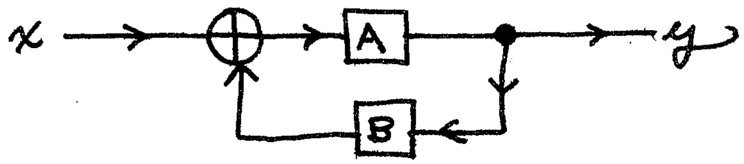
Linear Equations and Feedback

by Louis H. Kauffman

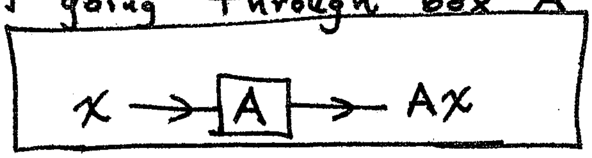
I. Introduction

One may not ordinarily think of a simple system of linear equations as a model for a feed-back process. Yet this can be the case! Consider the following diagram:

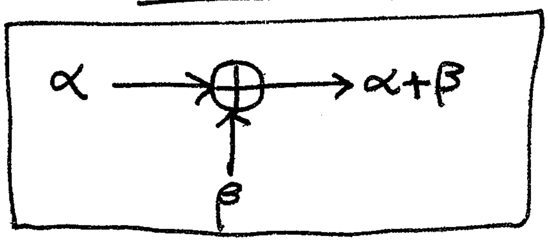


This diagram indicates the signal or information flow in a system whose input is x, whose output is y. The signal x is transformed by the black box A and fed-back through the box B to be recombined ($\oplus$) with the original input.

Suppose we represent the result of going through box A by Ax:



and combination ($\oplus$) by $+$:



②

Then we can understand the <u>behaviour</u> of the circuit: First x goes through and is fed-back. (We assume that the input x is held fixed in time.) Then the "new input" is $x + BAx$. This results in further new input of

$x + BAx + BA(x+BAx)$, and so on....

The system is said to be a <u>linear system</u> if the <u>transfer functions</u> A and B are linear so that $A(\alpha+\beta) = A\alpha + A\beta$ etc. Then we get the sequence of inputs and outputs:

input	output
x	Ax
$x+BAx$	$Ax+ABAx$
$x+BAx+BABAx$	$Ax+ABAx+ABABAx$
$x+BAx+BABAx+BABABAx$	$Ax+ABAx+ABABAx+ABABABAx$
...	...

If we let the system run for a very long time then we expect it to settle into a <u>stable state</u>. At the stable state, the input and output are no longer perturbed by the feedback. This means that

$$A(x + By) = y$$

since the feed-back is By and the new output is $A(x + \langle \text{feedback} \rangle)$.

<u>Thus</u>

$$A(x+By) = y$$

$$Ax + ABy = y$$

$$Ax = y(1-AB)$$

$$\therefore \quad y = \left(\frac{A}{1-AB}\right)x$$

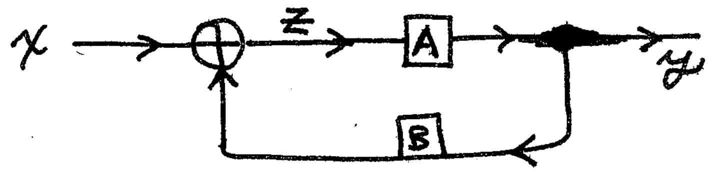
and formally,

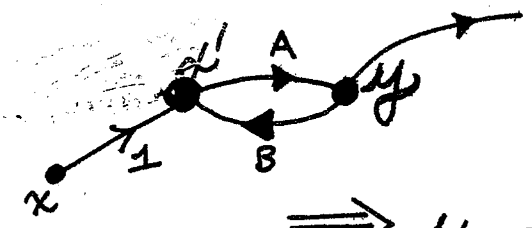
$$y = A(1+AB+ABAB+ABABAB+\cdots)x$$

$$y = Ax + ABAx + ABABAx + \cdots$$

This result coincides with our recursive analysis of the situation.

