Math 423 - Graph Theory
1. Homework Assignment Number 1.
Due Friday, January 22, 2010.
(shifted to Friday)
Read in Bondy and Murty sections 1.1, 1.2, 1.3.
page 4. problems 1.1.2 and 1.1.3.
page 5. problems 1.2.3, 1.2.4, 1.2.5, 1.2.6, 1.2.7
(Problem 1.2.12 is now moved to the next assignment!)

2. Homework Assignment Number 2.
Due Friday, January 29, 2010.
Read in Bondy and Murty 1.3, 1.4, 1.5, 1.6, 1.7.
Read http://www.math.uic.edu/~kauffman/DMA.pdf
(we will discuss this in class)
Read http://en.wikipedia.org/wiki/Group_(mathematics)
(Mainly read for the defintion of a group and a couple of examples. Again we will discuss more in class.)
page 5. problem 1.2.12.
page 8. problems 1.3.1, 1.3.2.
page 10. problem 1.4.1.
page 11. problem 1.5.4.
page 11, problem 1.6.2 (You do not need to hand this problem in, but try it and we will do much more with this idea later.)

3. Homework Assignment Number 3.
Due Wednesday, February 10, 2010.
Read the notes on Wang Algebra.

1. Read the notes on the website on Diagrammatic Matrix Algebra and ClassNotes #1. Do the exercise in the Diagrammatic Matrix Algebra notes. This exercise is repeated here for the record. (See below).

2. Find a book on linear algebra and learn about determinants and the formula for the inverse of a matrix in terms of the adjoint matrix. For example, see
<http://en.wikipedia.org/wiki/Adjugate_matrix>
and also find out about eigenvalues of matrices. See
<http://en.wikipedia.org/wiki/Eigenvalue,_eigenvector_and_eigenspace>

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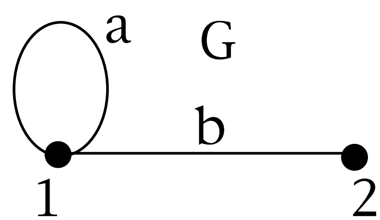
## Adjacency Matrix of a Graph

Given a graph G, we define the adjacency matrix A(G) to be an n xn matrix where n = #V(G) = the number of nodes of the graph G. Letting A = A(G), then A is defined by the equation

**$A_{ij}$ = the number of edges in G with endpoints i and j.**

For example, let G be the graph shown below with
V(G) = {1,2} and E(G) = {a,b} where a is a loop at 1 and b has enpoints 1 and 2.




Then we have

$$A = A(G) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

**Walk Theorem.**
*Let A = A(G) be the adjacency matrix of a graph G.*
*Let $B = A^m$ for some m = 1,2,3,.... Then $B_{ij}$ is equal to the number of walks of length m in G from node i to node j.*

**Proof.** Since

$$(A^m)_{ij} = \Sigma_{k1,k2,\ldots,k(m-1)} (A_{ik1} A_{k1k2} \ldots A_{k(m-1)j})$$

we see that each term in the sum for $(A^m)_{ij}$ counts the number of walks that could happen in the pattern

i ----> k1 ----> k2 ----> .... ----> k(m-1) -----> j.

These add up and count the total number of walks. The proves the Theorem. //

**Exercise.** (a) Let A be the adjacency matrix for the example G just before the statement of the Walk Theorem. (G has two nodes 1 and 2, a loop at 1 and an edge from 1 to 2.) Compute the first few powers $A^k$, and verify that the entries do count walks of length k on the graph G.
(b) Using the matrix A of part (a) find recursive formulas for the entries of $A^m$ as a function of m. Hint: Examine the first few powers of A to find patterns. Then prove your patterns by induction on m.
(c) Use the method of generating functions, via the finding the inverse matrix $(I - At)^{-1}$, to determine the values of the walks on G.
(d) Find the characteristic polynomial $C_A(t) = Det(A - tI)$ and use it to find the eigenvalues of A, and go through the eigenvalue calculations for the walks on G as in Class Notes #1.
**Remark:** You should find that the characteristic polynomial of A is $t^2 - t - 1$ and that if r and s denote the roots of this polynomial, then $(r, 1)^T$ and $(s,1)^T$ are eigenvectors for A. [ $(a,b)^T$ is the transpose of the row vector -- hence a column vector with the same entries.] Thus you can use a matrix P with these columns find that $P^{-1} A P$ is a diagonal matrix with entries r and s on the diagonal. With this you can work out a direct formula for the powers of A in terms of the roots r and s.
(e) Now do the same work as in steps (c) and (d) above for the adjacency matrix for the graph G(n,m,p) with vertices 1 and 2 and n loops at 1, m edges from 1 to 2 and p loops at 2. Write down the adjacency matrix for G(n,m,p) and find the generating series, the characteristic polynomial, the eigenvalues and find recursion relations for the walks from 1 to 1 of all lengths.
(f) Read the web site
<http://www.math.harvard.edu/archive/21b_fall_03/goodwill/>
(It is linked on the Graph Theory course webpage.)
Analyse the graph given there, using what you know about the adjacency matrix. Can you figure out the rest of what is going on mathematically on that page?

