SIGN AND SPACE

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1. *Introduction*

The complex phenomena of experience are deeply interconnected in levels of patterns. By looking for connections, we move through these levels and begin to see essential unity in diversity.

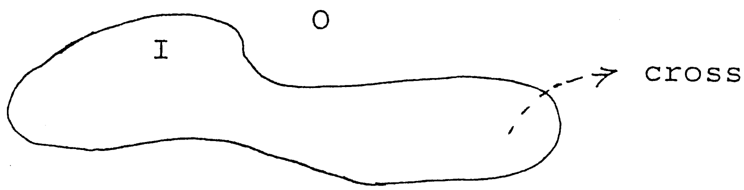
In this essay I examine spatial concepts and, by a process of descent, draw out what I believe are simple unifying patterns. These patterns connect space concepts with sign concepts.

Space is continuous -- signs are discrete. Yet these opposites meet. They more than meet. Sign and space are inseparable -- each creates the other.

The discussion of this topic is bound in circularity: symbols patterning a space created out of the very symbols themselves. Bateson says, "The pattern that connects is a pattern of patterns." [1]

Our method is as follows: Begin in the middle. Observe how new and simpler structures arise by throwing away excess descriptive baggage. Go back to the middle with these structures in mind. Descend again. This process continues, in its oscillatory way, binding the world of experience and the realm of pure pattern.

To illustrate, consider regions bounded by curves in the plane. The curves are simple closed curves -- without self-crossings and without free ends. A simple closed curve divides the plane into two pieces denoted inside (I) and outside (0).



A Flatlander in this arena will also wish to discuss the crossing of the boundary formed by the curve.

If X is the side you are on, let $\overline{X}\rceil$ denote the side you move to when you cross. Thus $\overline{0}\rceil$ = I and $\overline{I}\rceil$ = 0. 0 and I are *names* while $\rceil$ is an *operator* that indicates the crossing of the boundary. In this descriptive context names, operations, and the things that they stand for are held separate.

Since we are discussing only the inside and outside of a single boundary (a single distinction) it is sensible to adopt *condensation rules*: 00 = 0 and II = I. This eliminates repetition of the name. Condensation is already a form of descent, joining apparently distinct forms into one form.

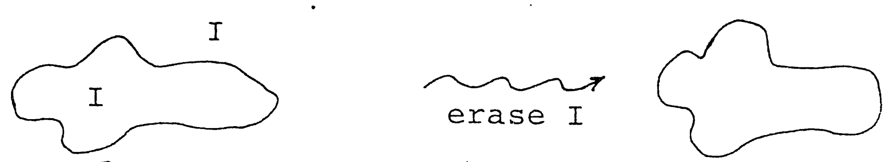
There are now four rules:

$$\overline{0}\rceil = I\ ,\quad \overline{I}\rceil = 0,$$
$$00 = 0\ ,\quad II = I$$

and the beginnings of a simple mathematical system. Complicated expressions now have geometrical interpretations. Thus $\overline{\overline{\overline{I}\rceil}\rceil}\rceil = \overline{\overline{0}\rceil}\rceil = \overline{I}\rceil = 0$ represents a journey that begins inside and crosses the boundary three times.

But the symbols have a life of their own. What is the (geometrical) meaning of 0I ? Perhaps it is an early cousin of the Chomsky sentence "Colorless green ideas sleep furiously."[2] One rule would set $0I = 0$. The 0 dominates I just as a high-voltage signal may dominate a low-voltage signal. With $0I = 0$ the system is *Boolean arithmetic* and it applies to many two-valued situations. For example, one can interpret 0 as True (T) and I as False (F), $\overline{X}\rceil$ as *not* X and 0I as T *or* F. Since the truth value of "T *or* F" is T, this vindicates $0I = 0$. An interpretation beyond the geometrical has suggested a new syntatic rule.

Further descent reveals another source for dominance. We have, all along, been using two names and two symbols (0 and I) for the two sides of the distinction. It would have been enough to simply mark one side, leaving the other unmarked. In other words, we can regard the *absence* of a symbol (the unmarked notational plane) as a reference to the inside (also unmarked). Guided by this idea, descent consists of *erasing* I from the equations:



$$\left\{\begin{array}{l} \overline{0}\rceil = I \\ \overline{I}\rceil = 0 \\ 00 = 0 \\ II = I \end{array}\right\} \longrightarrow \left\{\begin{array}{l} \overline{0}\rceil = \\ \rceil = 0 \\ 00 = 0 \\ \quad = \end{array}\right\}$$

Halfway down the rabbit-hole, the erased equations demand identification of operator $\rceil$ with operand 0! At the bottom of the hole there are *two* equations instead of four.

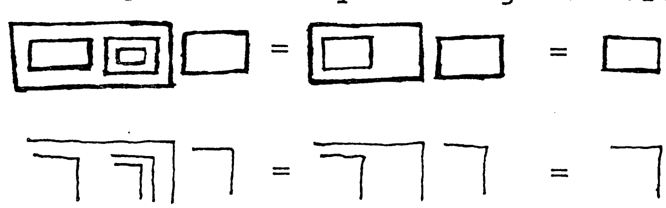
$\overline{\overline{\ \ }|\ }| =$ crossing

$\overline{\ \ }|\overline{\ \ }| = \overline{\ \ }|$ calling

$\overline{\ \ }|\overline{\ \ }| = \overline{\ \ }|$ (calling), $\overline{\overline{\ \ }|\ }| =$ (crossing).

These equations form the basis of G. Spencer-Brown's primary arithmetic. [3] In the primary arithmetic the mark, $\overline{\ \ }|$, is neither operator nor operand, but simply a sign that *itself* makes a distinction in the plane. The object-plane of the original distinction has condensed with the notation-plane of the forms of expression. The language is seen to be its own reference. Signifier and signified are one.

No paradox arises from the conflation. Primary arithmetic is a formal system with just enough structure to generate significant pattern. The mark may be see as shorthand for a rectangle □. Expressions in primary arithmetic consist of disjoint collections of rectangles; they are subject to the transformations generated by calling and crossing.



Calling and crossing constitute basic modes of simplification. Calling is the formal image of condensation, while in crossing, two forms fit together and cancel each other.

Each expression has a unique simplification, a determinant of value (marked or unmarked). Boolean arithmetic is now seen as a description of the properties of the primary arithmetic. By setting I = ⌉|, we recover the four Boolean equations from the primary arithmetic, and also the form of dominance since

0I = ⌉|⌉ = ⌉ = 0.

A space with a mark upon it cannot be unmarked.

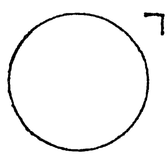
By falling into the primary arithmetic we have found a realm where operator and operand are no longer distinguished from one another and there is no longer a split between describer and described. Along with this descent into a world of patterns underneath Boolean arithmetic, there is a corresponding ascent into new views of logic, paradox, extra-logical values, complex numbers, and the formalisms of relativity and quantum physics. These relationships will be detailed as the paper continues.

In the next section we descend from the geometry of rotations in three-dimensional space through the algebra of quaternions

$$i^2 = j^2 = k^2 = ijk = -1$$

to the primary notion of reflection from which these complexities spring. The quaternions are the true middle ground about which all these musings turn. Section 3 finds quaternions again in the rotational symmetries of an object that is connected with its surroundings. Further descent (section 4) finds a condensation of the quaternionic pattern in the forms of superposition of periodic boundaries. Section 5 relates periodic and self-referential form with paradox and self-containing geometric form. Sections 6 and 7 treat complex numbers as the formalization of the possibility of multiple viewpoints of a periodic ground-form.

