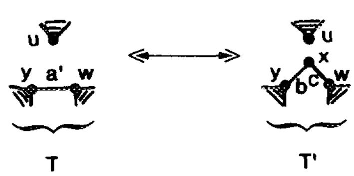

FIGURE 4.7

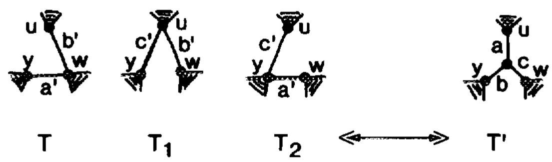

FIGURE 4.8

trees T_1 and T_2. Associate these three trees with the tree $T' = T - a' - b' + a + b + c$ in N_2; i.e., replace the triangle edges by the complete star (Fig. 4.8). Writing $w(T)$ in the form $a'b'w$, we then have $w(T_1) = b'c'w$, $w(T_2) = a'c'w$, and $w(T') = abcw$. By part (ii) of the lemma, $abc = DY$; so $w(T') = DYw$. But $w(T) + w(T_1) + w(T_2) = (a'b' + b'c' + c'a')w = Dw$ and, therefore, $w(T') = Y(w(T) + w(T_1) + w(T_2))$.

Combining all three cases, which include all trees contributing to $w(N_1)$ and $w(N_2)$, we have proved that $w(N_2) = Yw(N_1)$. The same reasoning applies to prove that $w'(N_2 + e') = Yw'(N_1 + e')$ (note that case 3 cannot occur if both terminals are nodes of the triangle). Hence the conductances are equal.

When the conductances on a star (or triangle) are ± 1 and not all of the same sign, then Theorem 4 says that the conductance of a tangle graph is invariant under the graphical Reidemeister move III.

We define the *conductance* $c(T)$ *of a special tunnel link* T *or of a tangle* T to be the conductance across the terminals of its two terminal signed graph, i.e., $c(T) = c(G(T), v, v')$. For the case of a tangle we have, by Remark (ii) at the beginning of this section,

$$c(T) = w(G(n(T)))/w(G(d(T))). \tag{12}$$

Since we have proved that two terminal signed graphs are invariant under all the graphical Reidemeister moves, we have our main theorem.

THEOREM 5. *The conductance of a special tunnel link or of a tangle is an ambient isotopy invariant.*

Not only do we have an invariant, but also some very powerful techniques for computing it. Using series, parallel, and star–triangle transformations, which usually transform the two-terminal graph to a more general two-terminal network with the same conductance, provides the general approach. For example we compute $c(v, v')$ for the Borommean tangle (Fig. 2.8). The series of transformations shown in Fig. 4.9, consisting of a star–triangle transformation followed by three parallel additions and

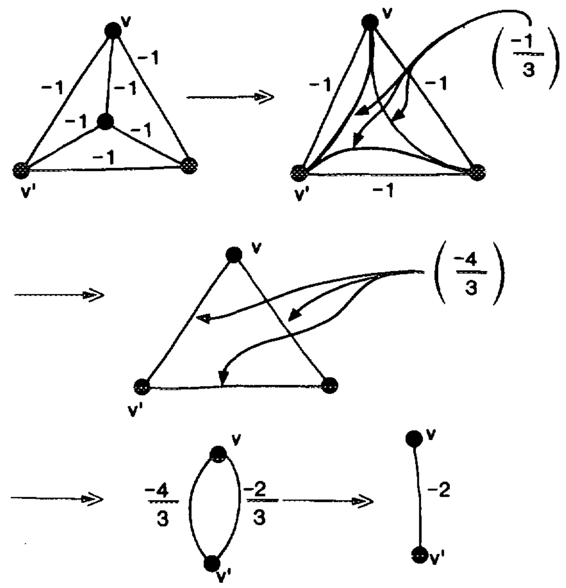

FIGURE 4.9

finally a series addition, convert the two-terminal graph to a two-terminal network with the same conductance. The final network consists of two terminals connected by one edge with conductance -2, and, therefore, $c(v, v') = -2$ by definition (Eq. (1)).

It is natural to ask whether we can always compute the conductance of a two-terminal signed network (or more special, a tangle or tunnel graph) using only the operations of series and parallel addition, star and triangle swaps, and deleting allowable hanging subnetworks. For classical planar electrical networks (positive conductance on all edges) a conjecture that this could always be done was made by A. Lehman in 1953 [20]. It was first proved true by Epifanov [10], with subsequent proofs by Truemper [28] and Feo and Proven [12]. In this case it is equivalent to using our operations to transform any connected planar electrical network to a network consisting of the two terminals connected by one edge.