And we obtained it by solving a simple linear equation. Lets look at this again:

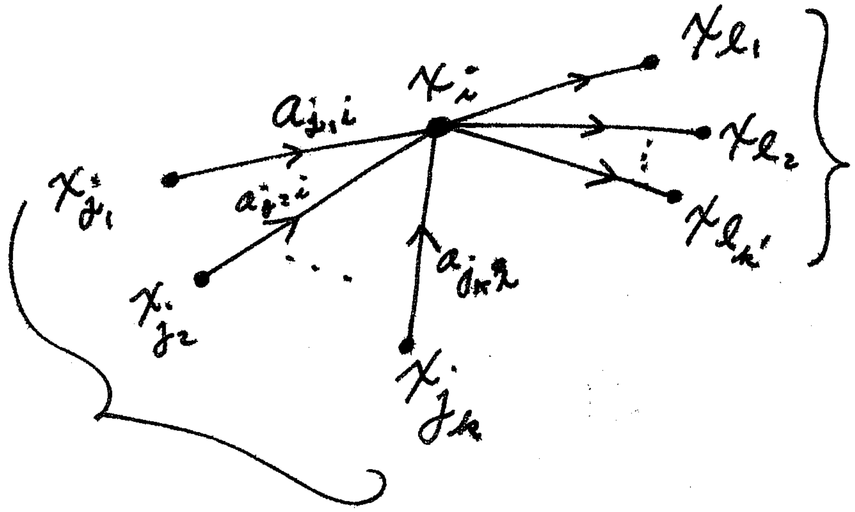




$$\boxed{\begin{aligned} x' &= 1x + By \\ y &= Ax' \end{aligned}}$$

$$\Longrightarrow y = A(x+By)$$

Here we abbreviate the linear system with a graph with <u>nodes</u> (•) that ~~represent~~ the summing (⊕) of the earlier notation. Each node is labeled by a letter x_i. Each arc (edge) is labelled by a letter A, B, C, ... representing a transfer function. The <u>nodal equation</u> expresses x_i as a sum of transfer functions applied to x_j's that are directed into the node labelled x_i:

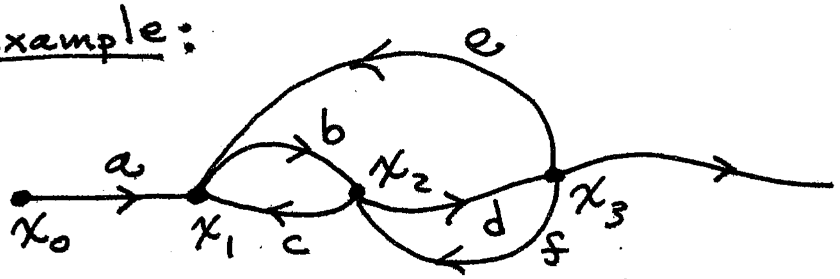


<u>Nodal Equation</u>

$$\boxed{x_i = a_{j_1 i}\, x_{j_1} + \cdots + a_{j_K i}\, x_{j_K}}$$

Solving this system of equations will give the stable state of the system.

Example:



Here the input is x_0 and the output is x_3. There are feed-backs from x_2 and x_3 back to x_1. The nodal equations are:

$$x_1 = a x_0 + c x_2 + e x_3$$
$$x_2 = b x_1 + f x_3$$
$$x_3 = d x_2$$

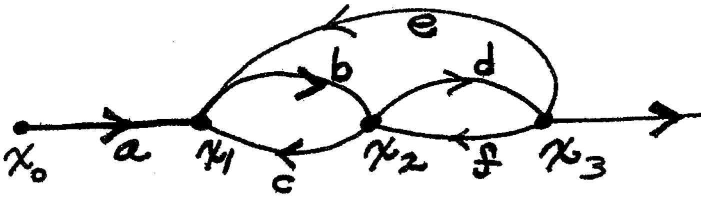
Since x_0 is regarded as input, we re-write in the form

$$\begin{aligned} -a x_0 &= -x_1 + c x_2 + e x_3 \\ 0 &= b x_1 - x_2 + f x_3 \\ 0 &= \phantom{b x_1 -{}} d x_2 - x_3 \end{aligned}$$

$$\begin{bmatrix} -a x_0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & c & e \\ b & -1 & f \\ 0 & d & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The linear algebra problem is, of course, to solve for x_1, x_2, x_3 in terms of x_0. (Particularly x_3, since we regard it as output)

Note the form of the matrix for this system.



