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Elementare Zahlentheorie
(*Vol.* 1, *Part* 1 of Zahlentheorie)

FOUNDATIONS OF ANALYSIS

THE ARITHMETIC OF WHOLE, RATIONAL, IRRATIONAL AND COMPLEX NUMBERS

A Supplement to Text-Books on the Differential and Integral Calculus

BY
EDMUND LANDAU

TRANSLATED BY
F. STEINHARDT
COLUMBIA UNIVERSITY

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PREFACE FOR THE STUDENT

1. Please don't read the preface for the teacher.

2. I will ask of you only the ability to read English and to think logically—no high school mathematics, and certainly no higher mathematics.

To prevent arguments: **a** number, **no** number, **two** cases, **all** objects of a given totality, and so on, are completely unambiguous phrases. Theorem 1, Theorem 2, . . . , Theorem 301, or 1), 2), etc. for distinguishing the various cases, are labels which distinguish the theorems and the cases; similarly for axioms, definitions, chapters and sections. These are more convenient to refer to than if we were to speak, say, of Theorem Light-blue, Theorem Dark-blue, and so on. As a matter of fact, the introduction of the so-called positive integers up to "301" would not offer any difficulty whatsoever. The first difficulty—overcome in Chapter I—lies in the *totality* of the positive integers

$$1, \ldots$$

with the mysterious series of dots after the comma (called natural numbers in Chapter I), in the definition of the arithmetical operations with these numbers, and in the proofs of the associated theorems.

I develop analogous material, first for the natural numbers in Chap. I; then for the positive fractions and positive rational numbers, in Chap. II; next for the positive (rational and irrational) numbers, in Chap. III; next for the real numbers (positive, negative, and zero), in Chap. IV; and finally for the complex numbers, in Chap. V. Thus I speak only of such numbers as you have already met with in high school.

Apropos:

3. Please forget everything you have learned in school; for you haven't learned it.

Please keep in mind at all times the corresponding portions of your school curriculum; for you haven't actually forgotten them.

4. The multiplication table will not occur in this book, not even the theorem

$$2 \cdot 2 = 4,$$

but I would recommend, as an exercise for Chap. I, § 4, that you define

$$2 = 1 + 1,$$
$$4 = (((1 + 1) + 1) + 1),$$

and then prove the theorem.

5. Forgive me for "theeing" and "thouing" you.* One reason for my doing so is that this book is written partly *in usum delphinarum*,† since, as is well known (cf. E. Landau, *Vorlesungen über Zahlentheorie*, Vol. 1, p. V), my daughters have been studying (chemistry) for several semesters, think they have learned differential and integral calculus in school, and yet even today don't know why

$$x \cdot y = y \cdot x$$

is true.

Berlin, December 28, 1929

EDMUND LANDAU

* In the German edition Professor Landau uses the familiar "du" (thou) throughout this preface. [*Trans.*]

† For Delphine use. The Delphin Classics were prepared by French scholars for the use of the Dauphin of France, son of King Louis XIV. [*Trans.*]

PREFACE FOR THE TEACHER

This little book is a concession to those of my colleagues (unfortunately in the majority) who do **not** share my point of view on the following question.

While a rigorous and complete exposition of elementary mathematics can not, of course, be expected in the high schools, the mathematical courses in colleges and universities should acquaint the student not only with the subject matter and results of mathematics, but also with its methods of proof. Even one who studies mathematics mainly for its applications to physics and to other sciences, and who must therefore often discover auxiliary mathematical theorems for himself, can not continue to take steps securely along the path he has chosen unless he has learned how to walk—that is, unless he is able to distinguish between true and false, between supposition and proof (or, as some say so nicely, between non-rigorous and rigorous proof).

I therefore think it right—as do some of my teachers and colleagues, some authors whose writings I have found of help, and most of my students—that even in his first semester the student should learn what the basic facts are, accepted as axioms, from which mathematical analysis is developed, and how one can proceed with this development. As is well known, these axioms can be selected in various ways; so that I do not declare it to be incorrect, but only to be almost diametrically opposite to my point of view, if one postulates as axioms for real numbers many of the usual rules of arithmetic and the main theorem of this book (Theorem 205, Dedekind's Theorem). I do not, to be sure, prove the consistency of the five Peano axioms (because that can not be done), but each of them is obviously independent of the preceding ones. On the other hand, were we to adopt a large number of axioms, as mentioned above, the question would immediately occur to the student whether some of them could not be proved (a shrewd one would add: or disproved) by means of the rest of them. Since it has been known for many decades that all these additional axioms can be proved, the student should really be allowed to acquaint himself with the proofs at the beginning of his course of study—especially since they are all quite easy.

