ROBBINS ALGEBRA

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Robbins Algebra

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I. Introduction.

It is a well-known problem in Boolean algebra to determine whether an algebra A with binary operation $+$ and unary operation $'$ is Boolean if it satisfies the axioms

1. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$
2. $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$
3. $\mathbf{x}=((\mathbf{x}+\mathbf{y})'+(\mathbf{x}+\mathbf{y}')')'$

(In each axiom it is intended that the equation be satisfied for all choices of x, y and z from A). See [3] for a statement and history of this problem – the **Robbins Problem**.

This is a delicate problem, and it remains unsolved as of this writing. The purpose of this paper is to show that the Robbins problem can be fruitfully investigated by using a simplified notation for formal algebras. In this notation we conjecture a non-standard model for Robbins algebra in terms of the language itself. (See section 2 of this paper for the model.)

This same language allows us to give very clean proofs of the following results of Winker [8]:

1. If there exists a in A, a Robbins algebra, such that $a+a=a$, then A is Boolean.
2. If in a Robbins algebra A there exist a, b such that $a+b=b$, then A is Boolean.
3. If in a Robbins algebra A there exist a, b such that $(a+b)'=b'$, then A is Boolean.
4. If a Robbins algebra is finite, then it is Boolean.

These results show that any algebra satisfying the Robbins axioms is very close to being Boolean. Finiteness, or an instance of absorbtion $(a+b=a)$, or an instance of idempotency $(a+a=a)$, will push A into being Boolean.

In section 2, I give the construction of the proposed non-Boolean model for Robbins Algebra. This section also details the different notations available for this model. The simplest way to express the model is in terms of disjoint collections of rectangles in the plane. This leads to algebras that describe modes of combining rectangular patterns. I call these Box Algebras, and explain how to translate both Boolean and Robbins algebras into the Box format. The section ends with a historical note about the relationship of the Robbins Problem and the axioms for Boolean algebra due to Huntington [4]. ≪The formalism that we are using dovetails also with the work of G. Spencer-Brown [7]. Section 2 explains this connection, and how the old work of Huntington can be applied and simplified to yield the Primary Algebra of Laws of Form from a single initial.≫ We must proceed in the Box algebra for the Robbins problem without assuming the presence of the empty word (a box-equivalent of zero in Boolean algebra). This issue of prohibiting void substitution is taken up in section 2. In section 3 we show (in Box notation) that a Robbins algebra containing an element a such that $a+a=a$ is necessarily Boolean. Section 4 discusses a quotient of the model in section 2. This algebra, The **Paradoxical Robbins Algebra - PR**, is a model for Robbins algebra that contains elements $\mathbf{J}$ such that $\mathbf{J}'=\mathbf{J}$. Hence **PR** is certainly non-Boolean **if** it is non-trivial. I conjecture that **PR** is non-trivial.

II. A Formal Algebra.

Let $\mathbf{P}$ denote the collection of all well-formed parenthetical expressions using left and right sharp angle brackets. Thus P is defined recursively by the rules:

1. $< >$ belongs to $\mathbf{P}$.
2. If a and b belong to $\mathbf{P}$, then ab (the juxtaposition of a and b belongs to $\mathbf{P}$.
3. If a belongs to $\mathbf{P}$, then $< a >$ belongs to $\mathbf{P}$.

A partial list of the elements of $\mathbf{P}$ begins:

$$<>, <<>>, <><>, <<<>>>, <<><>>, <<>><>,$$
$$<><><>, <><<>>, \ldots$$

Any element of $\mathbf{P}$ is uniquely specified by a sequence of binary bits denoted L and R (but not every such sequence defines an element of P). Thus $L=<$, and $R=>$ so that $<><<>>= LRLLRR$. Call two elements of P equal if they have identical sequences of bits.

It is clear that the operation of juxtaposition on P $(a, b \;----> ab)$ is associative, and non-commutative ($<><<>>$ is distinct from $<<>><>$.)

The candidate for a non-Boolean Robbins algebra is constructed as follows:

Definition 2.1. Let $\mathbf{R}$ denote the set of equivalence classes of elements of $\mathbf{P}$ under the equivalence relation generated by the elementary equivalences indicated below:

1. $ab = ba$
2. $a = << ab >< a < b >>>$

(valid whenever a and b belong to **P**).