When negative conductances are allowed on the edges, the question is more subtle and difficult. As we see in Fig. 4.10, a connected planar graph can be transformed to a disconnected one; hence the question is whether we can use our operations to transform any network to one where each connected component consists of one edge or one node. One difficulty is that star and triangle swaps can only be done when $S, D \neq 0$, whereas in the electrical case they can always be done. Series and parallel addition

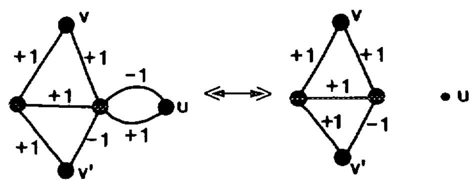

FIGURE 4.10

can be reduced to the classical case even when deletion and contraction occur (by allowing $c(e) = 0$ or ∞).

5. Topology: Mirror Images, Tangles and Continued Fractions

5.1. *Mirror Images*

The main purpose of this section is to present a set of examples which give some idea of the range of applicability of the conductance invariant. An important property of conductance is its ability to detect mirror images. The *mirror image* of a tunnel link or tangle with diagram K is the tunnel link or tangle whose diagram K^* is obtained from K by reversing all the crossings. If G is a network, G^* is the graph obtained from G by multiplying $c(e)$ by -1, for all e. Hence the tunnel or tangle graph of K^* is $(G(K)^*, v, v')$. A tunnel link or tangle is *achiral* if it is equivalent to its mirror image and *chiral* if not.

THEOREM. *If* $c(G, v, v') \neq 0, 0/0, \infty$, *then* $c(G^*, v, v') = -c(G, v, v')$. *If we also have* $G = G(K)$, *for some tangle or tunnel link* K, *then* $c(K^*) = -c(K) \neq c(K)$ *and* K *is chiral.*

Proof. By the definition of conductance (Eq. (4.1)), $c(G^*, v, v') = w(G^*)/w'(G^* + e')$. A spanning tree T of G corresponds to a spanning tree T^* of G^* (same edges—conductance multiplied by -1). If G and G^* have $n + 1$ vertices, then T and T^* have n edges. Therefore $w(T^*) = (-1)^n w(T)$ and $w(G^*) = (-1)^n w(G)$.

We have a similar correspondence between the spanning trees T of $G + e'$, containing e', and the spanning trees T^* of $G^* + e'$, containing e'. Since $c(e') = 1$, $w(T^*) = (-1)^{n-1} w(T)$ and $w'(G^* + e') = (-1)^{n-1} w'(G + e')$.

Hence, since $c(G, v, v') \neq 0, 0/0, \infty$ guarantees $w(G^*)$ and $w'(G^* + e')$ are not zero,

$$c(G^*, v, v') = \frac{(-1)^n w(G)}{(-1)^{n-1} w'(G + e')} = -c(G, v, v').$$

COROLLARY. *If the tunnel link or tangle K is* alternating *(the conductances of all the edges of G(K) have the same sign), then K is chiral.*

Proof. By the tree definition of conductance, $c(G, v, v') \neq 0, 0/0, \infty$, and we apply the theorem.

EXAMPLE 1. A tunnel link or tangle formed from the *trefoil* knot of Fig. 2.5 is alternating (with any choice of terminals) and the corollary applies. In particular, conductance detects the simplest type of knottedness.

EXAMPLE 2. The *figure eight* knot of Fig. 2.5 is achiral [16]. However, by the theorem, any way of turning this into a tunnel link or tangle (i.e., any choice of terminals in the corresponding graph) destroys this property since it has positive conductance (Fig. 5.1 shows one such example).

EXAMPLE 3. The *Hopf* link H (Fig. 2.2) has a graph consisting of two -1 edges in parallel.; thus there is only one choice of terminals, $G(H)$ is alternating and the corollary applies. In particular, we see that the conductance can detect the simple linking of two strands in tunnel links or tangles,

EXAMPLE 4. The *Borommean* tunnel link and tangle (Fig. 2.10) is alternating and thus chiral. Hence, we see that the conductance can detect the more subtle linking of this example.