I will refrain from speaking at length about the fact that often even Dedekind's fundamental theorem (or the equivalent theorem in the development of the real numbers by means of fundamental sequences) is not included in the basic material; so that such matters as the mean-value theorem of the differential calculus, the corollary of the mean-value theorem to the effect that a function having a zero derivative in some interval is constant in that interval, or, say, the theorem that a monotonically decreasing bounded sequence of numbers converges to a limit, are given without any proof or, worse yet, with a supposed proof which in reality is no proof at all. Not only does the number of proponents of this extreme variant of the opposite point of view seem to me to be decreasing monotonically, but the limit to which, in conformity with the above-mentioned theorem, this number converges, may even be zero.

Only rarely, however, is the foundation of the natural numbers taken as the starting point. I confess that while I myself have never failed to cover the (Dedekind) theory of real numbers, in my earlier courses I assumed the properties of the integers and of the rational numbers. But the last three times I preferred to begin with the integers. For the next Spring term (as once before) I have divided my course into two simultaneous courses one of which has the title "Grundlagen der Analysis" (Foundations of Analysis). This is a concession to those hearers who want, after all, to do differentiation right away, or who do not want to learn the whole explanation of the number concept in the first semester (or perhaps not at all). In the Foundations of Analysis course I begin with the Peano axioms for the natural numbers and get through the theory of the real and of the complex numbers. The complex numbers, incidentally, are not needed by the student in his first semester, but their introduction, being quite simple, can be made without difficulty.

Now in the entire literature there is no textbook which has the sole and modest aim of laying the foundation, in the above sense, for operations with numbers. The larger books which attempt that task in their introductory chapters leave (consciously or not) quite a bit for the reader to complete.

The present book should give to any of my colleagues of the other pedagogical faction (who therefore does not go through the foundations) at least the opportunity, provided he considers this book suitable, of referring his students to a source where the material he leaves out—and that material only—is treated in full. After the first four or five rather abstract pages the reading is quite easy if—as is actually the case—one is acquainted with the results from high school.

It is not without hesitation that I publish this little book, because in so doing I publish in a field where (aside from an oral communication of Mr. Kalmár) I have nothing new to say; but nobody else has undertaken this labor which in part is rather tedious.

But the immediate cause for venturing into print was furnished by a concrete incident.

The opposition party likes to believe that the student would eventually learn these things anyway during the course of his study from some lecture or from the literature. And of these honored friends and enemies, none would have doubted that everything needed could be found in, say, my lectures. I, too, believed that. And then the following gruesome adventure happened to me. My then assistant and dear colleague Privatdozent Dr. Grandjot (now Professor at the University of Santiago) was lecturing on the foundations of analysis and using my notebook as a basis for the lectures. He returned my manuscript to me with the remark that he had found it necessary to add further axioms to Peano's in the course of the development, because the standard procedure, which I had followed, had proved to be incomplete at a certain point. Before going into details I want to mention at once that

1. Grandjot's objection was justified.
2. Axioms which, because they depend on later concepts, cannot be listed at the very beginning, are very regrettable.
3. Grandjot's axioms can all be proved (as we could have learned from Dedekind), so that everything remains based on Peano's axioms (cf. the entire following book).

There were three places where the objection came in:

I. At the definition of $x + y$ for the natural numbers.

II. At the definition of $x \cdot y$ for the natural numbers.

III. At the definition of $\sum_{n=1}^{m} x_n$ and of $\prod_{n=1}^{m} x_n$, after one already has $x + y$ and $x \cdot y$, for some domain of numbers.

Since the situations in all three cases are analogous, I will speak here only about the case of $x + y$ for natural numbers x, y. When I prove some theorem on natural numbers, say in a lecture on number theory, by first establishing it as true for 1 and then deducing its validity for $x + 1$ from its validity for x, then occasionally some student will raise the objection that I have not first proved the assertion for x. The objection is not justified but it is excusable; the student just had never heard of the axiom of induction. Grandjot's objection sounds similar, with the difference that it was justified; so I had to excuse it also. On the basis of his five axioms, Peano defines $x + y$ for fixed x and all y as follows:

$$x + 1 = x'$$
$$x + y' = (x + y)',$$

and he and his successors then think that $x + y$ is defined generally; for, the set of y's for which it is defined contains 1, and contains y' if it contains y.

But $x + y$ has *not* been defined.