Definition 2.2. Endow **R** with a binary operation via $\{a\} + \{b\} = \{ab\}$ where $\{a\}$ denotes the **R**-equivalence class of an element a of **P**. In other words, the sum of two elements of **R** is the equivalence class of the juxtaposition of any two representatives of these elements in **P**.

Similarly, we endow **R** with the unary operation $\{a\}' = \{< a >\}$.

Proposition 2.3. **R** is a Robbins algebra.

Proof. Associativity and commutativity follow directly from the definitions. I shall verify the third axiom:

$$\begin{aligned} A &= \{a\} \\ &= \{<< ab >< a < b >>>\} \\ &= \{< ab >< a < b >>\}' \\ &= (\{< ab >\} + \{< a < b >>\})' \\ &= (\{ab\}' + \{a < b >\}')' \\ &= ((\{a\} + \{b\})' + (\{a\} + \{< b >\})')' \\ &= ((\{a\} + \{b\})' + (\{a\} + \{b\}')')' \\ &= ((A + B)' + (A + B')')'. \end{aligned}$$

This completes the proof of the Proposition.

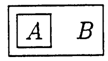
Conjecture 2.3. **R** is a non-Boolean Robbins algebra.

Discussion.

In terms of the parenthesis structures, this is a concrete conjecture about the equivalence classes generated by the equivalence

$$a = << ab >< a < b >>> .$$

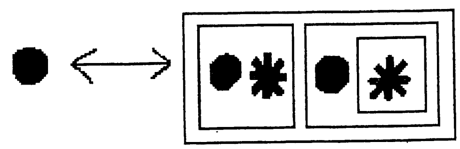
That is, we can put commutativity and associativity into the background and concentrate on the remaining equivalence that generates this model. One way to put commutativity and associativity into the background is to choose a notation for the parenthesis structures that facilitates seeing them. In particular we could replace each left/right pair of parentheses by a box or rectangle as shown below



$$<< A > B >$$

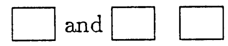
The advantage of the rectangles is that no search is needed to find the regions that are delimited by the parentheses. Secondly, **we can regard commutavity as a topological relation in this notation:** A given collection of disjoint rectangles in the plane is equivalent to another if there is a homeomorphism of the plane taking one collection to the other. In this way commutativity and associativity are truly in the background from the beginning. In rectangle form, the Robbins equivalence becomes the

Robbins Box Axiom:



where the disc and the star stand for any two ***non-empty*** collections of disjoint rectangles.

With commutativity and associativity in the background, the Robbins problem is thrown into sharp relief as the question whether the equivalence generated by this replacement on (topological) collections of rectangles reduces to a Boolean pattern. In particular we would like to show that it is never the case that



are equivalent.

It might seem that some simple counting argument would show this to be the case, and dispose of the Robbins problem at once! This does not yet seem to be the case.

The algebra BR.

To summarize, we have proposed an equivalence relation on the non-empty topological collections of disjoint rectangles in the plane. With this equivalence relation, generated by the Robbins Box Axiom, the equivalence classes have the structure of a Robbins Algebra. **Call this algebra BR.** *WE CONJECTURE THAT* **BR** *IS NOT BOOLEAN.*

Some History.

In looking at the Robbins problem it helps to recall its historical context. In a 1933 paper [4] Huntington pointed out that Boolean algebra could be axiomatized by three axioms: commutativity, associativity, and one more.

Huntington's Axioms for Boolean Algebra

1. $\mathbf{x + y = y + x}$
2. $\mathbf{x + (y + z) = (x + y) + z}$
3. $\mathbf{x = (x' + y)' + (x' + y')'}$

Here it is assumed that the algebra is endowed with binary operation $+$, and unary operation $'$. The axioms apply to all x, y and z in the set under discussion. It is not assumed that there is a zero element. That is, Huntington does not assume that there exists an element 0 such that $0 + x = x$ for all x. Part of his tour-de-force of derivation was to produce a zero element from this minimal set of axioms.

The Robbins problem replaces Huntington's third axiom with a plausible stand-in, and creates a subtle difficulty! What is the nature of this difficulty? A look at Huntington's derivation will help.

Lemma. Under Huntington's axioms, $\mathbf{x}+\mathbf{x}' = \mathbf{y}+\mathbf{y}'$ for any choice of $\mathbf{x}$ and $\mathbf{y}$.