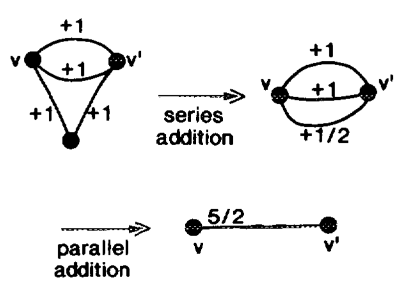

FIGURE 5.1

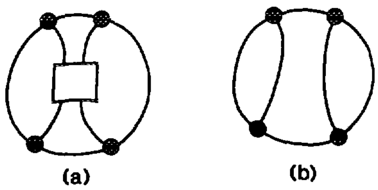

FIGURE 5.2

5.2 *Rational Tangles and Continued Fractions*

Let T be a *two-strand tangle* (only two strands are used to construct it) enclosed in the interior of a two-sphere, except for the endpoints of the four lines emanating from it which are on the boundary of the sphere (Fig. 5.2a). If there is an ambient isotopy taking T into the trivial ∞ tangle (Fig. 5.2b), with the restriction that the endpoints can only move on the sphere and the rest of the tangle remains inside the sphere during the deformation, then T is called a *rational tangle*. Intuitively, a rational tangle is a tangle that can "untwist" by moving the endpoints on the sphere. Every rational tangle is equivalent to a *canonical* rational tangle [6, 7, 11, 27], i.e., a tangle constructed as follows:

Start with two vertical strands (the ∞ tangle). Holding the top two endpoints fixed, twist the bottom two endpoints around each other some number of times in the positive or negative direction (the sign is the sign

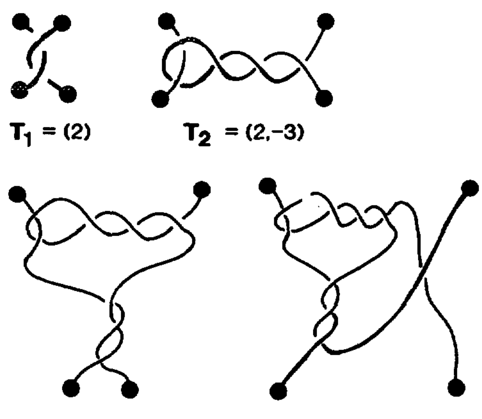

FIGURE 5.3

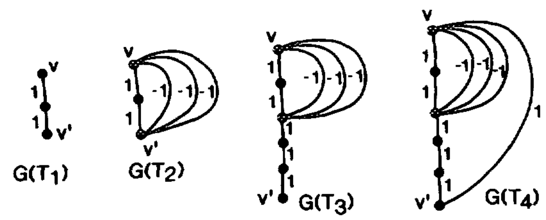

FIGURE 5.4

of the edges of the graph of this twist considered as a tangle T_1—see Figs. 5.3 and 5.4). Then hold the two left endpoints fixed and twist the two right endpoints some number of times in the positive or negative direction to obtain a new tangle T_2 (again, the sign is determined by the sign on the edges of $G(T_2)$ corresponding to this twist—Figs. 5.3 and 5.4). Continue by alternately twisting the two bottom endpoints and the two right endpoints, stopping either at the bottom or on the right after a finite number of twists.

For example, the sequence of twists $(2, -3, 3, 1)$ leads to the sequence of simple tangles shown in Fig. 5.3 and the corresponding tangle graphs of Fig. 5.4.

More generally, let T be the tangle given by the sequence of twists $(t_1, t_2, \ldots, t_r)$, $t_i \in \mathbf{Z}$. The graph of T is obtained by t_1 horizontal edges in series, connecting the top and bottom by t_2 edges in parallel, adding t_3 horizontal edges in series to the bottom, connecting the top and bottom by t_4 edges in parallel, and so on. The signs ($\neq 1$) on the edges are determined by the signs on the t_i's and the top and bottom points of the final graph are the terminals.

We compute the conductance of $G(T_4)$ by an alternating sequence of series and parallel additions which correspond to the twists (Fig. 5.5). So $c(G(T_4))$ is given by a continued fraction. In fact, using the sequence of twists to denote the corresponding simple tangle, we have, by the intermediate steps in Fig. 5.5,

$$c(T_1) = c((2)) = \tfrac{1}{2}, \qquad c(T_2) = c((2, -3)) = -3 + \tfrac{1}{2},$$

$$c(T_3) = c((2, -3, 3)) = \frac{1}{3 + 1/(-3 + 1/2)},$$

$$c(T_4) = c((2, -3, 3, 1)) = 1 + \frac{1}{3 + 1/(-3 + 1/2)}.$$

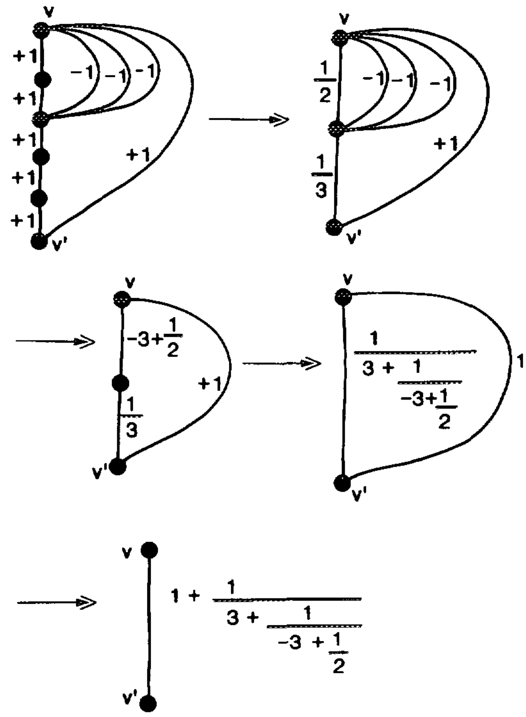

FIGURE 5.5

In general, we associate with the simple rational tangle $(t_1, \ldots, t_r)$ the continued fraction