All would be well if—and this is not done in Peano's method because order is introduced only after addition—one had the concept "numbers $\leqq y$" and could speak of the set of y's for which there is an $f(z)$, defined for $z \leqq y$, with the properties

$$f(1) = x,$$
$$f(z') = (f(z))' \qquad \text{for } z < y.$$

Dedekind's reasoning does follow these lines. With the kind help of my colleague von Neumann in Princeton I had worked out such a procedure, based on a previous introduction of ordering, for this book. This would have been somewhat inconvenient for the reader. At the last minute, however, I was informed of a much simpler proof by Dr. Kalmár in Szeged. The matter now looks so simple and the proof so similar to the other proofs in the first chapter, that not even the expert might have noticed this point had I not given above a detailed confession of crime and punishment. For $x \cdot y$ the same simple type of proof applies; however, $\sum_{n=1}^{m} x_n$ and $\prod_{n=1}^{m} x_n$ are possible only with the Dedekind procedure. But from Chap. I, § 3 on, one has the set of the $x \leqq y$ anyway.

To make it as easy as possible for the reader I have repeated in several chapters, or sometimes in all, certain (not very lengthy) phrases. For the expert it would of course be sufficient to say once and for all, for instance in the proof of Theorems 16 and 17: This reasoning holds for every class of numbers for which the symbols $<$ and $=$ are defined and have certain properties mentioned earlier. Such repeated deductive reasonings occurred in connection with theorems which had to be given in all the chapters concerned because the theorems are used later on. But it suffices to introduce $\sum_{n=1}^{m} a_n$ and $\prod_{n=1}^{m} a_n$ since they will then apply to the preceding types of numbers. I therefore defer their introduction to the chapter on complex numbers, and do the same for the theorems on subtraction and division; the former hold for the natural numbers, say, only if the minuend is larger than the subtrahend, the latter for the natural numbers, say, only if the division leaves no remainder.

My book is written, as befits such easy material, in merciless telegram style (**"Axiom," "Definition," "Theorem," "Proof,"** occasionally **"Preliminary Remark,"** rarely words which do not belong to one of these five categories).

I hope that I have written this book, after a preparation stretching over decades, in such a way that a normal student can read it in two days. And then (since he already knows the formal rules from school) he may forget its contents, with the exception of the axiom of induction and of Dedekind's fundamental theorem.

Should, however, any of my colleagues who holds the other point of view find the matter so easy that he presents it in his lectures for beginners (in the following or in any other way), I would have achieved a success which I do not even dare hope will be realized on any large scale.

Berlin, December 28, 1929

EDMUND LANDAU

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CHAPTER V

COMPLEX NUMBERS

CHAPTER I

NATURAL NUMBERS

§ 1

Axioms

We assume the following to be given:

A set (i.e. totality) of objects called natural numbers, possessing the properties—called axioms—to be listed below.

Before formulating the axioms we make some remarks about the symbols $=$ and $\neq$ which will be used.

Unless otherwise specified, small italic letters will stand for natural numbers throughout this book.

If x is given and y is given, then

either x and y are the same number; this may be written

$$x = y$$

($=$ to be read "equals");

or x and y are not the same number; this may be written

$$x \neq y$$

($\neq$ to be read "is not equal to").

Accordingly, the following are true on purely logical grounds:

1)
$$x = x$$
for every x.

2) If
$$x = y$$
then
$$y = x.$$

3) If
$$x = y, \quad y = z$$
then
$$x = z.$$

Thus a statement such as

$$a = b = c = d,$$

which on the face of it means merely that

$$a = b, \quad b = c, \quad c = d,$$

contains the additional information that, say,

$$a = c, \quad a = d, \quad b = d.$$

(Similarly in the later chapters.)

Now, we assume that the set of all natural numbers has the following properties:

Axiom 1: *1 is a natural number.*

That is, our set is not empty; it contains an object called 1 (read "one").

Axiom 2: *For each x there exists exactly one natural number, called the successor of x, which will be denoted by x'.*

In the case of complicated natural numbers x, we will enclose in parentheses the number whose successor is to be written down, since otherwise ambiguities might arise. We will do the same, throughout this book, in the case of $x + y$, xy, $x - y$, $-x$, x^y, etc.

Thus, if

$$x = y$$

then

$$x' = y'.$$

Axiom 3: *We always have*

$$x' \neq 1.$$

That is, there exists no number whose successor is 1.

Axiom 4: *If*

$$x' = y'$$

then

$$x = y.$$

That is, for any given number there exists either no number or exactly one number whose successor is the given number.