Proof.

$$\begin{aligned}
x + x' &= (x'+y')' + (x'+y'')' + x' \\
&= (x'+y')' + (x'+y'')' + (x''+y')' + (x''+y'')' \\
&= (y'+x')' + (y''+x')' + (y'+x'')' + (y''+x'')' \\
&= (y'+x')' + (y'+x'')' + (y''+x')' + (y''+x'')' \\
&= y + y'.
\end{aligned}$$

The first two lines use the third axiom; the next two use commutativity and associativity. Then the third axiom is applied in reverse. This completes the proof of the Lemma.

Huntington then goes on to define zero via

$$\underset{\sim}{0} = (a + a')' \text{ for any } a,$$

and he shows that this really is a zero, and that the algebra is Boolean. The crux of the matter appears to be the fact that Huntington's third axiom divides on the right hand side into two moveable pieces: $(x'+y)'$ and $(x'+y')'$. Robbins axiom does not have this property, and it is very difficult to start any process of derivation.

Another difficulty, even within this standard context is the complexity of the standard notation with its profusion of parentheses and other marks. For example, consider the appearance of Huntington's Lemma in the box notation:

Huntington Box Axiom

$$A = \boxed{\boxed{A}\ B}\quad \boxed{\boxed{A}\boxed{B}}$$

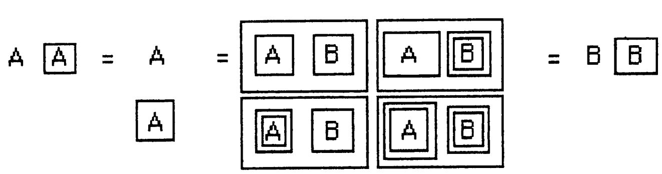
(plus implicit commutativity and associativity of juxtaposition)

The same Lemma then appears as follows:

Lemma 1. Assuming the Huntington Box Axioms, the following identity holds for any A and B in the algebra:

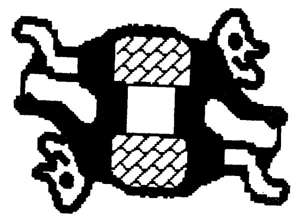
$$A\boxed{A} = B\boxed{B}.$$

Proof.



Thus we re-write the first expression in vertical form, and then expand each of its terms horizontally. We then read the two apparent vertical columns, and interpret each column as an instance of the Huntington Box Axiom.

This proof has the same perceptual effect as the double-take we experience on viewing the figure below:



In this figure, we see two men sitting - one vertical, the other inverted. Or we see two men lying horizontally - one upright, one upside down!

We have made a transition (notational transition) to a "box algebra" that is essentially equivalent to the usual form of Boolean-type algebra. The notation is useful in that it allows easy access to certain patterns. On the other hand, there are some non-standard features of this algebra just below the surface.

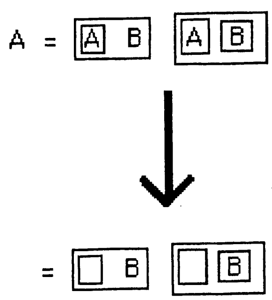
The first non-standard feature is that we have allowed the empty box as an element of the mathematical system under discussion. Since "putting a box around it" is the image in the box algebra of the unary operation a' in the ordinary algebra, this suggests that we regard an empty box as the result of applying the unary operation to the "void".

$$\boxed{A} \longleftrightarrow A'$$
$$\boxed{} \longleftrightarrow\ '.$$

The issue is illustrated in the diagram above. We have adopted the correspondence of "A with a box around it" with "A prime". Thus an empty box appears to correspond to a "lonely prime".

To put the matter simply, the box algebra admits the concept of an empty word, while it is not common to use an empty word in an axiomatization for a Boolean or quasi-Boolean algebra. In the ordinary algebra, the empty word is a bit awkward, leading to lonely primes and empty parentheses (e.g. $(\ )'$). In box algebra the void is always present within the smallest box. Let us call the empty word in box algebra the **void**.

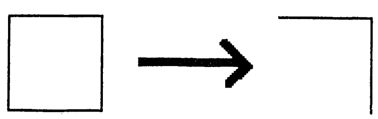
To allow the void into the box algebra is equivalent to having a zero element in the ordinary algebra. That is, we **could allow the substitution of a void in the axioms.** For example, in the Huntington Box Axiom we could allow as a special case:



This creates an equivalence between the void and particular combinations of symbols in the algebra.