G

$$\begin{bmatrix} +a x_0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} +1 & -c & -e \\ -b & +1 & -f \\ 0 & -d & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$M = \begin{bmatrix} +1 & -c & -e \\ -b & +1 & -f \\ 0 & -d & +1 \end{bmatrix}$$

Each diagonal entry of M equals $+1$.
The columns correspond to the nodes that are <u>not</u> inputs.

<u>Thus</u> $M_{ii} = +1$

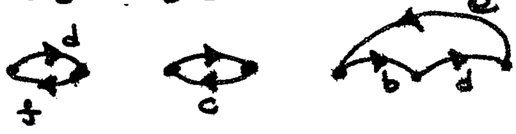
$M_{ij} = -\left(\begin{array}{l}\text{label on a line} \\ \text{directed } \underline{\text{from}}\ \underline{x_j}\ \underline{\text{to}}\ \underline{x_i} \\ (\text{for } i \neq j)\end{array}\right)$

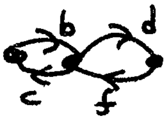
Now to solve the system, we can use Cramer's rule. This requires computing the detetermninant $|M|$:

$$|M| = + \begin{vmatrix} +1 & -f \\ -d & +1 \end{vmatrix} + (-c) \begin{vmatrix} -b & -f \\ 0 & +1 \end{vmatrix} - e \begin{vmatrix} -b & +1 \\ 0 & -d \end{vmatrix}$$

$$= +(1-df) + (-c)(-b) - e(bd)$$

$$= +1 - df - bc - ebd$$



Thus directed circuits in the graph (1) (i.e. feed-back loops) appear in the determinant expansion. Note that each circuit has no repeated vertices (thus  does not appear). There is a general result in back of this!

Before discussing the general result lets complete the computation:

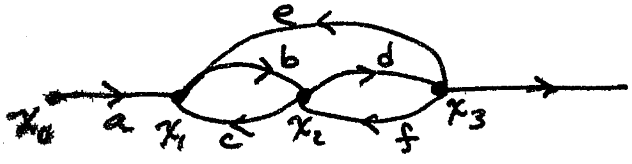
$$x_3 = \frac{\begin{vmatrix} 1 & -c & a x_0 \\ -b & 1 & 0 \\ 0 & -d & 0 \end{vmatrix}}{|M|}$$

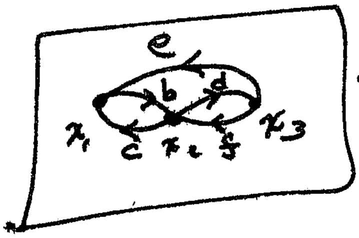
(By Cramer's Rule.)

$$\therefore \quad x_3 = \frac{a x_0 [\, b d \,]}{1 - df - bc - ebd}$$

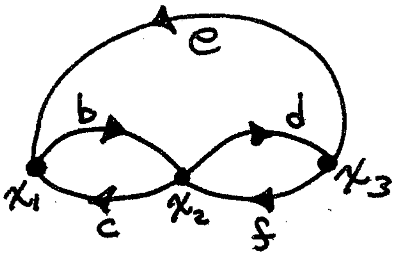
$$\boxed{\frac{x_3}{x_0} = \frac{abd}{1 - df - bc - ebd}}$$

abd is a directed path from x_0 to x_3.





$\leftrightarrow M$



Γ

$\longleftrightarrow$

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & 1 & -c & -e \\ x_2 & -b & 1 & -f \\ x_3 & 0 & -d & 1 \end{array}$$

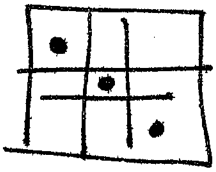
$M(\Gamma)$

This is how we associate a matrix $M(\Gamma)$ to a directed graph Γ.

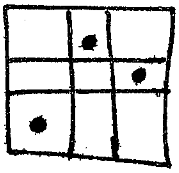
Now consider what is involved in computing $|M(\Gamma)|$.