$$t_r + \frac{1}{t_{r-1}} + \ddots + \frac{1}{t_1}$$

Then, the calculations of our examples immediately generalize and we have, for r even, that the continued fraction equals the conductance of the tangle, and, for r odd, it equals the resistance (the reciprocal of the conductance).

Conway taught us that every rational tangle is equivalent to a (not necessarily unique) simple tangle and that the rational number represented by the continued fraction of a simple tangle is a complete invariant for ambient isotopy of simple tangles [7, 6], so we have the remarkable fact that *Conway's continued fraction is either the conductance or resistance of the tangle.*

It would be interesting to have a proof of Conway's classification using our electrical ideas. The graphs of simple tangles (with arbitrary positive conductances on their edges) have long been known to electrical engineers as *ladder networks*, and they have used continued fractions to study them [30].

Our electrical concept can be extended to study Conway's notion of an *algebraic tangle* [7, 27]. We only indicate some ideas. The graphs of algebraic tangles correspond to series–parallel networks, well known in electrical theory [9]. Brylawski [4], in his studies of the matroids of series–parallel networks introduced an algebra for describing them. With some minor additions to this notation and some conventions on canonical embeddings of these networks in the plane, Brylawski's algebra can be extended to describe the graphs of algebraic tangles, with "monomials" in this algebra corresponding to simple rational tangles.

6. Classical Topology

In this section we show how our conductance invariant is related to the Alexander–Conway polynomial in the case of tangles. The conductance invariant for the more general class of special tunnel links appears to require further analysis in order to be related to classical topology (if indeed it is related).

Recall [14] that the Conway (Alexander) polynomial is an ambient isotopy invariant of oriented links. It is denoted by $\nabla_K(z) \in Z[z]$ and enjoys the following properties:

(i) If K is ambient isotopic to K', then $\nabla_K(z) = \nabla_{K'}(z)$.

(ii) $\nabla_U(z) = 1$ for the unknot U.

(iii) If K_+, K_-, and K_0 are three links differing at the site of one crossing, as shown in Fig. 6.1, then $\nabla_{K_+}(z) - \nabla_{K_-}(z) = z\nabla_{K_0}(z)$.

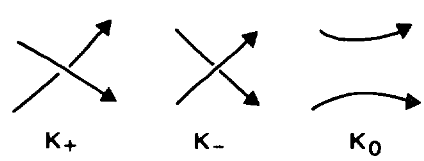

Figure 6.1

FIGURE 6.2

Recall also that the *writhe*, $wr(K)$, of an *oriented* link diagram (K can be a tangle as well) is equal to the sum of the *signs of the crossings* of K (defined in Fig. 6.2).

THEOREM. *Let F be a tangle, and $c(F)$ the conductance invariant, as defined in Section 4. Let $n(F)$ and $d(F)$ denote the numerator and denominator of F, as defined in Section 2. Then*

$$c(F) = -i^{\left[\frac{wr(d(F)) - wr(n(F))}{2}\right]} \frac{\nabla_{n(F)}(2i)}{\nabla_{d(F)}(2i)}.$$

Remark. It is assumed here that the numerator and denominator of F are oriented separately. If F itself can be oriented as in Fig. 6.3a, then both $n(F)$ and $d(F)$ can inherit orientation directly from F (Figs. 6.3b, c). Thus, in this case the theorem simply reads

$$c(F) = -i\frac{\nabla_{n(F)}(2i)}{\nabla_{d(F)}(2i)}.$$

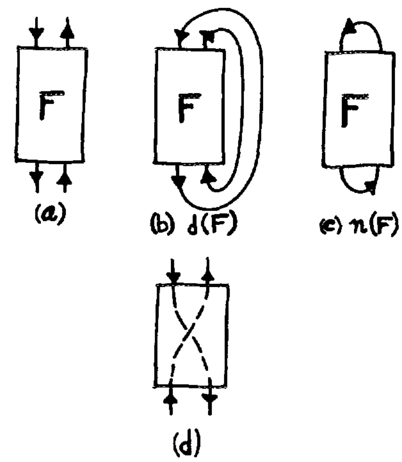

FIGURE 6.3

In general F can be oriented as in Fig. 6.3a or as in Fig. 6.3d, and then it is easy to reorient $d(F)$ so that $(wr(n(F)) - wr(d(F)))/2$ is the linking number between two components of $d(F)$. With this convention, the ratio of Conway polynomials is itself an invariant of the tangle.