By allowing void substitution, we obtain a simplified version of Huntington's axiomatics for Boolean algebra. By way of illustration of these methods, I shall continue in the rest of this section to derive some consequences from the Huntington Box Axioms, **with void substitution allowed.** In the next section we shall consider the Robbins problem in the context of the box algebra. There we must not allow void substitution, since this trivializes the problem.

The Final Change in Notation. A half box is obtained from a box by **erasing the left-hand and bottom edges of that box.**



The box with left edge and bottom edge removed will be called the **mark. Any question of usage for the mark is referred to the corresponding question for the box.** Thus there is no inherent ambiguity about whether a given expression is inside or outside a given mark. An expression in marks is well-formed if the corresponding boxed expression is well formed. The boxed expression is obtained by adding the left and bottom edges to each mark.

I shall use the marks notation throughout the rest of the paper. In particular the basic Huntington Box Axiom becomes

$$A = \overline{\overline{A}\rceil\ B}\rceil\ \ \overline{\overline{A}\rceil\ \ \overline{B}\rceil}\rceil \qquad \text{(H)}$$

in the form of the mark. I shall denote this axiom by **(H)**.

Lemma 1. $a\ \ \overline{a}\rceil = b\ \ \overline{b}\rceil$

Proof. This is a restatement of Lemma 1 as given above, into the mark notation.

Lemma 2. $\overline{\overline{a}\rceil}\rceil = a$

Proof.

$$\overline{\overline{a}\rceil}\rceil = \overline{\overline{\overline{\overline{a}\rceil}\rceil}\rceil\,\overline{a}\rceil}\rceil\ \ \overline{\overline{\overline{\overline{a}\rceil}\rceil}\rceil\,\overline{\overline{a}\rceil}\rceil}\rceil \qquad \text{(H)}$$

$$= \overline{\overline{\overline{\overline{a}\rceil}\rceil}\rceil\,\overline{a}\rceil}\rceil\ \ \overline{\overline{\overline{a}\rceil}\rceil\,\overline{a}\rceil}\rceil \qquad \text{(L1)}$$

$$= a \qquad \text{(H)}$$

Remark. In composing these derivations, I shall not (always) refer to uses of commutativity or associativity. Thus in the above demonstration, the first step is a direct application of (H); the second step uses Lemma 1 (L1) with $b = \overline{\overline{a}\rceil}\rceil$, $c = \overline{a}\rceil$ so that $\overline{b}\rceil\,b = \overline{c}\rceil\,c$; the last step is a reverse application of (H) with an implicit use of commutativity.

Note that neither Lemma 2 or Lemma 1 use void substitution.

Lemma 3. $a\,\overline{a}\rceil = \overline{\ \ }\rceil$

Proof. Since $a\,\overline{a}\rceil = b\,\overline{b}\rceil$ for any a and b, by Lemma 1, we conclude that

$$a\ \ \overline{a}\rceil = \overline{\ \ }\rceil$$

via void substitution for b.

Lemma 4. $\overline{a\ \ \overline{a}\rceil}\rceil =$

Proof. Cross both sides of the statement of Lemma 3. Apply Lemma 2 with void substitution for **a**.

Lemma 5. $aa = a$

Proof.

$$aa = \overline{\overline{aa}\rceil}\rceil \qquad \text{(L2)}$$

$$= \overline{\overline{a\,\overline{a}\rceil}\rceil\ \ \overline{aa}\rceil}\rceil \qquad \text{(L4)}$$

$$= \overline{\overline{\overline{\overline{a}\rceil}\rceil\,\overline{a}\rceil}\rceil\ \ \overline{\overline{\overline{a}\rceil}\rceil\,a}\rceil}\rceil \qquad \text{(L2)}$$

$$= \overline{\overline{a}\rceil}\rceil \qquad \text{(H)}$$

$$= a \qquad \text{(L2)}$$

Lemma 6. $\overline{\overline{a}\rceil\,b}\rceil = a$

Proof.