(See Class Notes #1)

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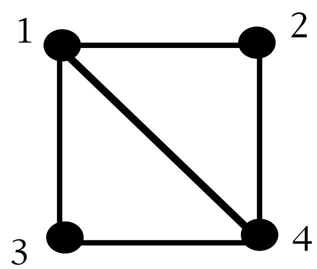
4. Homework Assignment Number 4.
Due Friday , February 19, 2010.
Read the notes on Wang Algebra.
Read (again) the notes on Diagrammatic Matrix Algebra, particularly the section about cross products and the "epsilon" matrix on three indices.
Read the notes "Electrical Graphs" and the notes "Matrix Tree Theorem".

(a) Let **G** be the graph below.

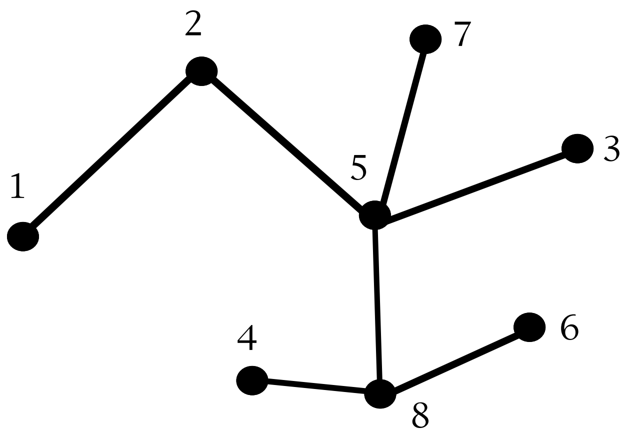



Use the Wang algebra to enumerate all the spanning trees of G.
Use the Matrix Tree Theorem to enumerate all the spanning trees of **G**.

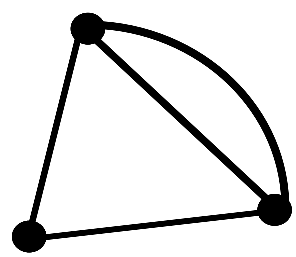
(b) Think about the Wang algebra method for enumerating spanning trees and see if you can prove that it works.

5. Homework Assignment Number 5.
Due Friday, February 26, 2010.
Read Chapter 2 (Trees). Sections 2.1, 2.2, 2.3 and 2.4.
(a) Work on producing a proof of the Matrix Tree Theorem via direct arguments about the terms in the determinant of the minor of the Kirchoff matrix. Report on the state of your proof.
(b) page 26. problems 2.1.1 and 2.1.4.
(c) page 30. problem 2.2.3 and problem 2.2.7 first graph only.
(d) page 36. problem 2.4.3.

6. Homework Assignment Number 6.
Due Wednesday, March 10, 2010.
Read Chapter 9, Sections 9.1,9.2,9.3.
(pp. 135 - 145).
(a) page 36. 2.4.1, 2.4.2, 2.4.4.
(b) Apply the Prufer method of Theorem 2.9 to the labeled tree shown below. After finding its Prufer code, apply the reverse procedure and retreive the tree from the code.




(c) Show that the following graph is isomorphic to its planar dual. Can you find other examples of self-dual plane



graphs?

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THERE WILL BE AN EXAM ON FRIDAY, MARCH 12.
TOPICS FOR THE EXAM INCLUDE

1. BASICS ABOUT GRAPHS -- SIMPLE GRAPHS, CONNECTIVITY, VERTEX DEGREE, EXAMPLES OF GRAPHS SUCH AS K_{3,3}, PETERSEN, K_{N}, NOTION OF PLANARITY, NOTION OF TREE, SPANNING TREES IN
A GRAPH.

2. WALKS ON A GRAPH. USE OF POWERS OF ADJACENCY MATRIX A(G) TO FIND NUMBER OF WALKS BETWEEN TWO NODES OF G. USE OF
MATRIX METHODS TO OBTAIN RECURSION FORMULA FOR WALKS. IN PARTICULAR, THE USE OF THE FACT THAT IF P(X) = DET(A(G) - XE)
WHERE E IS THE IDENTITY MATRIX (SO P(X) IS THE CHARACTERISTIC

POLYNOMIAL OF A(G)), THEN P(A) = 0 (CAYLEY-HAMILTON THEOREM)
GIVES A RECURSION RELATION FOR THE WALKS.

3. FOUR METHODS FOR FINDING SPANNING TREES IN G:
(A) WANG ALGEGRA.
(B) KIRCHOFF MATRIX - DETERMINANT OF A MINOR OF IT.
(C) CONTRACTION - DELETION FORMULA.
(D) PRUFER CODES FOR LABELLED TREES.