The properties of $c(F)$ with respect to mirror images of tangles are then seen to correspond to known properties of the Conway polynomial (see [14]), but it is remarkable that they can be elucidated with the much more elementary means of the conductance invariant.

The proof of this theorem will be based on the state model for the Conway polynomial given in [15], plus some properties of the state model for the Jones polynomial in bracket form [16–18]. We shall assume this background for the proof.

Proof of the theorem. We must identify the tree sums of Section 4 with factors in the Conway polynomials of the numerator and denominator of the tangle F. Accordingly, let K be any (oriented) link diagram that is connected. Recall the state sum for $\nabla_K(z)$ given in [15]. In this state sum we have $\nabla_K(z) = \Sigma_S\langle K|S\rangle$, where S runs over the Alexander states of the diagram K and $\langle K|S\rangle$ is the product of the vertex weights for the state S. These vertex weights are obtained from the crossings in the link diagram by the conventions shown in Fig. 6.4. Each state S designates one of the four quadrants of each crossing. The designated quadrant has a vertex weight of 1, $\pm t$, $\pm t^{-1}$ and $\langle K|S\rangle$ equals the product of these vertex weights. The states are in one–one correspondence with Jordan–Euler trails on the link diagram, and they are also in one–one correspondence with maximal trees in the graph of the diagram.

Now focus on the case $\nabla_K(2i)$. Here $z = t - t^{-1} = i - i^{-1}$, whence $t = i$. We write this symbolically in Fig. 6.5. Familiarity with the bracket polynomial state model [16–18] then shows that this reformulation of the vertex weights implies that, *as state sums*

$$\nabla_K(2i) = i^{wr(K)/2}\langle K\rangle(i^{1/2}). \tag{1}$$

Now recall that the clock theorem [15] implies that any two Alexander states can be connected by a series of "clocking" moves. In terms of $G(K)$, the graph of K, a clocking move always removes one edge e from a

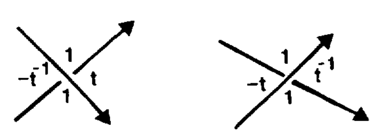

FIGURE 6.4

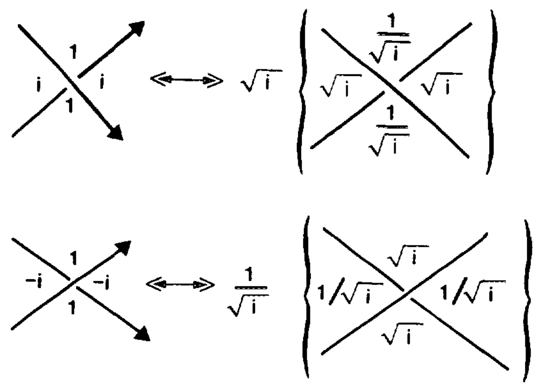

FIGURE 6.5

maximal tree T of $G(T)$ and replaces it with another edge e' to form a new tree T'. It is easy to verify for $\nabla_K(2i)$ that we have the equality $\langle K|S\rangle wt(e) = \langle K|S'\rangle wt(e')$, when S' is obtained from S by a single clocking move and $wt(e), wt(e')$ are the weights (± 1) assigned to the edges of $G(K)$ by our standard procedure in Section 2 (we used the notation $c(e), c(e')$ in Section 4). We leave the verification of this equation to the reader. It follows at once that for any state S, the product

$$p(K) = \langle K|S\rangle \prod_{e\in T(S)} wt(e)$$

is *invariant* under clocking (here $T(S)$ is the tree in $G(K)$ determined by the state S); hence it is a constant depending only on the diagram K. We can rewrite this equation as

$$\langle K|S\rangle = p(K) \prod_{e\in T(S)} wt(e), \tag{2}$$

since $(wt(e))^2 = 1$. Therefore,

$$\nabla_K(2i) = \sum_S \langle K|S\rangle = p(K) \sum_S \prod_{e\in T(S)} wt(e),$$

$$\nabla_K(2i) = p(K)w(G(K)),$$

where $w(G(K))$ is the tree sum (weight) defined at the beginning of Section 4.