Here pattern-evaluation and numerical measurement begin to intermix as the real numbers sound within their complex counterparts. Section 8 details the matrix formulation of complex numbers. By viewing matrices as shorthand for periodic arrays or "waveforms" of real numbers, we obtain a direct relationship with the approach of section 7 and a new interpretation of matrix multiplication as a form of pattern combination. Section 9 studies the form of an event. The most primitive event is an act of observation -- a splitting of the world into observer and observed -- a distinction being made (or unmade). In first approximation this may be symbolized by a curve marking a distinction in the plane.



Further articulation would replace the boundary by an oscillation and the sides by conjugate imaginary values.

a b a
b b $\bar{z}$
a z a
b b
a a
b

Here we realize that there really can be no distinctions at all. Full descent condenses all into one. But how *would* a distinction appear if such a mythical beast could indeed exist? This is a matter of experience and it is the form of an event. The summary form of an event is the form of the crossing, a 2 x 2 hermitian matrix

$$\begin{bmatrix} a & z \\ \bar{z} & b \end{bmatrix}.$$

Here our imagery touches the formalism of relativity and quantum theory. Depending upon one's viewpoint, such a matrix represents either a point in Minkowski spacetime or the mathematical analog of an observer of a two-state quantum system (the quantum mechanical analog of a simple distinction). By mixing these viewpoints we find that an event observing itself can observe its own time and length. An event on the light-cone that operates upon itself produces a radial series of events in Minkowski space, in time-ratios of the original event's time raised to powers of Fibonacci numbers. These last results may have no present physical interpretation, but they provide a beguiling link with the mythology.

I believe that what we see here is the beginning of an arcing connection between myth and science. Both find their commonality in the culture in which we are embedded and the language that we use. It should not be too astonishing that a descent from physics through its mathematical formalism to pattern and form leads inevitably to the mythology of world arising -- with the observer as imaginary mirror of the internal and the external (which are really one).

Section 10 addresses projection, coding, wholes and parts, through the metaphor of knotting and linking in three-space. A summary of the major points, with illustrations, concludes the paper.

I would like to take this opportunity to thank the many people without whose time, conversation and love this attempt at synthesis would never have seen the light of day. Particular thanks go to Kyoko Inoue, David Solzman, Alex Comfort, Barney Weissbourd, Milton Singer, and Francisco Varela.

2. *Rotations - Reflections - Quaternions*

Rotations in three-dimensional space have remarkable properties. Consider the following problem. *Rotate a solid cube by* 90° *about a vertical axis, followed by* 90° *about a horizontal axis. Describe the resulting motion of the cube as a rotation by some angle about a single axis. Determine the angle and the axis.*

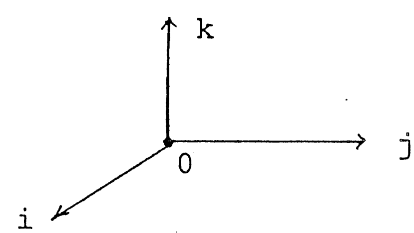
The problem can be solved by experiment, or by a series of pictures. For the pictures, an axis of rotation will by represented by an arrow:

This arrow can be regarded as a unit vector in three-space, R^3. We take R^3 as the set $R^3 = \{(a,b,c) \mid a, b, \text{ and } c \text{ are real numbers}\}$. This is the coordinate model for space. Each point is specified by a triple of numbers. In this interpretation the three basic mutually perpendicular directions are

$i = (1, 0, 0),$

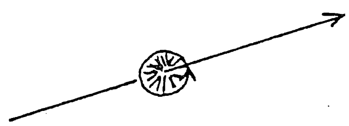
$j = (0, 1, 0),$

$k = (0, 0, 1).$

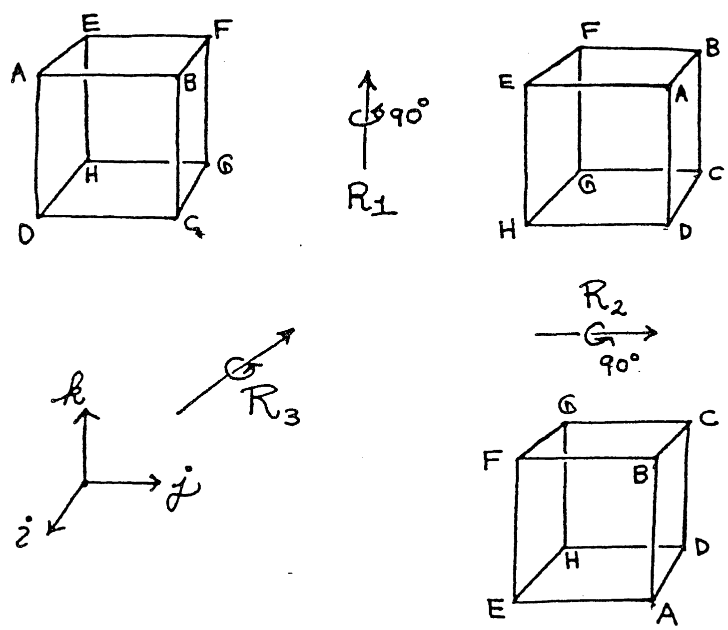


By defining $(a,b,c) + (d,e,f) = (a+d, b+e, c+f)$ and $r(a,b,c) = (ra,rb,rc)$, where the lower-case letters stand for real numbers, we see that R^3 is the set of points of form $ai + bj + ck$ (a,b,c real numbers).

A rotation is specified by an angle θ and an axis U. We adopt the *right-hand rule* to determine the sense of rotation about the axis: The diagram below indicates a positive rotation (positive angle) about the axis U. By placing a right hand around the axis with thumb pointing in the axis direction, the fingers will curl around the axis in the positive sense.

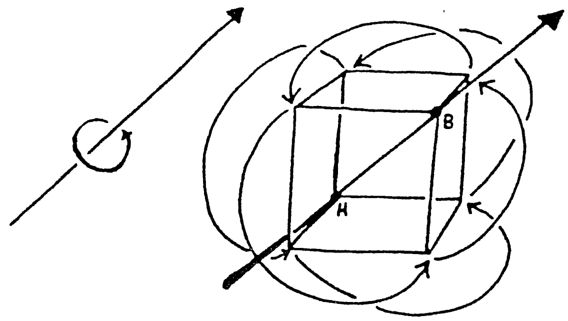


With these conventions, consider the rotation problem:



$R_3 = R_2 R_1$

Denoting the two 90° rotations by R_1 and R_2 as in this diagram, we write $R_3 = R_2R_1$ for the rotation obtained by first doing R_1 and then doing R_2. Examine R_3 directly. R_3 fixes the corners B and H.

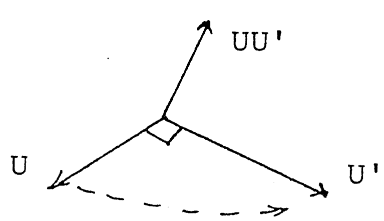


Thus R_3 is a 120° (a third of a full rotation) about the diagonal axis through B and H!

A remarkable symbolic schema, devised in the 1800's by the mathematician William Rowan Hamilton,[4] gives the pattern underlying these compositions of rotations. The schema is an algebra associated with i, j, k that Hamilton called the *quaternions*. It is based upon a series of rules for multiplying i, j, and k among themselves. These rules are:

$$\left\{\begin{array}{ll} i^2 = j^2 = k^2 = -1 & \\ ij = k & ji = -k \\ jk = i & kj = -i \\ ki = j & ik = -j \end{array}\right\}.$$

The rules are easy to remember. The square of each unit vector is −1 . If two unit vectors U, U' are at 90° to one another, then their product UU' is perpendicular to both of them. This product points in an axial direction so that a rotation about it in the sense of rotating U into U' conforms to the right-hand rule.



Actually, we are dealing here with the *four-dimensional space* $\mathbb{H} = \{ai + bj + ck + d \mid a,b,c,d, \text{ real}\}$, since $i^2 = -1$ does not belong to R^3. Hamilton's search for this system was impeded for many years until he stepped off into that fourth dimension.