$$\overline{\overline{a}\rceil\,b}\rceil\,a = \overline{\overline{a}\rceil\,b}\rceil\,\overline{\overline{a}\rceil\,b}\rceil\,\overline{\overline{a}\rceil\,\overline{b}\rceil}\rceil \qquad \text{(H)}$$

$$= \overline{\overline{a}\rceil\,b}\rceil\,\overline{\overline{a}\rceil\,\overline{b}\rceil}\rceil \qquad \text{(5)}$$

$$= a \qquad \text{(H)}$$

Lemma 7. $\overline{a}\rceil\,b = \overline{ab}\rceil\,b$

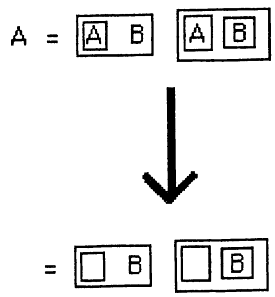
$$\overline{a}\rceil\,b = \overline{\overline{\overline{a}\rceil}\rceil\,b}\rceil\ \ \overline{\overline{\overline{a}\rceil}\rceil\,\overline{b}\rceil}\rceil\,b \qquad \text{(H)}$$

$$= \overline{\overline{\overline{a}\rceil}\rceil\,b}\rceil\,b \qquad \text{(L6)}$$

$$= \overline{ab}\rceil\,b \qquad \text{(L2)}$$

Lemma 8. $\overline{\overline{ac}\rceil\,\overline{bc}\rceil}\rceil = \overline{\overline{a}\rceil\,\overline{b}\rceil}\rceil c$

as a special case:



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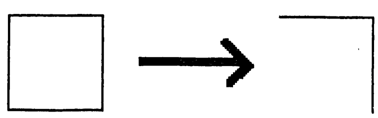

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in the form of the mark. I shall denote this axiom by **(H)**.

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Proof. This is a restatement of Lemma 1 as given above, into the mark notation.

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Proof.

$$\begin{aligned} \overline{\overline{a}|}| &= \overline{\overline{\overline{\overline{a}|}|}|\,\overline{a}|}|\ \ \overline{\overline{\overline{\overline{a}|}|}|\,\overline{\overline{a}|}|}| && \text{(H)} \\ &= \overline{\overline{\overline{\overline{a}|}|}|\,\overline{a}|}|\ \ \overline{\overline{\overline{a}|}|\,\overline{a}|}| && \text{(L1)} \\ &= a && \text{(H)} \end{aligned}$$

Remark. In composing these derivations, I shall not (always) refer to uses of commutativity or associativity. Thus in the above demonstration, the first step is a direct application of (H); the second step uses Lemma 1 (L1) with $b = \overline{\overline{a}|}|$, $c = \overline{a}|$ so that $\overline{b}|\,b = \overline{c}|\,c$; the last step is a reverse application of (H) with an implicit use of commutativity.

Note that neither Lemma 2 or Lemma 1 use void substitution.

Lemma 3. $a\ \overline{a}| = \overline{\ \ }|$

Proof. Since $a\ \overline{a}| = b\ \overline{b}|$ for any a and b, by Lemma 1, we conclude that

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via void substitution for b.

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Proof. Cross both sides of the statement of Lemma 3. Apply Lemma 2 with void substitution for a.

Lemma 5. $aa = a$

Proof.

$$\begin{aligned} aa &= \overline{\overline{aa}|}| && \text{(L2)} \\ &= \overline{\overline{a\,\overline{a}|}|\ \ \overline{aa}|}| && \text{(L4)} \\ &= \overline{\overline{\overline{\overline{a}|}|\,\overline{a}|}|\ \ \overline{\overline{\overline{a}|}|\,\overline{a}|}|}| && \text{(L2)} \\ &= \overline{\overline{a}|}| && \text{(H)} \\ &= a && \text{(L2)} \end{aligned}$$

Lemma 6. $\overline{\overline{a}|\,b}| = a$

Proof.

$$\begin{aligned} \overline{\overline{a}|\,b}|\,a &= \overline{\overline{a}|\,b}|\,\overline{\overline{a}|\,b}|\,\overline{\overline{a}|\,\overline{b}|}| && \text{(H)} \\ &= \overline{\overline{a}|\,b}|\,\overline{\overline{a}|\,\overline{b}|}| && \text{(5)} \\ &= a && \text{(H)} \end{aligned}$$

Lemma 7. $\overline{a}|\,b = \overline{ab}|\,b$

$$\begin{aligned} \overline{a}|\,b &= \overline{\overline{\overline{a}|}|\,b}|\ \ \overline{\overline{\overline{a}|}|\,\overline{b}|}|\,b && \text{(H)} \\ &= \overline{\overline{\overline{a}|}|\,b}|\,b && \text{(L6)} \\ &= \overline{ab}|\,b && \text{(L2)} \end{aligned}$$

Lemma 8. $\overline{\overline{ac}|\,\overline{bc}|}| = \overline{\overline{a}|\,\overline{b}|}|\,c$

Proof.