4. DUAL GRAPHS FOR PLANE GRAPHS.

5. EULER FORMULA AND ITS APPLICATIONS TO NON-PLANARITY.
(SEE PROBLEM 2 OF HW#7 BELOW.)

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7. Homework Assignment Number 7
Due Wednesday, March 31, 2010.
PLEASE NOTE THE DUE DATE FOR THIS HOMEWORK!
THIS IS AN ASSIGNMENT THAT GOES ACROSS THE SPRING BREAK AND IS DUE WEDNESDAY OF THE WEEK AFTER THE BREAK.
Read Chapter 4, pp. 51 - 60.
Read the links on our webpage about Graph Automorphisms.
Read as much of Chapter 9 of Bondy and Murty as you can.
We will not cover everything in that chapter (e.g. we will probably not try to do the proof of the Kuratowski Theorem in class) but it is worth reading it.

1. Let G be a simple connected plane graph such that every node has degree 4.
(a) Show, by using the Euler Formula that G must have at least one region with exactly three faces.
Hint: First note the 4v = 2e by the assumption that every node has degree 4. Write
$f = f_3 + f_4 + \ldots$ where $f_n$ denotes the number of faces of G that have n edges. Show that
$2e = 3f_3 + 4f_4 + 5f_5 + \ldots$
and use these facts in conjuction with v - e + f = 2 to prove that
$f_3 = 8 + f_5 + 2f_6 + 3f_7 + 4f_8 + \ldots$
The assertion you wish to prove follows from this. In fact, as you see you will have proved that G has at least 8 triangles!
(b) Give an example of such a G with exactly 8 three-sided regions.

(c) Explore examples of graphs of this type. Report on the phenomena of having 4-sided regions. Note that the result you proved does not say anything about the 4-sided regions.

2. Let G be a connected finite graph with girth $g > 2$, and E edges and V nodes. Write out a clear proof of the formula

$$k >= E - (V-2)g /(g -2)$$

where k is the crossing number of G (the least number of extra degree four crossings needed to place G in the plane).
*Use this formula to show that the following graphs are non-planar and do your best to determine the minimal crossing number in each case.* (For $K_n$ you will get an inequality as above that is a function of n. Can you see the first place where it fails to locate the minimal crossing number?):

(a) $K_{3,3}$

(b) Petersen Graph

(c) $K_n$ for $n > 4$.

(d) Find a non-planar graph whose non-planarity is not detected by this formula. Such examples exist and you should be able to make infinitely many such examples by subdividing edges (for example) in given non-planar graphs. You should also be able to find examples where there are no degree 2 nodes in the graph.

3. Let G be the graph $K_3$, the complete graph on three nodes 1, 2, 3.

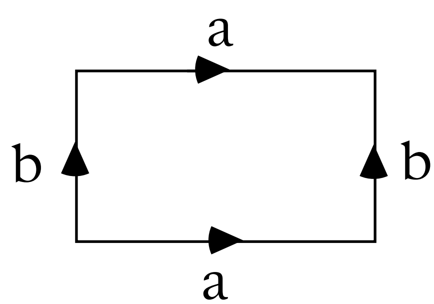
(a) Give the adjacency matrix A for G.

(b) Show by direct calculation that $A^2 = A + 2E$ where E is a 3 x 3 identity matrix.

(c) Find a recursion relation for g(n) = "the number of walks of length n from node 1 to node 1 in G". Determine the values of g(n) for n=1,2,3,4,5.

(d) Determine the characteristic polynomial of A and show that its roots are (-1) with multiplicity two, and (2) with multiplicity one. Find a and b such that $g(n) = a(-1)^n + b\,2^n$ for all n.

4. Page 139 in Bondy and Murty. problem number 9.1.4. Figure 9.4 and the discussion related to it should enable you to think about graphs in the torus. You need to understand that a torus is the result of identifying the opposite sides of a rectangle.




In this figure you will get a torus from a flexible rectangle if you identify a to a and b to b in the directions show. This is what is meant by the appearance of P's, Q's and R's in Figure 9.4.

5. Now that you are acquainted with the torus, try examining some graphs on the torus. As in the plane, graphs on the torus divide it into regions. Assume that all the regions have the shape of discs. What do you find for v-e+f on the torus?

6. Page 142 in Bondy and Murty. Problem 9.2.2.

8. Homework Assignment Number 8.
Due Wednesday, April 14, 2010.

1. Consider regular graphs on a sphere with g handles.
That is we have a graph with n edges per vertex and m edges per face tightly embedded in $S_g$. Call this a graph of type (n, m, g).
Show that if G is a graph of type (n,m,g) then, assuming that G has v nodes, e edges (not zero) and f faces then
$(1/n) + (1/m) = (1/2) + (1-g)/e$ .

2. Take the situation of problem 1 in the case g = 1 (the torus) and find out everything you can about graphs of type (n,m, 1). Compare your results with what you already know about graphs of type (n,m, 0).

3. Given a finite (abstract) graph G, the *genus of G* , denoted by g(G), is the smallest genus for a sphere with handles , $S_g$, in which G can be tightly embedded. Following your class notes, write up an exposition with examples of how to determine the genus of a graph.