$|M(\Gamma)|$ = a sum of products of elements of $M(\Gamma)$ corresponding to a permutation of 1, 2, 3 times the sign of the permutation.

Thus 123 $\longleftrightarrow$



312 $\longleftrightarrow$



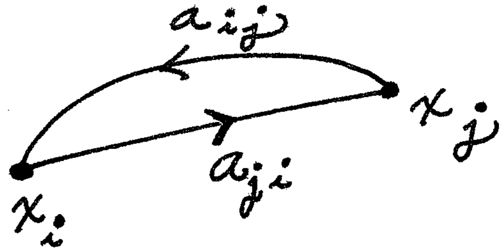
Let $\varepsilon(p)$ denote the sign of a permutation. Then if M is an $n \times n$ matrix, then

$$|M| = \sum_{\sigma} \varepsilon(\sigma) M_{1\sigma_1} M_{2\sigma_2} \cdots M_{n\sigma_n}$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a permutation of $(1, 2, \ldots, n)$.

$$\begin{array}{c|cccccc} & x_1 & x_2 & x_3 & & & x_n \\ \hline x_1 & 1 & -a_{12} & -a_{13} & & & \\ x_2 & -a_{21} & 1 & & & & \\ x_3 & -a_{31} & & 1 & & & \\ x_4 & & & & & & \\ \vdots & & & & \ddots & & \\ & & & & & 1 & \\ x_n & & & & & & 1 \end{array} = M(\Gamma)$$

$$M_{ii} = 1$$
$$M_{ij} = -a_{ij} \quad (i \neq j)$$



In Γ

$$|M(\Gamma)| = \sum_{\sigma} \varepsilon(\sigma) M_{1\sigma_1} M_{2\sigma_2} \cdots M_{n\sigma_n}$$

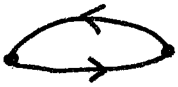
A permutation is a product of cycles. These correspond to the loops in the graph!

example: $n = 5$, $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$
$= (3, 2, 1, 4, 5)$

$1 \to 3 \to 1$

$2 \to 2$
$4 \to 4$
$5 \to 5$

$$\varepsilon(\sigma) M_{1\sigma_1} M_{2\sigma_2} M_{3\sigma_3} M_{4\sigma_4} M_{5\sigma_5}$$
$$\|$$
$$\varepsilon(\sigma) M_{13} M_{31} \; M_{22} M_{44} M_{55}$$
$$\|$$
$$\varepsilon(\sigma) \left(-(a_{31})\right)(-a_{13}) \; (1)(1)(1)$$
$$\|$$
$$\varepsilon(\sigma) \, a_{31} a_{13}$$



Here $\varepsilon(\sigma) = -1$

exemple: $\underline{n = 6}$

$$\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$$

$$= (2, 3, 4, 1, 6, 5)$$

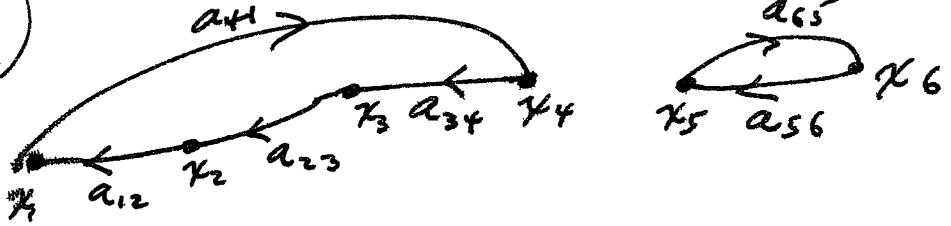
$$1 \to 2 \to 3 \to 4 \qquad 5 \to 6$$

Note that the sign of a cycle of length K is $(-1)^{K+1}$, and in the expansion, we will get K (-1)'s from the individual terms. Thus <u>each cycle contributes (-1) to the eventual sign of a term</u>.