$$\overline{\overline{ac|}\,\overline{bc|}\,|} = \overline{\overline{\overline{ac|}\,\overline{bc|}\,c|}\;\overline{\overline{ac|}\,\overline{bc|}\,\overline{c|}|}|} \qquad \text{(H,L2)}$$

$$= \overline{\overline{\overline{ac|}\,c\,\overline{bc|}\,c|}\;\;\overline{\overline{\overline{c|}|}\,\overline{a|}\,\overline{c|}\,\overline{\overline{c|}|}\,\overline{b|}\,\overline{c|}|}|} \qquad \text{(L5,L2)}$$

$$= \overline{\overline{a|}\,\overline{c\,\overline{b|}\,c|}\,\overline{\overline{c|}\,c|}|} \qquad \text{(L6,L7)}$$

$$= \overline{\overline{a|}\,\overline{\overline{b|}\,c|}\,\overline{c|}|} \qquad \text{(L5)}$$

$$= \overline{\overline{a|}\,\overline{\overline{b|}\,c|}|}\,c \qquad \text{(L2)}$$

$$= \overline{\overline{a|}\,\overline{\overline{b|}|}|}\,c \qquad \text{(L7)}$$

This sequence of Lemmas completes the verification that the Huntington Box Algebra with void substitution corresponds to a standard Boolean algebra. To see this we can use the following standard axiomatization of Boolean algebra:

1. $x + y = y + x$ for all x and y.
2. $(x + y) + z = x + (y + z)$ for all x, y and z.
3. There exists an element $\underline{0}$ such that $\underline{0} + x = x$ for all x.
4. $\underline{0} = (x + x')'$ for all x.
5. $((x + z)' + (y + z)')' = (x' + y')' + z$ for all x, y and z.

In this version of Boolean algebra we introduce the second binary operation via the definition.

$$ab = (a' + b')'.$$

The last axiom then is seen to be a distributive law and we define the element 1 via $1 = a + a'$ for any a. I leave it as an exercise for the reader to see that these axioms generate ordinary Boolean algebra.

In box or mark notation, with 0 corresponding to the void, the first three axioms are implicit. The last two are the contents of Lemma 4 and 8, respectively. This completes the verification that the Huntington Box Axiom with void substitution generates Boolean algebra.

The same structure also verifies that the primary algebra of Laws of Form [7] can be derived from a single initial (In the context of Laws of Form an initial is an axiom added to implicit commutativity and associativity. Void substitution is allowed in Laws of Form.). That the primary algebra of Laws of Form can be derived from only one initial was first observed by Dr. Rodney Johnson [1], and independently proved by Mr. Graham Ellsbury [2]. The present proof is due to the author [6], and was found in the process of comparing the Huntington approach with Laws of Form.

The Robbins Axiom. We now look at the Robbins Axiom in mark form:

$$a = \overline{\overline{\overline{a|}\,b|}\;\overline{\overline{ab|}|}|}. \qquad \text{(R)}$$

The original Robbins Problem did not allow void substitution. In fact it is easy to see that the **Robbins Box Algebra is Boolean if we allow void substitution**:

1. Let the letter a be replaced by void in R (R stands for the Robbins Axiom). Then we have

$$(\text{void}) = \overline{\overline{b|}\,\overline{\overline{b|}|}|}$$

for any b. Since any element p of the algebra is of the form $\overline{b|}$ (by R) we conclude that

$$(\text{void}) = \overline{\overline{pp|}|} \text{ for any element } p.$$

Use 1. and R to show that $\overline{\overline{a|}|} = a$ for any a:

$$\overline{\overline{a|}|} = \overline{\overline{\overline{\overline{a|}|}\,a|}\;\overline{\overline{\overline{a|}|}\,\overline{a|}|}|} \qquad \text{(R)}$$

$$= \overline{\overline{\overline{\overline{a|}|}\,a|}\qquad\quad|} \qquad \text{(1.)}$$

$$= \overline{\overline{\overline{\overline{a|}|}\,a|}\;\overline{\overline{a|}\,a|}|} \qquad \text{(1.)}$$

$$= a. \qquad \text{(R)}$$

Once we have that $\overline{\overline{a|}|} = a$, we can deduce the Huntington Box Axiom from the Robbins Box Axiom. We have already verified that the Huntington Box algebra is Boolean if void substitution is allowed. Hence in the presence of void substitution, the Robbins Axiom gives a Boolean algebra.