Axiom 5 (Axiom of Induction): *Let there be given a set $\mathfrak{M}$ of natural numbers, with the following properties:*

I) *1 belongs to $\mathfrak{M}$.*

II) *If x belongs to $\mathfrak{M}$ then so does x'.*

Then $\mathfrak{M}$ contains all the natural numbers.

§ 2

Addition

Theorem 1: *If*

$$x \neq y$$

then

$$x' \neq y'.$$

Proof: Otherwise, we would have

$$x' = y'$$

and hence, by Axiom 4,

$$x = y.$$

Theorem 2: $$x' \neq x.$$

Proof: Let $\mathfrak{M}$ be the set of all x for which this holds true.

I) By Axiom 1 and Axiom 3,

$$1' \neq 1;$$

therefore 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then

$$x' \neq x,$$

and hence by Theorem 1,

$$(x')' \neq x',$$

so that x' belongs to $\mathfrak{M}$.

By Axiom 5, $\mathfrak{M}$ therefore contains all the natural numbers, i.e. we have for each x that

$$x' \neq x.$$

Theorem 3: *If*

$$x \neq 1,$$

then there exists one (hence, by Axiom 4, exactly one) *u such that*

$$x = u'.$$

Proof: Let $\mathfrak{M}$ be the set consisting of the number 1 and of all those x for which there exists such a u. (For any such x, we have of necessity that

$$x \neq 1$$

by Axiom 3.)

I) 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then, with u denoting the number x, we have

$$x' = u',$$

so that x' belongs to $\mathfrak{M}$.

By Axiom 5, $\mathfrak{M}$ therefore contains all the natural numbers; thus for each

$$x \neq 1$$

there exists a u such that

$$x = u'.$$

Theorem 4, and at the same time **Definition 1**: *To every pair of numbers x, y, we may assign in exactly one way a natural number, called $x + y$* (+ to be read "plus"), *such that*

1) $x + 1 = x'$ *for every* x,

2) $x + y' = (x + y)'$ *for every* x *and every* y.

$x + y$ is called the sum of x and y, or the number obtained by addition of y to x.

Proof: A) First we will show that for each fixed x there is at most one possibility of defining $x + y$ for all y in such a way that

$$x + 1 = x'$$

and

$$x + y' = (x + y)' \quad \text{for every } y.$$

Let a_y and b_y be defined for all y and be such that

$$a_1 = x', \quad b_1 = x',$$
$$a_{y'} = (a_y)', \quad b_{y'} = (b_y)' \quad \text{for every } y.$$

Let $\mathfrak{M}$ be the set of all y for which

$$a_y = b_y.$$

I)
$$a_1 = x' = b_1;$$

hence 1 belongs to $\mathfrak{M}$.

II) If y belongs to $\mathfrak{M}$, then

$$a_y = b_y,$$

hence by Axiom 2,

$$(a_y)' = (b_y)',$$

therefore

$$a_{y'} = (a_y)' = (b_y)' = b_{y'},$$

so that y' belongs to $\mathfrak{M}$.

Hence $\mathfrak{M}$ is the set of all natural numbers; i.e. for every y we have

$$a_y = b_y.$$

B) Now we will show that for each x it is actually possible to define $x + y$ for all y in such a way that

$$x + 1 = x'$$

and

$$x + y' = (x + y)' \quad \text{for every } y.$$

Let $\mathfrak{M}$ be the set of all x for which this is possible (in exactly one way, by A)).

I) For

$$x = 1,$$

the number

$$x + y = y'$$

is as required, since

$$x + 1 = 1' = x',$$
$$x + y' = (y')' = (x + y)'.$$

Hence 1 belongs to $\mathfrak{M}$.

II) Let x belong to $\mathfrak{M}$, so that there exists an $x + y$ for all y. Then the number

$$x' + y = (x + y)'$$

is the required number for x', since

$$x' + 1 = (x + 1)' = (x')'$$

and

$$x' + y' = (x + y')' = ((x + y)')' = (x' + y)'.$$

Hence x' belongs to $\mathfrak{M}$.

Therefore $\mathfrak{M}$ contains all x.

Theorem 5 (Associative Law of Addition):

$$(x + y) + z = x + (y + z).$$

Proof: Fix x and y, and denote by $\mathfrak{M}$ the set of all z for which the assertion of the theorem holds.

I) $(x + y) + 1 = (x + y)' = x + y' = x + (y + 1)$;

thus 1 belongs to $\mathfrak{M}$.