Any two quaternions are multiplied by treating them as ordinary numbers, but subject to the quaternionic rules. Thus

$$\begin{aligned}(2 + 3i)(5 + 7j) &= 2(5 + 7j) + 3i(5 + 7j) \\ &= 10 + 14j + 15i + 21ij \\ \therefore (2 + 3i)(5 + 7j) &= 10 + 14j + 15i + 21k.\end{aligned}$$

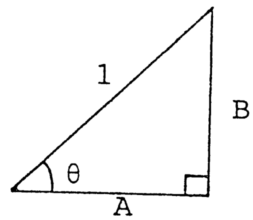
Some quaternionic facts

1. Let $U = ai + bj + ck$ where $\|U\| = a^2 + b^2 + c^2 = 1$. Then $U^2 = -1$ (unit directions have square minus one).
2. Suppose P and Q belong to R^3 with $P = (a,b,c)$ and $Q = (d,e,f)$. Define the *inner product* $P \cdot Q = ad + be + cf$. P and Q are said to be *perpendicular* ($P \perp Q$) if $P \cdot Q = 0$. This conforms to the usual notion of perpendicularity. Then the quaternionic product of P and Q is given by the formula $PQ = -P \cdot Q + PxQ$ where $PxQ = (bf - ce)i + (cd - af)j + (ae - bd)k$. PxQ is called the *vector cross-product* of P and Q. It is perpendicular to both P and to Q.
3. There is a three-dimensional sphere of quaternions of unit length. These may be expressed as $A + BU$, where U is a unit vector in three-space (hence a pure imaginary quaternion) and A and B are real numbers with $A^2 + B^2 = 1$. Define the *conjugate* quaternion $\overline{A + BU} = A - BU$, and note that

$$(A+BU)(\overline{A+BU}) = A^2-ABU+BAU-B^2U^2 = A^2+B^2 = 1.$$

Thus the conjugate of a unit quaternion is its multiplicative inverse.

4. Recall the definitions of sine and cosine as lengths of sides of a right triangle with unit hypotenuse.



$A = \cos(\theta)$

$B = \sin(\theta)$

Thus unit quaternions may be written in the form

$$e^{U\theta} = \cos(\theta) + U\sin(\theta),$$

where $U \in R^3$, $\|U\| = 1$.

Here is the relation between quaternion multiplication and the composition of rotations:

Theorem 2.1. Let $R(U,\theta)$ denote a rotation about axis U by angle θ. Given two rotations $R(U,\theta)$ and $R(V,\phi)$, then $R(U,\theta)R(V,\phi) = R(W,\psi)$, where

$$e^{U\theta/2}e^{V\phi/2} = e^{W\psi/2}.$$

Thus the rotation $R(U,\theta)$ corresponds to the quaternion

$$e^{U\theta/2} = \cos(\theta/2) + U\sin(\theta/2).$$

Example. In the cube problem $R_1 = R(k,\pi/2)$ and $R_2 = R(j,\pi/2)$. Hence R_1 corresponds to

$$e^{k\pi/4} = \cos(\pi/4) + k\sin(\pi/4) = \frac{\sqrt{2}}{2} + k\frac{\sqrt{2}}{2},$$

and R_2 corresponds to

$$e^{j\pi/4} = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}.$$

Thus

$$e^{j\pi/4}\, e^{k\pi/4} = \left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2} + k\frac{\sqrt{2}}{2}\right)$$

$$= \frac{1}{2}(1 + j)(1 + k) = \frac{1}{2}(1 + j + k + jk).$$

$$= \frac{1}{2}(1 + i + j + k) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{i + j + k}{\sqrt{3}}\right).$$

$$\therefore\ e^{j\pi/4}\, e^{k\pi/4} = e^{[(i + j + k)/\sqrt{3}]\ [2\pi/3]\ /2}$$

$$\underset{\text{diagonal axis}}{\uparrow} \qquad \underset{120^\circ}{\uparrow}$$

This calculation corresponds precisely to our geometric-diagrammatic work, giving the correct diagonal axis and rotational angle of 120°. Thus each rotation corresponds to a quaternion in such a way that successive rotations are described by the algebraic product of the quaternions. Theorem 2.1 codifies this remarkable relationship.

Here is a first explanation of why this theorem works. The crux of the matter is that to each $g = A + BU$ we can define a mapping $g^{\#} : R^3 \longrightarrow R^3$ by the formula

$$g^{\#}(p) = gp\bar{g}.$$

One then verifes that $g^{\#}$ is a rotation about the U-axis by the angle θ when $g = e^{U\theta/2}$. Since $(gg')^{\#} = g^{\#}g'^{\#}$, this shows that composition of rotations corresponds to multiplication of quaternions, proving the theorem.

Thus a quaternion of length one corresponds to a rotation about its imaginary part U by the angle associated to it in the form $e^{U\theta/2}$. Since $e^{U\pi/2} = U$, we see that *each unit direction* U *in* R^3 *corresponds to a 180° turn about itself.* Indeed,

$$i^{\#}(i) = ii\bar{i} = ii(-i) = -1(-i) = i,$$

$$i^{\#}(j) = ij\bar{i} = k\bar{i} = -ki = -j,$$

$$i^{\#}(k) = ik\bar{i} = -ij = -k.$$

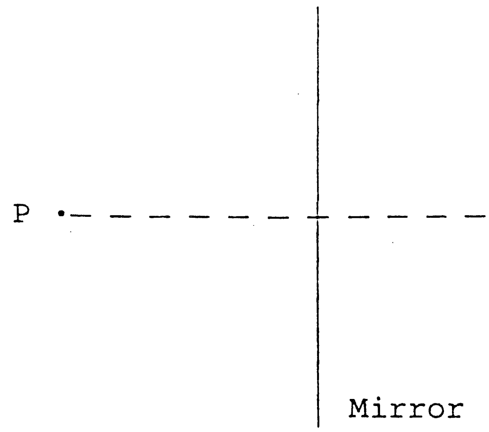
Thus

$$i^{\#}(ai + bj + ck) = ai - bj - ck,$$

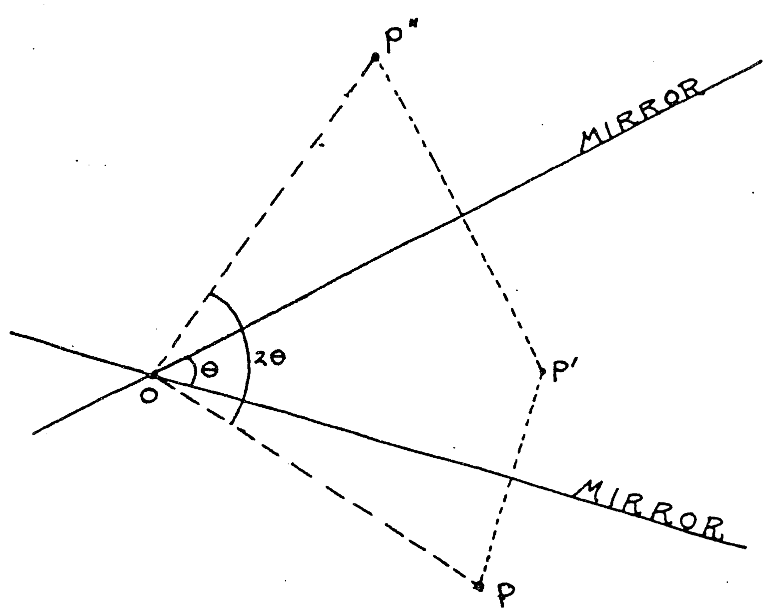
and this is a 180° turn about the i-axis.

But to stop here would be to leave the mystery unsolved. Why does this clever algebra work so well? Why half angles? How do the rules arise? In order to go deeper, a key fact is needed: *Every rotation is the composition of two reflections in two planes that make an angle with each other that is one half the rotation angle.*

To see this, view a plane that is perpendicular to the two reflection planes. Represent the reflection planes by their lines of intersection with the perpendicular plane. Take the plane of the paper as the perpendicular plane. Recall that a reflection is performed by drawing a line orthogonal (perpendicular) to the mirror, and continuing it by the same distance across the other side of the mirror. (See the diagram on the following page.)