I leave it to the reader to check that in standard algebra notation, these same ideas give a proof that a Robbins algebra with a zero element is Boolean. A zero element, 0, is an element such that $0 + a = a$ for every a in the algebra.

In the next section we consider the Robbins Box Algebra without void substitution.

III. Robbins Box Algebra.

Following the conventions set out in section 2, we shall examine the Robbins Box Algebra without void substitution. Thus we assume the single Robbins Box Axiom (R), (in mark notation)

$$A = \overline{\overline{\overline{A|}\,B|}\;\overline{\overline{A\,B|}|}|}$$

plus implicit commutativity and associativity as explained in the previous section. (I shall refer to this implicit work as **rearrangement**.) In this section **we will not allow void substitution.** Thus the scope of the axiom is restricted to expressions that are non-empty. In these terms the Robbins Problem is equivalent to the question whether the Robbins Box Algebra admits a non-Boolean model. We have suggested such a model at the beginning of section 2.

In all further discussions, the **Robbins Box Algebra** shall refer to the algebra with the axiom (R), and void substitution forbidden. Here is a proof of Winker's Theorem [8].

Theorem A. If there is an element a in the Robbins Box Algebra such that $aa = a$, then there exists an element f in the algebra such that $fx = x$ for all x. Hence the algebra is Boolean.

Proof. The proof of this Theorem will proceed by a series of Lemmas. Under the hypothesis of the Theorem, let

$$f = \overline{a\,\overline{a}|}|.$$

Let

$$e = \overline{a\,a}|.$$

We will show that this definition of **f** fits the bill.

L1. $ea = a$

Dem.

$ea = \overline{a\,a}|\,a$ (def of e)

$= \overline{a\,a\,a}|$ (rearrange)

$= \overline{a\,a}|$ $(a\,a = a)$

$= e$ (def of e.)

L2. $\overline{ea}| = e\,e$

Dem.

$e\,\overline{a}| = \overline{a\,a}|\,\overline{a}|$ (def)

$= \overline{a\,a\,a}|\,\overline{a}|$ $(a\,a = a)$

$= \overline{a\,a}|\,\overline{a\,a}|$ (rearrange)

$= e\,e$ (def)

L3. $a = \overline{\overline{a}|\,f}|$

Dem.

$a = \overline{\overline{aa}|\,\overline{a\,\overline{a}|}|}|$ (R)

$= \overline{\overline{aa}|\,f}|$ (def)

$= \overline{\overline{a}|\,f}|$ $(a\,a = a.)$

L4. $\overline{a}| = \overline{\overline{ee}|\,a}|$

Dem.

$\overline{a}| = \overline{\overline{\overline{a}|\,e}|\,\overline{\overline{a}|\,\overline{e}|}|}|$ (R)

$= \overline{\overline{\overline{a}|\,e}|\,\overline{\overline{a}|\,f}|}|$ $(\overline{e}| = f)$

$= \overline{\overline{\overline{a}|\,e}|\,a}|$ (L3.)

$= \overline{\overline{ee}|\,a}|$ (L2.)

L5. $a = \overline{\overline{ee}|\,\overline{a}|}|$

Dem.

$a = \overline{\overline{aee}|\,\overline{\overline{aee}|}|}|$ (R)

$= \overline{\overline{aee}|\,\overline{a}|}|$ (L4.)

$= \overline{\overline{ee}|\,\overline{a}|}|$ (L1.)

L6. $\overline{ee}| = \overline{\overline{a}|\,a}| = f$

Dem.

$\overline{ee}| = \overline{\overline{\overline{ee}|\,a}|\,\overline{\overline{ee}|\,\overline{a}|}|}|$ (R)

$= \overline{\overline{a}|\,a}|$ (L4., L5.)

L7. $\overline{fa}| = \overline{a}|$

Dem.

$\overline{fa}| = \overline{\overline{ee}|\,a}|$ (L6.)

$= \overline{a}|$ (L4.)

L8. $fa = a$

Dem.

$fa = \overline{\overline{fa\,a}|\,\overline{fa\,\overline{a}|}|}|$ (R)

$= \overline{\overline{fa}|\,\overline{f\,a\,\overline{a}|}|}|$ $(a\,a = a)$

$= \overline{\overline{a}|\,\overline{fa\,\overline{a}|}|}|$ (L7.)