Returning to our tangle F, we recall that the conductance invariant is given by Eq. 4.12, viz.,

$$c(F) = w(G(n(F)))/w(G(d(F))).$$

Therefore, the theorem will follow from a verification of the formula

$$p(n(F))/p(d(F)) = i.i^{(wr(n(F)) - wr(d(F)))/2}.$$

We omit the details of this verification, but point out that it is an application of Eqs. (1) and (2). These equations give a specific expression for $p(K)$, namely,

$$p(K) = i^{wr(K)/2}\left[\prod_{e \in T} wt(e) i^{wt(e)/2} \prod_{e \in T^*} i^{wt(e)/2}\right].$$

Here T is any maximal tree in $G(K)$ and T^* is the corresponding maximal tree in $G(K)^*$, where $G(K)^*$ denotes the planar graph dual to $G(K)$. By $e \in T$ (or $e \in T^*$) we mean that e is an edge of the tree T (or T^*). The formula for $p(n(F))/p(d(F))$ follows from this expression for $p(K)$. This completes the proof.

Remark. It is remarkable to note that, for tangles, our conductance invariant is related to *both* the Alexander–Conway and the Jones polynomials. This follows here because of the relationship (proved in the course of the theorem above) $\nabla_K(2i) = i^{wr(K)/2}\langle K\rangle(\sqrt{i})$; $\langle K\rangle$ is an unnormalized form of the Jones polynomial. In the usual form $V_K(t)$ for the Jones polynomial [13, 17], this identity becomes $\nabla_K(2i) = V_K(-1)$. This point of coincidence of the Alexander–Conway and the Jones polynomials is of independent interest.

Since the Jones polynomial can be expressed in terms of the signed Tutte polynomial [17], the conductance is also expressible as a special evaluation of the quotient of two signed Tutte polynomials.

Appendix: From Electricity to Trees

It is worthwhile recalling how the calculation of conductance occurs in electrical network theory, based on Ohm's and Kirchoff's laws. Ohm's law states that the electrical potential (voltage drop) between two nodes in a network is equal to the products of the current between these two nodes and the resistance between the two points. This law is expressed as $E = IR$, where E denotes the potential, I is the current, and R is the resistance. Kirchoff's law states that the total sum of currents into and out of any node is zero.

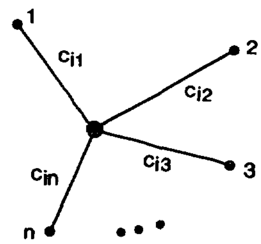

FIGURE A.1

These laws give the local rules for combining conductance. Conductance C is defined to be the reciprocal of resistance, viz., $C = 1/R$. Thus $E = I/C$ and $C = I/E$. Note that one also uses the principle that the voltage drop or potential along a given path in the network is equal to the sum of the voltage drops from node to node along the path.

With these ideas in mind, consider a node i with conductances $c_{i1}, c_{i2}, \ldots, c_{in}$ on the edges incident to this node (Fig. A.1). Here c_{ij} labels an edge joining nodes i and j. Then let E_i denote the voltage at node i with respect to some fixed reference node (the *ground*) in the network (the voltages are only determined up to a fixed constant). Then the current in the edge labeled c_{ik} is $(E_i - E_k)c_{ik}$. ($EC = I$, where C is the conductance.) Hence we have by Kirchoff's law,

$$\sum_{k=1}^{n} (E_i - E_k)c_{ik} = 0.$$

This equation will be true without exception at all nodes but *two*. These special nodes v, v' are the ones where we have set up a current source on a special edge between them as shown in Fig. A.2. We can assume that the

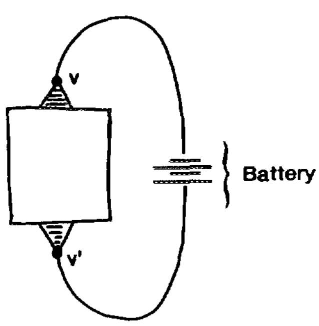

FIGURE A.2

battery edge delivers a fixed current I_0 and voltage E_0. Then the set of equations for voltages and currents takes the form

$$\begin{pmatrix} -I_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = M \begin{pmatrix} E_0 \\ E_1 \\ E_2 \\ \vdots \\ E_m \end{pmatrix},$$

where the nodes in the graph are labeled $0, 1, \ldots, m, 0'$, with 0 labeling v and $0'$ labeling v'. We take $0'$ to be the ground, whence $E_{0'} = 0$. We take M to be the matrix for the system of equations for nodes $0, 1, 2, \ldots, m$. Then, if M is invertible, we have

$$\begin{pmatrix} E_0 \\ E_1 \\ E_2 \\ \vdots \\ E_m \end{pmatrix} = M^{-1} \begin{pmatrix} -I_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

M is the famous Kirchoff matrix. It is a remarkable fact that the determinant of M enumerates the spanning trees in G (no battery edge) in the sense that, up to sign, $\det(M)$ is the sum over these trees of the products of the conductances along the edges of the tree. This is the matrix tree theorem (see [29, 25]).