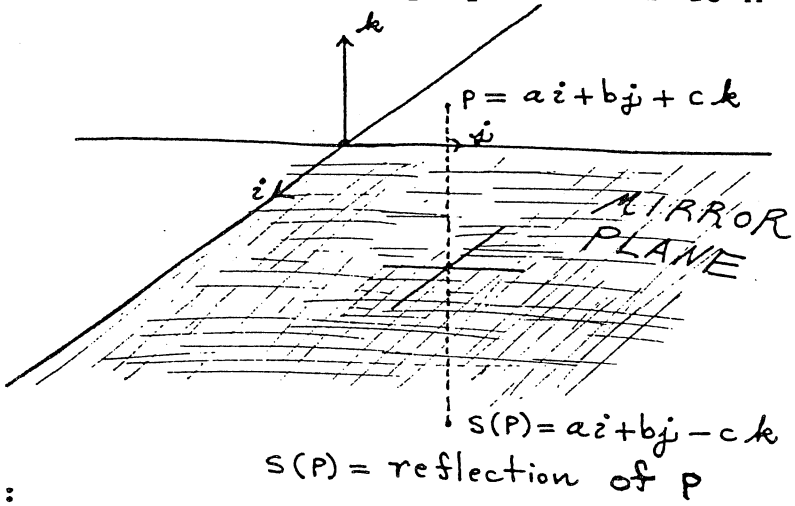


In the next diagram the two lines represent the two intersecting mirrors. The point 0 is a point on the line of intersection of the mirrors. Successive reflections in the two mirrors $P \mapsto P' \mapsto P''$ give a rotation $P \mapsto P''$ about 0. The angle of this rotation is twice the angle between the two mirrors.



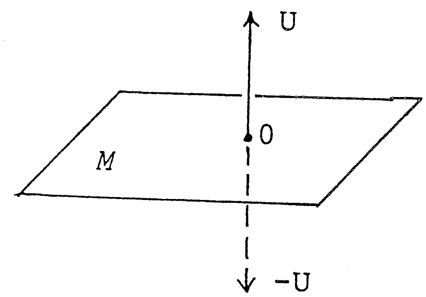
Rotations are secondary phenomena. Reflections are primary. The algebra that describes reflections will also describe rotations. Consider reflection in the i-j plane. This is given by the formula $S(ai + bj + ck) = ai + bj - ck$. The i-j plane is where $c = 0$; it is invariant under the reflection.

$k = U$, unit direction perpendicular to M



Observe:

$$kik = -k^2 i = i,$$
$$kjk = -k^2 j = j,$$
$$kkk = +k^2 k = -k.$$



Thus

$$S(P) = kPk.$$

In general, given a mirror plane M and a unit direction U that is orthogonal to M, reflection in M is given by the formula $S(P) = UPU$. $U^2 = -1$ makes sure that U is reflected, and the anti-commutativity of U with vectors orthogonal to it keeps the mirror plane invariant. All features of the quaternion algebra have direct geometric interpretation in the mirror world.

If U and U' are directions orthogonal to the planes M and M', then $R(P) = U'UPUU'$ is the formula for performing successive reflections in M and M'. Letting $g = U'U$, we have $\overline{U} = -U$, $\overline{U'} = -U'$, and $\overline{U'U} = UU'$. Thus $R(P) = gP\overline{g}$. This recovers the quaternionic formula for rotations.

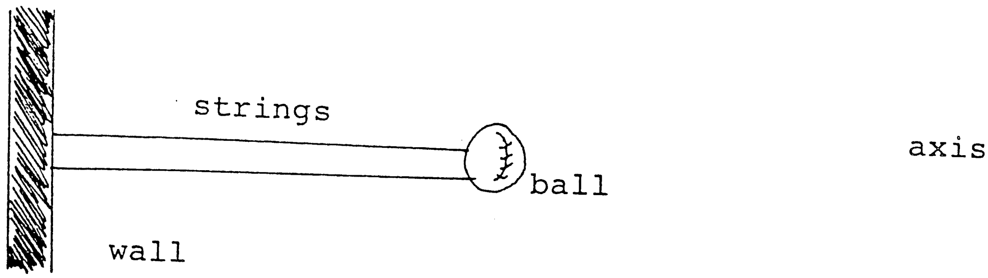
The structure of three-space rotation is purely based on mirrors. By associating to each mirror M an imaginary quantity U, orthogonal to it with $U^2 = -1$, reflection becomes $P \longmapsto UPU$. Thus in the algebra *P becomes a* (notational) *mirror for U. Mirror and mirrored are exchanged in the crossing from geometry to algebra!*

3. *Quaternionic Tangles*

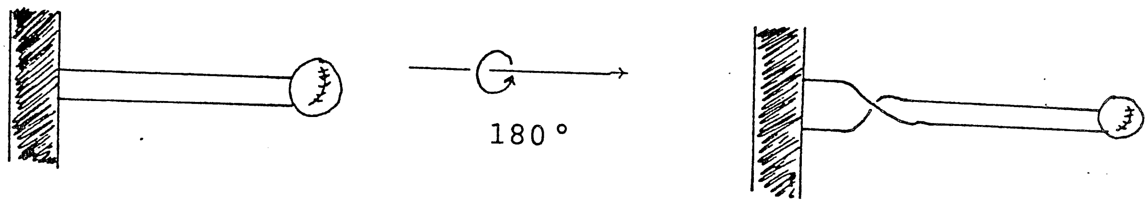
The quaternionic pattern also arises from topological considerations. Recall that i is interpreted in quaternions as a 180° rotation about itself. Thus $i^2 = -1$ represents a 360° rotation. But we are not back to start with -1! It requires 720° or i^4 to return to +1. This algebraic fact has a geometric counterpart.

It is an idealization to imagine an object in space that is free to rotate as it pleases, disconnected from the rest of space and ony other objects in it. Therefore let the object being rotated be connected by strings to a fixed background or reference. Then, as it rotates, the strings become tangled, exhibiting the track of rotation. Amazingly, however, after two full rotations (720°) the strings can be disentangled without moving object or reference. This is the geometric analog of $i^2 = -1$.

In order to see the simplest instance of this phenomena, imagine a ball connected by two strings to a wall.

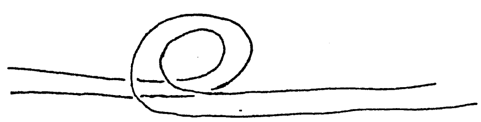


The strings will become twisted about each other if the ball is rotated about an axis perpendicular to the wall.



Let I denote the operation of rotating the ball by 180°. This puts a *single crossing*  or half-twist on the strings. I^2 corresponds to 360°, and puts a full twist  on the strings.