<u>Here</u> $\varepsilon(\sigma) = (-1)^{4+1}(-1)^{2} = (-1)(-1) = +1$

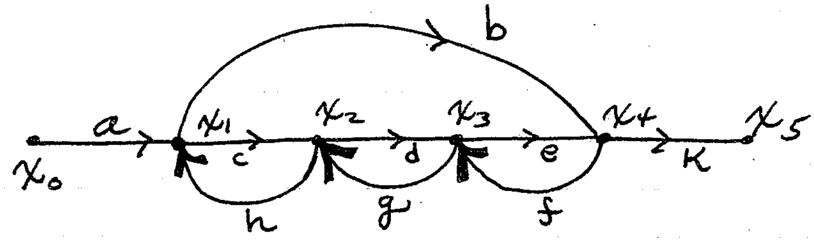
$$\varepsilon(\sigma)\, M_{1\sigma_1} \cdots M_{6\sigma_6}$$
$$\parallel$$
$$(+1)(-a_{12})(-a_{23})(-a_{34})(-a_{41})(-a_{56})(-a_{65})$$
$$\parallel$$
$$\underbrace{(a_{12}a_{23}a_{34}a_{41})}\ \underbrace{(a_{56}a_{65})}$$

These products are called <u>cycle gains</u>



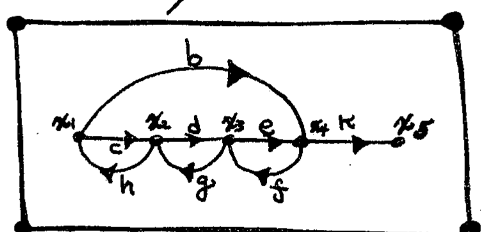
$$\boxed{\begin{array}{l}\underline{\text{Conclusion}}:\ |M(\Gamma)| = 1 - L_1 + L_2 - L_3 \pm \cdots + (-1)^n L_n \\ \text{where } n = \#\text{ of vertex-disjoint circuits in } \Gamma. \\ L_1 = \text{products of labels on single circuits.} \\ L_2 = \text{products of labels on vertex disjoint pairs.} \\ \vdots \\ L_n = \text{product of labels on vertex disjoint } n\text{-tuples.}\end{array}}$$

Now lets continue the analysis a bit further. Take the example:



So x_0 is input, and x_5 is output.

$$\left\{\begin{aligned} x_1 &= a x_0 + h x_2 \\ x_2 &= c x_1 + g x_3 \\ x_3 &= d x_2 + f x_4 \\ x_4 &= b x_1 + e x_3 \\ x_5 &= K x_4 \end{aligned}\right\}$$



Γ

$$\begin{bmatrix} a x_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -h & 0 & 0 & 0 \\ -c & 1 & -g & 0 & 0 \\ 0 & -d & 1 & -f & 0 \\ -b & 0 & -e & 1 & 0 \\ 0 & 0 & 0 & -K & 1 \end{bmatrix}}_{M(\Gamma)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Now by Cramer's rule,

$$x_5 = \begin{vmatrix} 1 & -h & 0 & 0 & a x_0 \\ -c & 1 & -g & 0 & 0 \\ 0 & -d & 1 & -f & 0 \\ -b & 0 & -e & 1 & 0 \\ 0 & 0 & 0 & -K & 0 \end{vmatrix} \Big/ \underline{\underline{|M(\Gamma)|}}$$

The determinant of the numerator

is $$ax_0 \begin{vmatrix} -c & 1 & -g & 0 \\ 0 & -d & 1 & -f \\ -b & 0 & -e & 1 \\ 0 & 0 & 0 & -K \end{vmatrix} = (ax_0)(K) \begin{vmatrix} -c & 1 & -g \\ 0 & -d & 1 \\ -b & 0 & -e \end{vmatrix}$$

$$= (ax_0)(-K)\left[-c \begin{vmatrix} -d & 1 \\ 0 & -e \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -b & -e \end{vmatrix} - g \begin{vmatrix} 0 & -d \\ -b & 0 \end{vmatrix}\right)$$

$$= (ax_0)(-K)\left(-cde - b + gbd\right)$$

$$= x_0\left(+acdeK + abK - agbdK\right)$$

$$= x_0\left(+acdeK + abK[1 - gd]\right)$$

If **you** now look back at the graph you will see that there is a pattern to this.

P_1: $acdeK \Leftrightarrow$ path from x_0 to x_5.
P_2: $abK \Leftrightarrow$ second path from x_0 to x_5.