EXAMPLE. Consider the equations for the network of Fig. A.3:

$$\begin{aligned}
&(\text{node } 0), -I_0 = (E_1 - E_0)a + (E_2 - E_0)b \\
&(\text{node } 1), \quad 0 = (E_0 - E_1)a \qquad + (E_2 - E_1)c + (E_{0'} - E_1)d \\
&(\text{node } 2), \quad 0 = \qquad (E_0 - E_2)b + (E_1 - E_2)c \qquad + (E_{0'} - E_2)e.
\end{aligned}$$

Thus

$$\begin{pmatrix} -I_0 \\ 0 \\ 0 \end{pmatrix} = M \begin{pmatrix} E_0 \\ E_1 \\ E_2 \end{pmatrix},$$

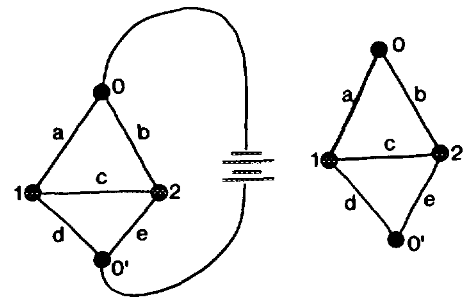

FIGURE A.3

where

$$M = \begin{pmatrix} -a-b & a & b \\ a & -a-c-d & c \\ b & c & -b-c-e \end{pmatrix}.$$

The matrix M is obtained from a matrix K defined for the graph G without the special edge

$$0 \cdot \text{———}\|\|\text{———} \cdot 0'.$$

The matrix K is a node—node matrix with nondiagonal entry k_{ij} equal to the sum of the labels (conductances) on the edges connecting nodes i and j, if such edges exist, and 0 if there is no edge. The diagonal entries k_{ii} are the negative sum of the labels of edges incident to node i. Hence

$$K = \begin{array}{c} 0 \\ 1 \\ 2 \\ 0' \end{array} \overset{\begin{array}{cccc} 0 & 1 & 2 & 0' \end{array}}{\begin{pmatrix} -a-b & a & b & 0 \\ a & -a-c-d & c & d \\ b & c & -b-c-e & e \\ 0 & d & e & -d-e \end{pmatrix}}.$$

Thus we see that M is obtained from K by removing the $0'$ row and the $0'$ column.

Now go back to our system of equations and solve for E_0:

$$E_0 = \frac{\operatorname{Det}\begin{bmatrix} -I_0 & a & b \\ 0 & -a-c-d & c \\ 0 & c & -b-c-e \end{bmatrix}}{\operatorname{Det} M}$$

$$= -I_0 \frac{\operatorname{Det}\begin{bmatrix} -a-c-d & c \\ c & -b-c-e \end{bmatrix}}{\operatorname{Det} M}.$$

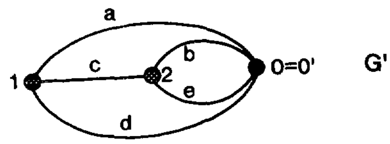

FIGURE A.4

Since $I_0 = CE$, where C is the conductance from v to v' (0 to 0′), we have

$$C = \frac{-\operatorname{Det} M}{\operatorname{Det}\begin{bmatrix} -a-c-d & c \\ c & -b-c-e \end{bmatrix}}.$$

Let G' be the graph obtained by identifying v with v' (Fig. A.4). The Kirchoff matrix K' is given by

$$K' = \begin{matrix} & 0 & 1 & 2 \\ 0 & -a-b-d-e & a+d & b+e \\ 1 & a+d & -a-c-d & c \\ 2 & b+e & c & -b-c-e \end{matrix}$$

and we take

$$M' = \begin{pmatrix} -a-c-d & c \\ c & -b-c-e \end{pmatrix}.$$

Thus $\operatorname{Det}(M')$ enumerates the trees in G'.

Therefore, we see that the conductance is given by the ratio of tree summations

$$C(G, v, v') = |\operatorname{Det}(M)/\operatorname{Det}(M')| = w(G)/w(G').$$

This is the full electrical background to our combinatorics and topology.

Remark. That the determinants of minors of the Kirchoff matrix enumerate spanning trees in the graph has been the subject of much study. In the electrical context, it is worth mentioning the Wang algebra [8] and the work of Bott and Duffin [9] that clarified some of these issues in terms of Grassman algebra.