Call a string form of the following type a *curl*:



Curl

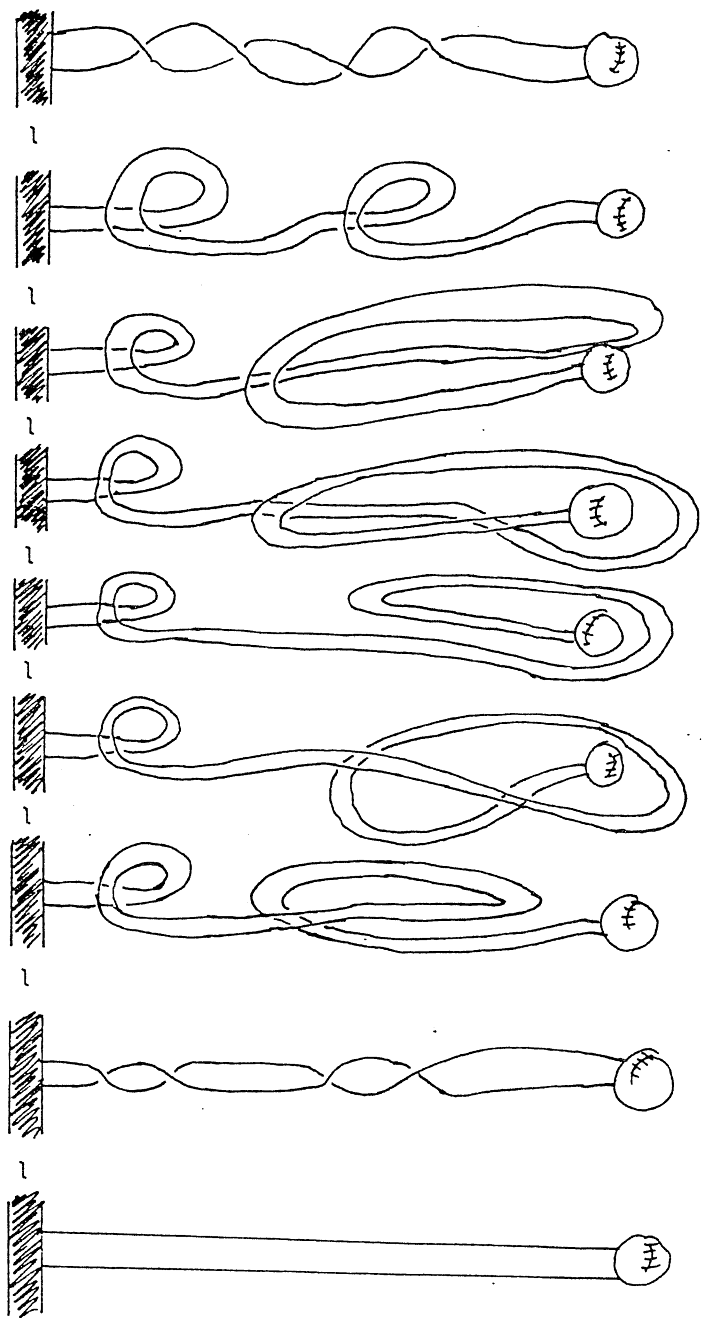
≀ (topological equivalence)

≀

≀

360° Twist

A curl is topologically equivalent to a 360° twist. Thus a 720° turn will put two curls of the same type on the strands. The deformations of the strings occur without moving the ball or the wall. The following diagrams show how the 720° tangle is unraveled.



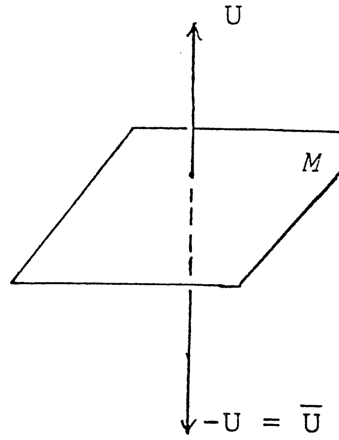
720° twist

two curls (same type)

two curls (opposite type)

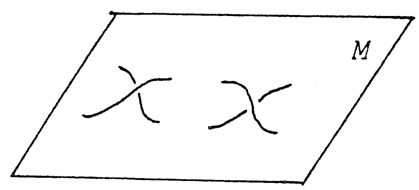
(twists unravel)

<u>0° twist</u>



$U^2 = \overline{U}^2 = -1$

GEOMETRY

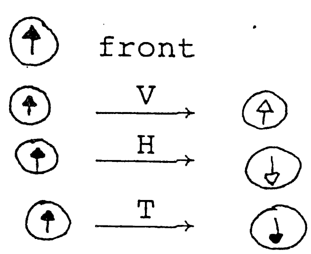


$(\;)^2 = (\;)^2 = -1$

TOPOLOGY

In this sense and are analogous to +1: $(\;)^2 =$ $\longleftrightarrow$ 360° twist $\longleftrightarrow$ -1. Both and constitute "imaginary" third dimensional values associated with a crossing in the plane. They are topological analogs of the unit vectors associated with the mirror plane.

The entire structure of the quaternion group Q = {+1, +i, +j, +k} arises through this weaving trick. In order to see this, first consider the symmetries of a disc. Then add strings connecting it to the surroundings. One has a disc with labelled sides, and the motions:



back

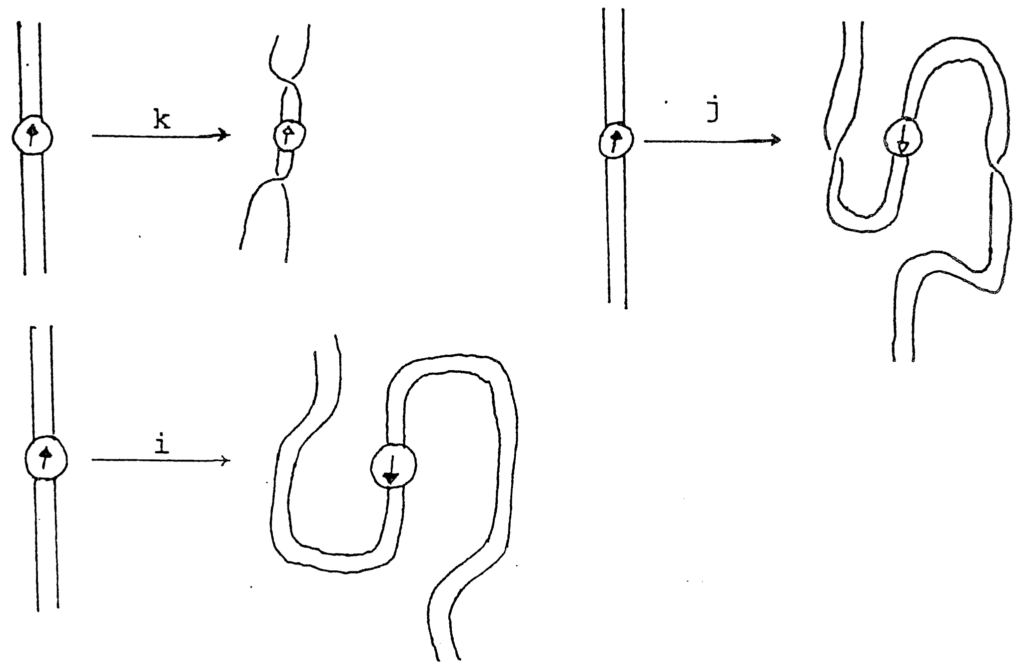
180 turn about (vertical)

180 turn about (horizontal)

180 turn about (axis $\perp$ to page).

Then it is easy to see that $V^2 = H^2 = T^2 = 1$, $VH = T = T = HV$. These turns generate the *Klein 4-Group*, $\mathbb{K}$. $\mathbb{K}$ has the form of the quaternions with -1 set equal to $+1$. If we add strings to the disks and work out how these motions drag the strings around, then the Klein group is replaced by the quaternion group.

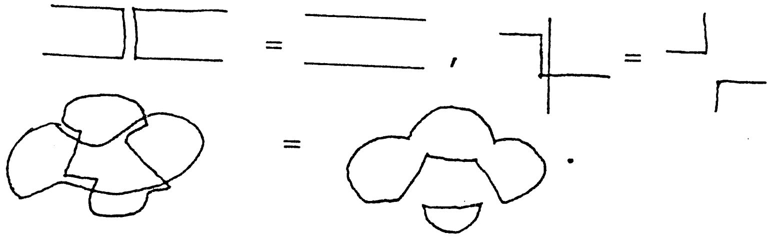
Rotation may deform the space surrounding the rotating object, but space is restored to its original condition (topologically) after every two turns!