II) Let z belong to $\mathfrak{M}$. Then

$$(x + y) + z = x + (y + z),$$

hence

$$(x + y) + z' = ((x + y) + z)' = (x + (y + z))' = x + (y + z)' = x + (y + z'),$$

so that z' belongs to $\mathfrak{M}$.

Therefore the assertion holds for all z.

Theorem 6 (Commutative Law of Addition):

$$x + y = y + x.$$

Proof: Fix y, and let $\mathfrak{M}$ be the set of all x for which the assertion holds.

I) We have

$$y + 1 = y',$$

and furthermore, by the construction in the proof of Theorem 4,

$$1 + y = y',$$

so that

$$1 + y = y + 1$$

and 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then

$$x + y = y + x,$$

therefore

$$(x + y)' = (y + x)' = y + x'.$$

By the construction in the proof of Theorem 4, we have

$$x' + y = (x + y)',$$

hence

$$x' + y = y + x',$$

so that x' belongs to $\mathfrak{M}$.

The assertion therefore holds for all x.

Theorem 7: $y \neq x + y.$

Proof: Fix x, and let $\mathfrak{M}$ be the set of all y for which the assertion holds.

I)

$$1 \neq x',$$
$$1 \neq x + 1;$$

1 belongs to $\mathfrak{M}$.

II) If y belongs to $\mathfrak{M}$, then

$$y \neq x + y,$$

hence

$$y' \neq (x + y)',$$
$$y' \neq x + y',$$

so that y' belongs to $\mathfrak{M}$.

Therefore the assertion holds for all y.

Theorem 8: *If*

$$y \neq z$$

then

$$x + y \neq x + z.$$

Proof: Consider a fixed y and a fixed z such that

$$y \neq z,$$

and let $\mathfrak{M}$ be the set of all x for which

$$x + y \neq x + z.$$

I)

$$y' \neq z',$$
$$1 + y \neq 1 + z;$$

hence 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then

$$x + y \neq x + z,$$

hence

$$(x + y)' \neq (x + z)',$$
$$x' + y \neq x' + z,$$

so that x' belongs to $\mathfrak{M}$.

Therefore the assertion holds always.

Theorem 9: *For given x and y, exactly one of the following must be the case:*

1) $$x = y.$$

2) *There exists a u* (exactly one, by Theorem 8) *such that*

$$x = y + u.$$

3) *There exists a v* (exactly one, by Theorem 8) *such that*

$$y = x + v.$$

Proof: A) By Theorem 7, cases 1) and 2) are incompatible. Similarly, 1) and 3) are incompatible. The incompatibility of 2) and 3) also follows from Theorem 7; for otherwise, we would have

$$x = y + u = (x + v) + u = x + (v + u) = (v + u) + x.$$

Therefore we can have at most one of the cases 1), 2) and 3).

B) Let x be fixed, and let $\mathfrak{M}$ be the set of all y for which one (hence by A), exactly one) of the cases 1), 2) and 3) obtains.

I) For $y = 1$, we have by Theorem 3 that either

$$x = 1 = y \qquad \text{(case 1))}$$

or

$$x = u' = 1 + u = y + u \qquad \text{(case 2))}.$$

Hence 1 belongs to $\mathfrak{M}$.

II) Let y belong to $\mathfrak{M}$. Then

either (case 1) for y)

$$x = y,$$

hence

$$y' = y + 1 = x + 1 \qquad \text{(case 3) for } y');$$

or (case 2) for y)

$$x = y + u,$$

hence if

$$u = 1,$$

then

$$x = y + 1 = y' \qquad \text{(case 1) for } y');$$

but if

$$u \neq 1,$$

then, by Theorem 3,

$$u = w' = 1 + w,$$

$$x = y + (1 + w) = (y + 1) + w = y' + w$$

$$\text{(case 2) for } y');$$

or (case 3) for y)

$$y = x + v,$$

hence

$$y' = (x + v)' = x + v'$$

$$\text{(case 3) for } y').$$

In any case, y' belongs to $\mathfrak{M}$.
Therefore we always have one of the cases 1), 2) and 3).

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§ 3

Ordering

Definition 2: *If*

$$x = y + u$$

then

$$x > y.$$

($>$ to be read "is greater than.")

Definition 3: *If*

$$y = x + v$$

then

$$x < y.$$

($<$ to be read "is less than.")

Theorem 10: *For any given x, y, we have exactly one of the cases*

$$x = y, \quad x > y, \quad x < y.$$

Proof: Theorem 9, Definition 2 and Definition 3.