References

1. H. Au-Yang and J. H. H. Perk, Onsager's star–triangle equation: Master key to integrability, *in* "Integrable Systems in Quantum Field Theory and Statistical Mechanics," Advanced Studies in Pure Mathematics, Vol. 19 pp. 57–94, Academic Press, Boston, 1989.
2. R. J. Baxter, "Exactly Solved Models in Statistical Mechanics," Academic Press, New York/London, 1982.
3. B. Bollobas, "Graph Theory: An Introductory Course," Graduate Texts in Math, Vol. 63, Springer-Verlag, New York, 1979.
4. T. H. Brylawski, A combinatorial model for series–parallel networks, *Trans. Am. Math. Soc.* **154** (1971), 1–22.
5. T. H. Brylawski, A determinantal identity for resistive networks, *SIAM J. Appl. Math.* **32**, No. 2 (1977), 443–449.
6. G. Burde and H. Zieschang, "Knots," deGruyter Berlin, 1986.
7. J. H. Conway, An enumeration of knots and links, and some of their algebraic properties, *in* "Computational Problems in Abstract Algebra, Proc. Conference in Oxford 1967," pp. 329–358, Pergammon, Elmsford, NY, 1970.
8. R. J. Duffin, An analysis of the Wang algebra of networks, *Trans. Am. Math. Soc.* **93** (1959), 114–131.
9. R. J. Duffin, Topology of series parallel networks, *J. Math. Anal. Appl.* **10** (1965), 303–318.
10. G. V. Epifanov, Reduction of a plane graph to an edge by a star–triangle transformation, *Soviet. Math. Dokl.* **166** (1966), 13–17.
11. C. Ernst and D. W. Summers, A calculus for rational tangles—Applications to DNA recombination, *Math. Proc. Cambridge Philos. Soc.* **108** (1990), 489–515.
12. T. A. Feo and J. S. Proven, Delta–wye transformations and the efficient reduction of two-terminal planar graphs, Univ. of North Carolina, Chapel Hill, 1988, preprint.
13. V. F. R. Jones, A polynomial invariant for links via von Neumann algebras, *Bull. Amer. Math. Soc.* **129** (1985), 103–112.
14. L. H. Kauffman, The Conway polynomial, *Topology* **26** (1980), 101–108.
15. L. H. Kauffman, "Formal Knot Theory," Princeton Univ. Press, Mathematical Notes #30, (1983).
16. L. H. Kauffman, "On Knots," Annals of Math. Studies, Vol. 115, Princeton Univ. Press, Princeton, New Jersey, 1987.
17. L. H. Kauffman, New invariants in the theory of knots, *Amer. Math. Monthly* **95**, No. 3 (1988), 195–242.
18. L. H. Kauffman, "Knots and Physics," World Scientific, Singapore, 1991.
19. A. E. Kennelly, The equivalence of triangles and three pointed stars in conducting networks, *Elec. World Eng.* **34** (1889), 413–414.
20. A. Lehman, Wye–delta transformations in probabilistic networks, *J. SIAM* **11** (1963), 773–805.
21. K. Reidemeister, "Knot Theory," BSC Associates, Moscow, ID, 1983; transl. of "Knotentheorie," Springer, Berlin, 1932.
22. T. L. Saaty and P. C. Kainen, "The Four Color Theorem: Assaults and Conquest," McGraw–Hill, NY, 1977.
23. S. Seshu and M. B. Reed, "Linear Graphs and Electrical Networks," Addison–Wesley, Reading, MA, 1961.
24. C. E. Shannon, A symbolic analysis of relay and switching circuits, *Trans. Am. Inst. Elec. Eng.* **57** (1938), 713–723.

25. D. Stanton and D. White, "Constructive Combinatorics," Undergraduate Texts Math., Springer-Verlag, New York, 1986.
26. D. W. Sumners, Untangling DNA, *Math. Intelligencer* **12**, No. 3 (1990), 71–80.
27. M. Thistlethwaite, Knot tabulations and related topics, *in* "Aspects of Topology in Memory of Hugh Dowker" (I. M. James and E. H. Kronheimer, Eds.), London Math. Soc. Lecture Notes Series, No. 93, pp. 1–76, Cambridge Univ. Press, Cambridge, 1984.
28. K. Truemper, On the delta–wye reduction for planar graphs, *J. Graph Theory* **13** (1989), 141–148.
29. W. Tutte, "Graph Theory," Encyclopedia of Mathematics and Its Applications, Vol. 21, Cambridge Univ. Press, Cambridge, 1984.
30. M. E. Van Valkenburg, "Introduction to Modern Network Synthesis," Wiley, New York, 1960.
31. T. Yajima and S. Kinoshita, On the graphs of knots, *Osaka Math. J.* **9** (1957), 155–163.