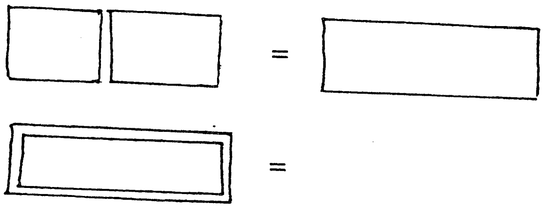
4. *The Klein 4-Group Revisited*

We have just seen the Klein 4-Group as symmetries of a rotating disk. This pattern also arises in the superposition of the simplest periodic waveforms. To explain this we first make a side descent that condenses calling and crossing. The key is Spencer-Brown's notion of *idemposition*.[5]

Axiom of Idemposition. Superimposed boundaries of the same "color" cancel each other. Thus



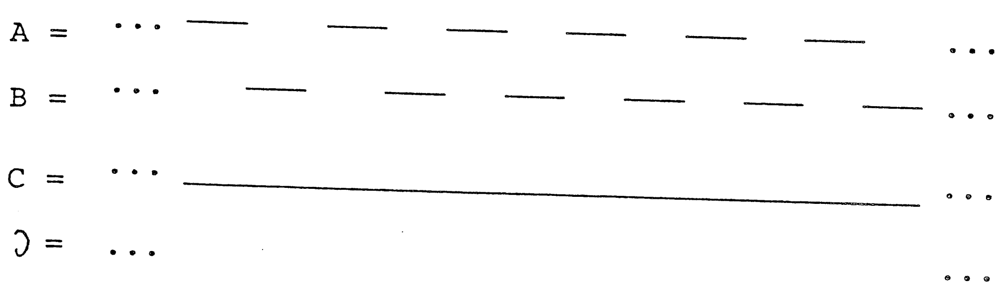
As a corollary of this principle we obtain the following versions of calling and crossing (see section 1).



(call)

(cross).

To obtain the Klein 4-Group consider the following wave-trains.



(Ↄ is the empty wave.)

Let XY denote the superposition of X and Y. Then idemposition implies that

AA = BB = CC = Ↄ ,
AB = C,
BC = A,
CA = B.

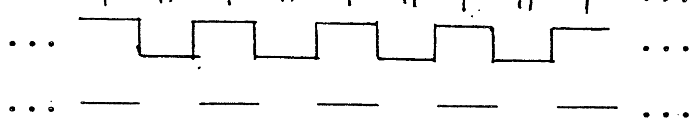
Hence the Klein Group appears from superposition of simple wave-trains. In this sense, the quaternions are directly related to the primary arithmetic. (Quaternions condense to the 4-group.) The next section examines wave-trains and self-reference.

5. *Wave Trains and Self-Referential Form*

The equation $F = \overline{F}|$ has no solution in the primary arithmetic. The attempt to solve it leads to a (paradoxical?) oscillation of values:

$$F = \urcorner \Rightarrow F = \overline{F}| = \overline{\urcorner}| \Rightarrow F = \overline{F}| = \overline{\overline{\urcorner}|}| = \urcorner \Rightarrow F = \overline{\urcorner}| \Rightarrow F = \urcorner \Rightarrow \cdots$$

$$\cdots \quad \urcorner \quad \overline{\urcorner}| \quad \urcorner \quad \overline{\urcorner}| \quad \urcorner \quad \overline{\urcorner}| \quad \urcorner \quad \overline{\urcorner}| \quad \urcorner \quad \cdots$$



The paradox is its own solution. View the wave-train either as a repetition of $\urcorner\ \overline{\urcorner}|$ or as a repetition of $\overline{\urcorner}|\ \urcorner$. Let [a,b] denote the pair a,b with specified order.

$$I = [\urcorner, \overline{\urcorner}|\,] \text{ and } J = [\overline{\urcorner}|\,, \urcorner]$$

denote the two views of the pattern. In general, let [a,b] denote one view of ...abababababa... and define $[a,b][c,d] = [ac\,,\,bd]$, $\overline{[a\,,\,b]}| = [\overline{b}|\,,\,\overline{a}|\,]$.

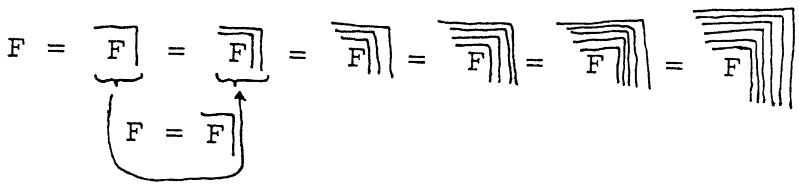
Then

$$IJ = [\urcorner, \overline{\urcorner}|\,][\overline{\urcorner}|\,, \urcorner] = [\urcorner\overline{\urcorner}|\,, \overline{\urcorner}|\urcorner] = [\urcorner, \urcorner] = \urcorner,$$

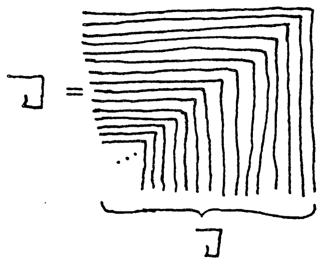
$$\overline{I}| = \overline{[\urcorner, \overline{\urcorner}|\,]}| = [\overline{\overline{\urcorner}|}|\,, \overline{\urcorner}|\,] = [\urcorner, \overline{\urcorner}|\,] = I \text{ and } \overline{J}| = J.$$

This creates a waveform arithmetic in which $F = \overline{F}|$ has solutions. Note that the product adopted here is different from idemposition (as in section 4). In section 7 a similar procedure will produce the complex numbers. Imaginary numbers and logical values beyond True and False have the same underlying form. [6,7]

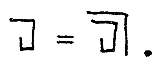
The wave is not the only way to understand $F = \overline{F|}$. Take this equation as a prescription to *re-enter* F into its own indicative space:



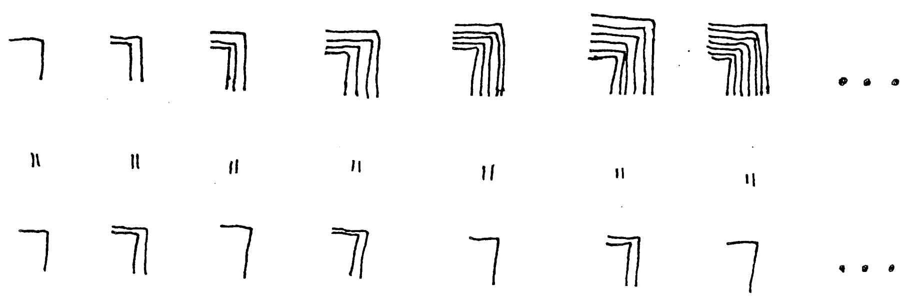
In the limit there is an infinite descending chain of crosses:

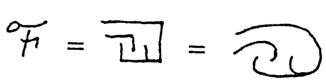


The chain is denoted by the *re-entering mark*  (or ),[8] since it literally contains a copy of itself.

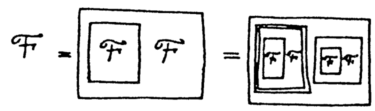


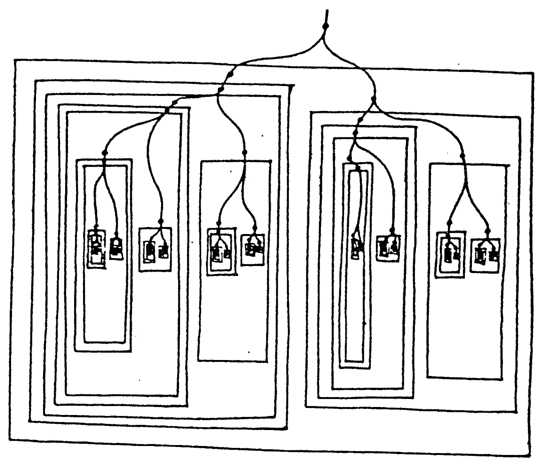
Of  we cannot say that is is marked or that it is unmarked. It is itself. In the form of its construction the waveform is regenerated.