$= \overline{\overline{a\,a}|\,\overline{fa\,\overline{a}|}|}|$ $(a\,a = a)$

$= \overline{\overline{a\,\overline{f\,\overline{a}|}|}|\,a\,\overline{f\,\overline{a}|}|}|$ (L3.)

$= a$ (R.)

L9. $ef = e$

Dem.

$ef = \overline{aa}|\,f$ (def)

$= \overline{aa}|$ $(a\,f = a$ by L8.)

$= e$ (def.)

L10. $f\,\overline{a}| = \overline{a}|$

Dem.

$f\,\overline{a}| = \overline{\overline{f\,\overline{a}|\,a}|\,\overline{f\,\overline{a}|\,\overline{a}|}|}|$ (R)

$= \overline{\overline{fe}|\,\overline{f\,\overline{a}|\,\overline{a}|}|}|$ (def)

$= \overline{f\,\overline{f\,\overline{a}|\,\overline{a}|}|}|$ (L9. and def)

$= \overline{\overline{\overline{a}|\,f\,a}|\,f}|$ (rearrange)

$= \overline{\overline{\overline{a}|\,f\,a}|\,\overline{\overline{a}|\,a}|}|$ (def)

$= \overline{\overline{\overline{a}|\,f\,a}|\,\overline{\overline{a}|\,\overline{f\,a}|}|}|$ (L3.)

$= a$ (R.)

L11. $f\,f = f$

Dem.

$f\,f = \overline{\overline{f\,fa}|\,\overline{f\,f\,\overline{a}|}|}|$ (R)

$= \overline{\overline{a}|\,\overline{f\,\overline{a}|}|}|$ (L8., L10.)

$= \overline{\overline{a}|\,a}|$ (L3.)

$= f$ (def.)

L12. $xf = x$ for any x.

Dem.

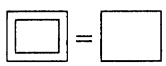
$$\begin{aligned} xf &= \overline{\overline{xfe}\;\overline{xf\,\overline{e}}} && \text{(R)} \\ &= \overline{\overline{xe}\;\overline{x\,f\,f}} && \text{(def, L9.)} \\ &= \overline{\overline{xe}\;\overline{x\,f}} && \text{(L11.)} \\ &= \overline{\overline{xe}\;\overline{x\,\overline{e}}} && \text{(def)} \\ &= x && \text{(R)} \end{aligned}$$

This completes the proof of Theorem A.

By an extension of these techniques we can prove the Box-Theoretic analogs of Winker's other results [8] about the Robbins Problem. That is, we can show that a Robbins Box Algebra containing a pair of elements a and b such that $a + b = b$ is necessarily Boolean. It is then immediate that a non-Boolean Robbins algebra must be infinite. The exposition of these results will be given in a sequel to this paper.

IV. Epilogue on Paradoxical Values.

For the reader who sees some plausibility in the conjecture that the box algebra **BR** of section 2 is a non-Boolean Robbins algebra, I will here offer a modification of that algebra that still appears to be a non-Boolean model. Call this model **PR** - the **Paradoxical Robbins Algebra**. As before, the elements of **PR** are equivalence classes of topological classes of disjoint collections of rectangles in the plane. The equivalence relation is generated by the Robbins Box Axiom (as before) plus the extra equivalence relation generated by the specific relation:



It must be understood that this relation is to be taken quite literally – each situation must occur isolated within a topological disk in the plane of any larger expression. Thus, a box around an empty box is equivalent to a box. We can conclude that a nest of boxes is equivalent to a single box – by applying the relation repeatedly to the two inner-most boxes. However, we can conclude nothing about a box surrounding the juxtaposition of two or more boxes.

The algebra **PR** contains elements **J** such that $\mathbf{J}' = \mathbf{J}$. (Let **J** denote the equivalence class of the empty box. Recall that the unary operation in the model is obtained by "putting a box around it".) Thus **PR** contains "paradoxical" elements, and is necessarily non-Boolean if it is non-trivial. *I conjecture that* **PR** *is indeed non-trivial.*

One final remark – We can not have a non-standard Robbins Algebra with $x''' = x'$ for all x in the algebra. The axioms for the Robbins algebra imply that any element of the algebra is of the form y' for some y in the algebra. Thus $x = y'$ and $y' = y'''$ for all y implies that $x'' = y''' = y' = x$. Hence $x'' = x$ for all x, and this makes the Robbins algebra Boolean.

The upshot is that Robbins algebra does not make a good context for an intuitionistic type logic, but it does seem to provide room for paradoxical elements invariant under the analogue of negation.

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