Theorem 11: *If*

$$x > y$$

then

$$y < x.$$

Proof: Each of these means that

$$x = y + u$$

for some suitable u.

Theorem 12: *If*

$$x < y$$

then

$$y > x.$$

Proof: Each of these means that

$$y = x + v$$

for some suitable v.

Definition 4: $x \geqq y$

means

$$x > y \quad or \quad x = y.$$

($\geqq$ to be read "is greater than or equal to.")

Definition 5: $x \leqq y$

means

$$x < y \quad or \quad x = y.$$

($\leqq$ to be read "is less than or equal to.")

Theorem 13: *If*

$$x \geqq y$$

then

$$y \leqq x.$$

Proof: Theorem 11.

Theorem 14: *If*

$$x \leqq y$$

then

$$y \geqq x.$$

Proof: Theorem 12.

Theorem 15 (Transitivity of Ordering): *If*

$$x < y, \quad y < z,$$

then

$$x < z.$$

Preliminary Remark: Thus if

$$x > y, \quad y > z,$$

then

$$x > z,$$

since

$$z < y, \quad y < x,$$
$$z < x;$$

but in what follows I will not even bother to write down such statements, which are obtained trivially by simply reading the formulas backwards.

Proof: With suitable v, w, we have

$$y = x + v, \quad z = y + w,$$

hence

$$z = (x + v) + w = x + (v + w),$$
$$x < z.$$

Theorem 16: *If*

$$x \leqq y, \quad y < z \quad or \quad x < y, \quad y \leqq z,$$

then

$$x < z.$$

Proof: Obvious if an equality sign holds in the hypothesis; otherwise, Theorem 15 does it.

Theorem 17: *If*

$$x \leqq y, \quad y \leqq z,$$

then

$$x \leqq z.$$

Proof: Obvious if two equality signs hold in the hypothesis; otherwise, Theorem 16 does it.

A notation such as

$$a < b \leqq c < d$$

is justified on the basis of Theorems 15 and 17. While its immediate meaning is

$$a < b, \quad b \leqq c, \quad c < d,$$

it also implies, according to these theorems, that, say

$$a < c, \quad a < d, \quad b < d.$$

(Similarly in the later chapters.)

Theorem 18: $x + y > x.$

Proof: $x + y = x + y.$

Theorem 19: *If*

$$x > y, \; or \; x = y, \; or \; x < y,$$

then

$$x + z > y + z, \; or \; x + z = y + z, \; or \; x + z < y + z,$$

respectively.

Proof: 1) If

$$x > y$$

then

$$x = y + u,$$
$$x + z = (y + u) + z = (u + y) + z = u + (y + z) = (y + z) + u,$$
$$x + z > y + z.$$

2) If

$$x = y$$

then clearly

$$x + z = y + z.$$

3) If

$$x < y$$

then

$$y > x,$$

hence, by 1),

$$y + z > x + z,$$
$$x + z < y + z.$$

Theorem 20: *If*

$$x + z > y + z, \; or \; x + z = y + z, \; or \; x + z < y + z,$$

then

$$x > y, \text{ or } x = y, \text{ or } x < y, \text{ respectively.}$$

Proof: Follows from Theorem 19, since the three cases are, in both instances, mutually exclusive and exhaust all possibilities.

Theorem 21: *If*

$$x > y, \quad z > u,$$

then

$$x + z > y + u.$$

Proof: By Theorem 19, we have

$$x + z > y + z$$

and

$$y + z = z + y > u + y = y + u,$$

hence

$$x + z > y + u.$$

Theorem 22: *If*

$$x \geqq y, \ z > u \text{ or } x > y, \ z \geqq u,$$

then

$$x + z > y + u.$$

Proof: Follows from Theorem 19 if an equality sign holds in the hypothesis, otherwise from Theorem 21.

Theorem 23: *If*

$$x \geqq y, \ z \geqq u,$$

then

$$x + z \geqq y + u.$$

Proof: Obvious if two equality signs hold in the hypothesis; otherwise Theorem 22 does it.

Theorem 24: $x \geqq 1.$

Proof: Either

$$x = 1$$

or

$$x = u' = u + 1 > 1.$$

Theorem 25: *If*

$$y > x$$

then

$$y \geqq x + 1.$$

Proof:

$$y = x + u,$$

$$u \geqq 1,$$

hence

$$y \geqq x + 1.$$

Theorem 26: *If*

$$y < x + 1$$

then

$$y \leqq x.$$

Proof: Otherwise we would have

$$y > x$$

and therefore, by Theorem 25,

$$y \geqq x + 1.$$

Theorem 27: *In every non-empty set of natural numbers there is a least one* (i.e. one which is less than any other number of the set).