There are no loops in the graph that do not touch P_1.

There is the loop $\underline{\underline{gd}}$ that does not touch P_2.

$$\boxed{\text{This is a general pattern!}}$$

The numerator determinant for the "output" will be a sum of the form $\sum_K T_K \Delta_K$ where T_K is the product of edge labels (hence the transfer function for) for a directed path from input node to output node. And Δ_K is the signed loop-sum for the <u>complement</u> of this path.

Thus we can say in general that if we have source node x and output node y for graph ① then

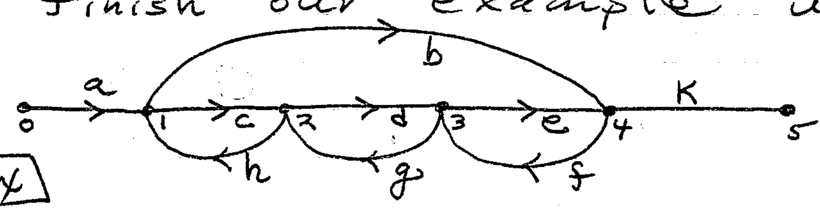
$$y/x = \sum_K T_K \Delta_K \Big/ \Delta$$

where Δ is the loop-sum for G (− input) and $\sum_K T_K \Delta_K$ is as defined above.

This is called MASON'S FORMULA.

To finish our example we have

$\boxed{x_0 = x}$



$\boxed{x_5 = y}$

$$y/x = \frac{acdeK + abK(1 - dg)}{\begin{pmatrix} 1 - (ch) - (dg) - (ef) + (ch)(ef) \\ - (bfgh) \end{pmatrix}}$$

On the next page we have explicitly calculated the loop-sum $\Delta = |M(M)|$ for comparison with the denominator in this formula for Mason's rule.

$$\Delta = |M(r)| = \begin{vmatrix} 1 & -h & 0 & 0 & 0 \\ -c & 1 & -g & 0 & 0 \\ 0 & -d & 1 & -f & 0 \\ -b & 0 & -e & 1 & 0 \\ 0 & 0 & 0 & -k & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -g & 0 & 0 \\ -d & 1 & -f & 0 \\ 0 & -e & 1 & 0 \\ 0 & 0 & -k & 1 \end{vmatrix} + h \begin{vmatrix} -c & -g & 0 & 0 \\ 0 & 1 & -f & 0 \\ -b & -e & 1 & 0 \\ 0 & 0 & -k & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -f & 0 \\ -e & 1 & 0 \\ 0 & -k & 1 \end{vmatrix} + g \begin{vmatrix} -d & -f & 0 \\ 0 & 1 & 0 \\ 0 & -k & 1 \end{vmatrix} + h \left[-c \begin{vmatrix} 1 & -f & 0 \\ -e & 1 & 0 \\ 0 & -k & 1 \end{vmatrix} + g \begin{vmatrix} 0 & -f & 0 \\ -b & 1 & 0 \\ 0 & -k & 1 \end{vmatrix} \right]$$

$$= 1 + f \begin{vmatrix} -e & 0 \\ 0 & 1 \end{vmatrix} - gd \begin{vmatrix} 1 & 0 \\ -k & 1 \end{vmatrix} + h \left[-c \left[1 + f \begin{vmatrix} -e & 0 \\ 0 & 1 \end{vmatrix} \right] + gf \begin{vmatrix} -b & 0 \\ 0 & 1 \end{vmatrix} \right]$$

$$= 1 - ef - gd - ch[(1 - ef)] + ghf(-b)$$

$$= 1 - ef - gd - ch - gbhf + chef$$

$$\therefore \Delta = 1 - (ef) - (gd) - (ch) - (bfgh) + (ch)(ef)$$

This checks with the application of the signed loop-sum.

Thus we have seen that by "graphing" a system of linear equations it can be seen as a problem of input/output with feedback. The graph provides a simple pattern for the combinatorics of the solution.

MASON'S FORMULA $y/x = \Sigma T_K \Delta_K / \Delta$

summarizes this version of Cramer's Rule.

LK Jan 1, 1986