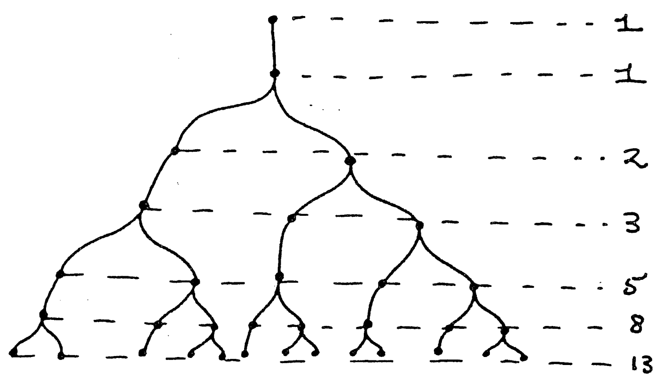


More complex self-referential forms appear in the same way. For example, let  .

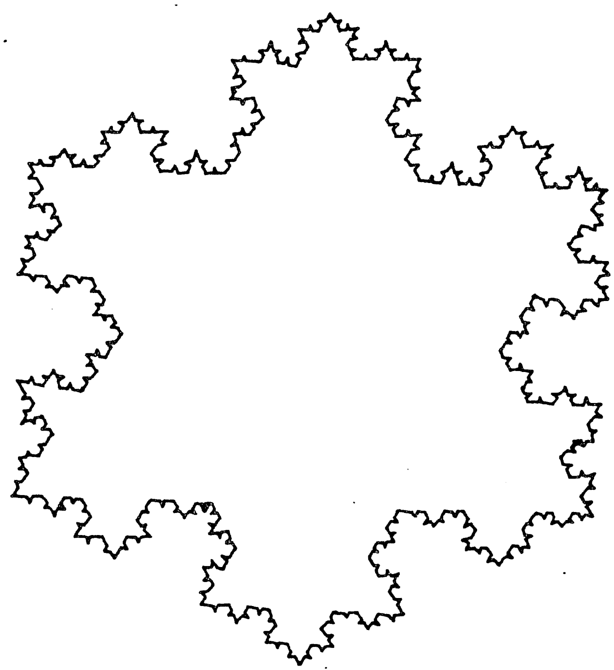
This is the *Fibonacci Form.* The number of divisions of $\mathcal{F}$ at depth n is the n^{th} Fibonacci number f_n: $f_{n+1} = f_n + f_{n-1}$. $f_o = f_1 = 1$. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,...







Unfolded in this way, self-referential form becomes infinite self-similar geometric form -- the precursor to space-filling curves, fractals,[9] and to the multitude of common forms (coastlines, mountain ranges, branching systems of trees and rivers, the human circulatory system,...) that exhibit self-similarity at different scales of observation. (Fractal is a term coined by Benoit Mandelbrot.[9] A fractal is a geometric form of fractional (!) dimension. Such forms are often self-similar as in the "snowflake" depicted below. It is convenient to use the word fractal when referring to infinite self-similar geometric form.)

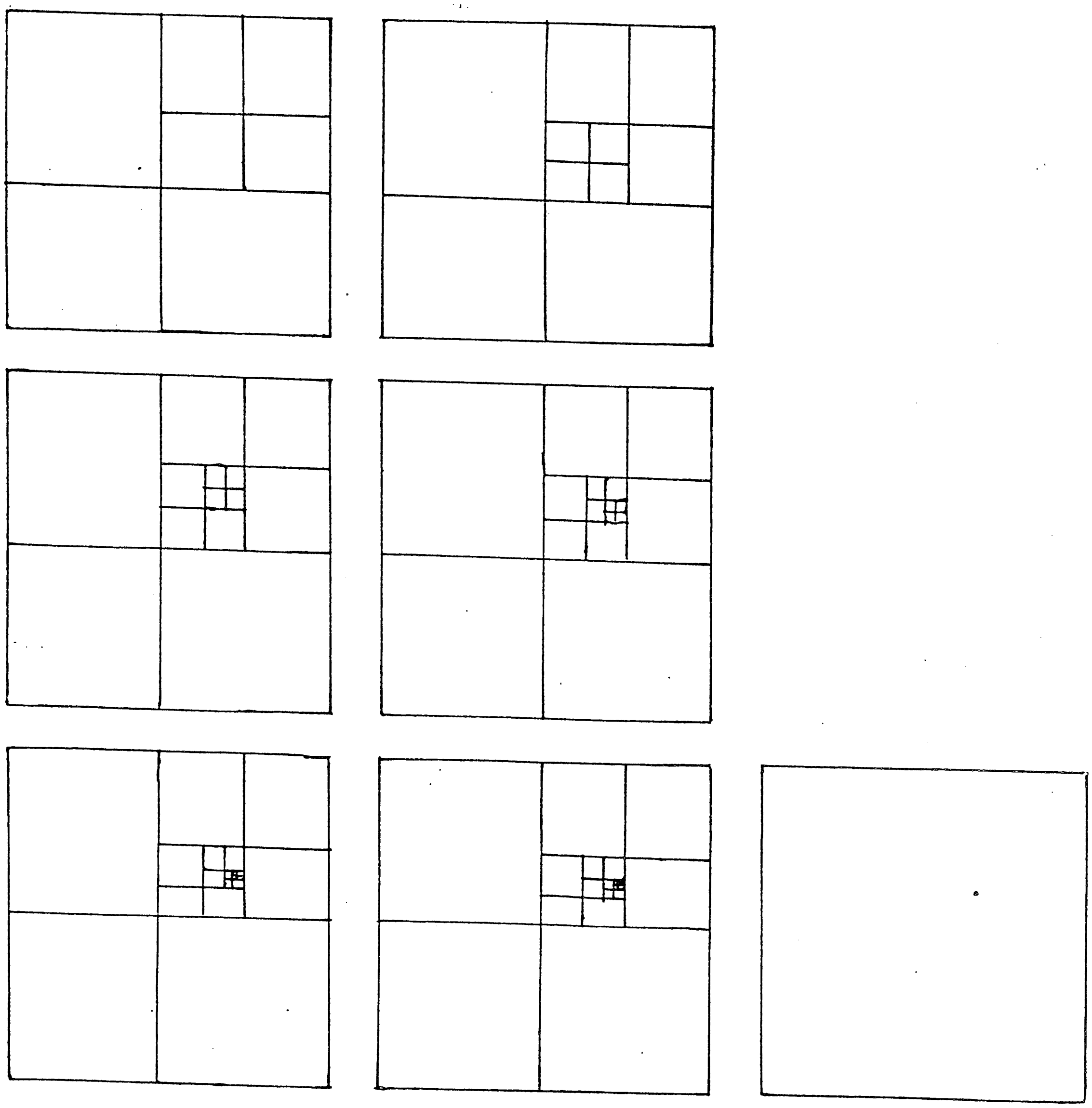


6. *Real Numbers*

The real numbers R consist of calculating forms like 11.379821... , an integer followed by a (possibly) infinite decimal expansion. Real numbers (and more generally *points* in any space) are shorthand for procedures to make a decision, establish a location. Think of points in the Cartesian plane R x R. You can specify a specific point by a sequence of choices of four values

2	1
3	4

in a concatenating series of grids.