Proof: Let $\mathfrak{N}$ be the given set, and let $\mathfrak{M}$ be the set of all x which are $\leqq$ every number of $\mathfrak{N}$.

By Theorem 24, the set $\mathfrak{M}$ contains the number 1. Not every x belongs to $\mathfrak{M}$; in fact, for each y of $\mathfrak{N}$ the number $y + 1$ does not belong to $\mathfrak{M}$, since

$$y + 1 > y.$$

Therefore there is an m in $\mathfrak{M}$ such that $m + 1$ does not belong to $\mathfrak{M}$; for otherwise, every natural number would have to belong to $\mathfrak{M}$, by Axiom 5.

Of this m I now assert that it is $\leqq$ every n of $\mathfrak{N}$, and that it belongs to $\mathfrak{N}$. The former we already know. The latter is established by an indirect argument, as follows: If m did not belong to $\mathfrak{N}$, then for each n of $\mathfrak{N}$ we would have

$$m < n,$$

hence, by Theorem 25,

$$m + 1 \leqq n;$$

thus $m + 1$ would belong to $\mathfrak{M}$, contradicting the statement above by which m was introduced.

§ 4

Multiplication

Theorem 28 and at the same time **Definition 6:** *To every pair of numbers x, y, we may assign in exactly one way a natural number, called* $x \cdot y$ ($\cdot$ to be read "times"; however, the dot is usually omitted), *such that*

1) $x \cdot 1 = x$ *for every* x,
2) $x \cdot y' = x \cdot y + x$ *for every* x *and every* y.

$x \cdot y$ *is called the product of* x *and* y, *or the number obtained from multiplication of* x *by* y.

Proof (*mutatis mutandis*, word for word the same as that of Theorem 4): A) We will first show that for each fixed x there is at most one possibility of defining xy for all y in such a way that

$$x \cdot 1 = x$$

and

$$xy' = xy + x \text{ for every } y.$$

Let a_y and b_y be defined for all y and be such that

$$a_1 = x, \quad b_1 = x,$$
$$a_{y'} = a_y + x, \quad b_{y'} = b_y + x \text{ for every } y.$$

Let $\mathfrak{M}$ be the set of all y for which

$$a_y = b_y.$$

I)
$$a_1 = x = b_1;$$

hence 1 belongs to $\mathfrak{M}$.

II) If y belongs to $\mathfrak{M}$, then

$$a_y = b_y,$$

hence

$$a_{y'} = a_y + x = b_y + x = b_{y'},$$

so that y' belongs to $\mathfrak{M}$.

Hence $\mathfrak{M}$ is the set of all natural numbers; i.e. for every y we have

$$a_y = b_y.$$

B) Now we will show that for each x, it is actually possible to define xy for all y in such a way that

$$x \cdot 1 = x$$

and

$$xy' = xy + x \text{ for every } y.$$

Let $\mathfrak{M}$ be the set of all x for which this is possible (in exactly one way, by A)).

I) For

$$x = 1,$$

the number

$$xy = y$$

is as required, since

$$x \cdot 1 = 1 = x,$$
$$xy' = y' = y + 1 = xy + x.$$

Hence 1 belongs to $\mathfrak{M}$.

II) Let x belong to $\mathfrak{M}$, so that there exists an xy for all y. Then the number

$$x'y = xy + y$$

is the required number for x', since

$$x' \cdot 1 = x \cdot 1 + 1 = x + 1 = x'$$

and

$$x'y' = xy' + y' = (xy + x) + y' = xy + (x + y') = xy + (x + y)'$$
$$= xy + (x' + y) = xy + (y + x') = (xy + y) + x' = x'y + x'.$$

Hence x' belongs to $\mathfrak{M}$.

Therefore $\mathfrak{M}$ contains all x.

Theorem 29 (Commutative Law of Multiplication):

$$xy = yx.$$

Proof: Fix y, and let $\mathfrak{M}$ be the set of all x for which the assertion holds.

I) We have

$$y \cdot 1 = y,$$

and furthermore, by the construction in the proof of Theorem 28,

$$1 \cdot y = y,$$

hence

$$1 \cdot y = y \cdot 1,$$

so that 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then

$$xy = yx,$$

hence

$$xy + y = yx + y = yx'.$$

By the construction in the proof of Theorem 28, we have

$$x'y = xy + y,$$

hence

$$x'y = yx',$$

so that x' belongs to $\mathfrak{M}$.

The assertion therefore holds for all x.

Theorem 30 (Distributive Law):

$$x(y+z) = xy + xz.$$

Preliminary Remark: The formula

$$(y+z)x = yx + zx$$

which results from Theorem 30 and Theorem 29, and similar analogues later on, need not be specifically formulated as theorems, nor even be set down.

Proof: Fix x and y, and let $\mathfrak{M}$ be the set of all z for which the assertion holds true.

I) $$x(y+1) = xy' = xy + x = xy + x \cdot 1;$$

1 belongs to $\mathfrak{M}$.

II) If z belongs to $\mathfrak{M}$, then

$$x(y+z) = xy + xz,$$

hence

$$x(y+z') = x((y+z)') = x(y+z)+x = (xy+xz)+x$$
$$= xy+(xz+x) = xy+xz',$$

so that z' belongs to $\mathfrak{M}$.

Therefore, the assertion always holds.

Theorem 31 (Associative Law of Multiplication):

$$(xy)z = x(yz).$$

Proof: Fix x and y, and let $\mathfrak{M}$ be the set of all z for which the assertion holds true.

I) $$(xy) \cdot 1 = xy = x(y \cdot 1);$$

hence 1 belongs to $\mathfrak{M}$.

II) Let z belong to $\mathfrak{M}$. Then

$$(xy)z = x(yz),$$

and therefore, using Theorem 30,

$$(xy)z' = (xy)z + xy = x(yz) + xy = x(yz + y) = x(yz'),$$

so that z' belongs to $\mathfrak{M}$.

Therefore $\mathfrak{M}$ contains all natural numbers.

Theorem 32: *If*

$$x > y, \text{ or } x = y, \text{ or } x < y,$$

then

$$xz > yz, \text{ or } xz = yz, \text{ or } xz < yz, \text{ respectively.}$$

Proof: 1) If

$$x > y$$

then

$$x = y + u,$$
$$xz = (y+u)z = yz + uz > yz.$$

2) If

$$x = y,$$

then clearly

$$xz = yz.$$

3) If

$$x < y$$

then

$$y > x,$$

hence by 1),

$$yz > xz,$$
$$xz < yz.$$

Theorem 33: *If*

$$xz > yz, \text{ or } xz = yz, \text{ or } xz < yz,$$

then

$$x > y, \text{ or } x = y, \text{ or } x < y, \text{ respectively.}$$

Proof: Follows from Theorem 32, since the three cases are, in both instances, mutually exclusive and exhaust all possibilities.

Theorem 34: *If*

$$x > y, \; z > u,$$

then

$$xz > yu.$$

Proof: By Theorem 32, we have

$$xz > yz$$

and

$$yz = zy > uy = yu,$$

hence

$$xz > yu.$$

Theorem 35: *If*

$$x \geqq y,\ z > u \text{ or } x > y,\ z \geqq u,$$

then

$$xz > yu.$$

Proof: Follows from Theorem 32 if an equality sign holds in the hypothesis; otherwise from Theorem 34.

Theorem 36: *If*

$$x \geqq y,\ z \geqq u,$$

then

$$xz \geqq yu.$$

Proof: Obvious if two equality signs hold in the hypothesis; otherwise Theorem 35 does it.

CHAPTER II

FRACTIONS

§ 1

Definition and Equivalence

Definition 7: *By a fraction* $\frac{x_1}{x_2}$ (read "x_1 over x_2") *is meant the pair of natural numbers* x_1, x_2 (in this order).

Definition 8:

$$\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$$

($\sim$ to be read "equivalent") *if*

$$x_1 y_2 = y_1 x_2.$$

Theorem 37:

$$\frac{x_1}{x_2} \sim \frac{x_1}{x_2}.$$

Proof:

$$x_1 x_2 = x_1 x_2.$$

Theorem 38: *If*

$$\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$$

then

$$\frac{y_1}{y_2} \sim \frac{x_1}{x_2}.$$

Proof:

$$x_1 y_2 = y_1 x_2,$$

hence

$$y_1 x_2 = x_1 y_2.$$

Theorem 39: *If*

$$\frac{x_1}{x_2} \sim \frac{y_1}{y_2},\quad \frac{y_1}{y_2} \sim \frac{z_1}{z_2}$$

then

$$\frac{x_1}{x_2} \sim \frac{z_1}{z_2}.$$

Proof: $x_1 y_2 = y_1 x_2,\quad y_1 z_2 = z_1 y_2,$