7. *Complex Numbers*

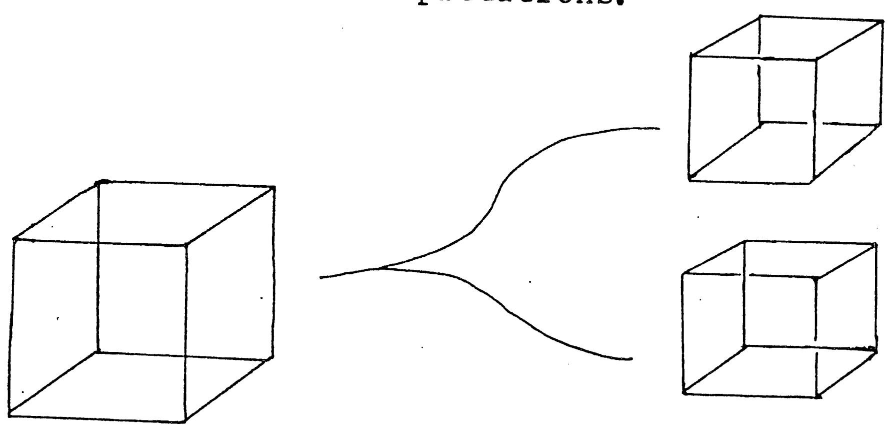
Consider a period pattern of period two.

... ABABABA ...

[A,B]   [B,A]

This can be seen as a repetition of AB or as a repetition of BA. Let [A, B] and [B, A] denote these two viewpoints of the waveform. Let $\overline{[A, B]} = [B, A]$ and call this reversal operation *conjugation*.

The A's and the B's could be anything at all. In general we consider a ground-form that is susceptible to multiple interpretation. In this sense the situation is quite analogous to the Necker cube illusion -- a drawing of a cube with two three-dimensional interpretations.



Now suppose that A and B represent real numbers. Then it is natural to add and multiply waveforms by combining corresponding terms. (If these are temporal sequences, then we are combining parts of the wave that appear at the same time.)

W = ... A B A B A B A B A ...

T = ... C D C D C D C D C ...

$W * T = \ldots\ AC\ \ BD\ \ AC\ \ BD\ \ AC\ \ BD\ \ AC\ \ BD\ \ AC \ldots$

$W + T = \ldots\ A{+}C\ \ B{+}D\ \ A{+}C\ \ B{+}D\ \ A{+}C\ \ B{+}D\ \ A{+}C\ \ B{+}D\ \ A{+}C \ldots$

$[A,B] * [C,D] = [AC,BD],$

$[A,B] + [C,D] = [A + C, B + D].$

We identify A with [A,A] and set

$$A[C,D] = A * [C,D] = [A,A] * [C,D] = [AC,AD].$$

Let $\mathbb{D}$ denote the set of pairs of real numbers with this structure. $\mathbb{D}$ will be called the *counter-complex numbers*. Let $i = [+1, -1]$. Every counter-complex number is of the form $a + bi$ for real numbers a and b.

$$a + bi = [a, a] + b[1, -1],$$

$$\therefore a + bi = [a + b, a - b].$$

Thus $a + bi$ denotes an oscillation between $a + b$ and $a - b$. We say that $[a, a]$ is the *real part* of $a + bi$ and that $[b, -b]$ is the *imaginary part*. Thus

$$R(\alpha) = \tfrac{1}{2}(\alpha + \overline{\alpha}) = \text{real part } \alpha,$$

$$I(\alpha) = \tfrac{1}{2}(\alpha - \overline{\alpha}) = \text{imaginary part } \alpha.$$

The basic imaginary waveform i satisfies $i * i = +1$. In the complex numbers $\mathbb{C} = \{a + bi \mid a,b \text{ real}\}$, i is characterized by $ii = -1$. What is the relationship between $\mathbb{D}$ and $\mathbb{C}$? By defining a new multiplication on $\mathbb{D}$ in terms of the old multiplication $*$, we can obtain the structure of the complex numbers: Let the new product of α and β be denoted $\alpha\beta$. Then

$$\alpha\beta = R(\alpha) * \beta + I(\alpha) * \overline{\beta}.$$

Thus $ii = 0*i + i*\overline{i} = i*\overline{i} = -1$, and

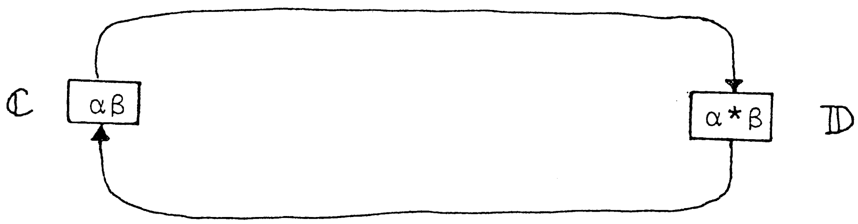
$$(a + bi)(c + di) = a * (c + di) + bi * (c - di)$$
$$= ac + adi + bci - bdi * i,$$
$$\therefore (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

It is easy to verify that

$$[\mathbb{D} \text{ defines } \mathbb{C}]\ \alpha\beta = \frac{1}{2}(\alpha * \beta + \bar{\alpha} * \beta + \alpha * \bar{\beta} - \bar{\alpha} * \bar{\beta}),$$

$$[\mathbb{C} \text{ defines } \mathbb{D}]\ \alpha * \beta = \frac{1}{2}(\alpha\beta + \bar{\alpha}\beta + \alpha\bar{\beta} - \overline{\alpha\beta}).$$

Hence $\mathbb{C}$ and $\mathbb{D}$ are mirror images in respect of their multiplicative structure. Each defines the other in the same pattern.



Spacetime Structure

From this mirror-imaging reference the structure of Minkowski spacetime is born. From the outside this is the personal myth of forms breaking forth into duality only to return once again into void and dissolution in an endless round. The poles of each duality support each other, contain each other, and ultimately, *in the place of light* are seen to bear no difference. By choosing a symbolism to describe a difference that may be no difference, the mathematical structure will emerge.

Let $M = \mathbb{D} \times \mathbb{C}$. This is a four-dimensional space. Define an interval $I: M \longrightarrow \mathbb{R}$ by the formula $I(\alpha, \beta) = \alpha * \bar{\alpha} - \beta\bar{\beta}$ (see note 6). Let $L = \{(\alpha, \beta) \in M \mid I(\alpha, \beta) = 0\}$. Since L is that locus in the four-space where the evaluations in the separate mirror planes agree, it will be called the place of agreement, the image of the place of light.

From the inside, this is the formalism of Minkowski spacetime , for spacetime is given by coordinates (T, X, Y, Z) (time, space, space, space) and with the speed of light set equal to 1 for convenience, the spacetime interval is given by $T^2-X^2-Y^2-Z^2$, a quantity independent of the observer (for observers moving at constant velocity with respect to one another). The *light-cone* is the locus $T^2-X^2-Y^2-Z^2 = 0$. Represent spacetime coordinates in M by the scheme $(\alpha, \beta) = (T + iX, Y + iZ)$. Then $I(\alpha, \beta) = T^2-X^2-Y^2-Z^2$. Hence the light-cone corresponds to that place of light. Macroscopic geometry is governed by the wavelike structure of conceptual microspace. Waves of pattern crystallizing the void.

8. *Complex Numbers as Matrices*

By a (2 x 2) matrix we mean an array

$$\begin{bmatrix} a & c \\ d & b \end{bmatrix}.$$

Matrices with numerical entries are commonly added and multiplied by the rules

$$\begin{bmatrix} a & c \\ d & b \end{bmatrix} + \begin{bmatrix} e & g \\ h & f \end{bmatrix} = \begin{bmatrix} a + e & c + g \\ d + h & b + f \end{bmatrix},$$

$$\begin{bmatrix} a & c \\ d & b \end{bmatrix} \begin{bmatrix} e & g \\ h & f \end{bmatrix} = \begin{bmatrix} ae + ch & ag + cf \\ de + bh & dg + bf \end{bmatrix},$$

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} \quad \text{(K a constant)}$$

$$\therefore K \begin{bmatrix} a & c \\ d & b \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} a & c \\ d & b \end{bmatrix} = \begin{bmatrix} Ka & Kc \\ Kd & Kb \end{bmatrix}.$$

Observe that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1.$$

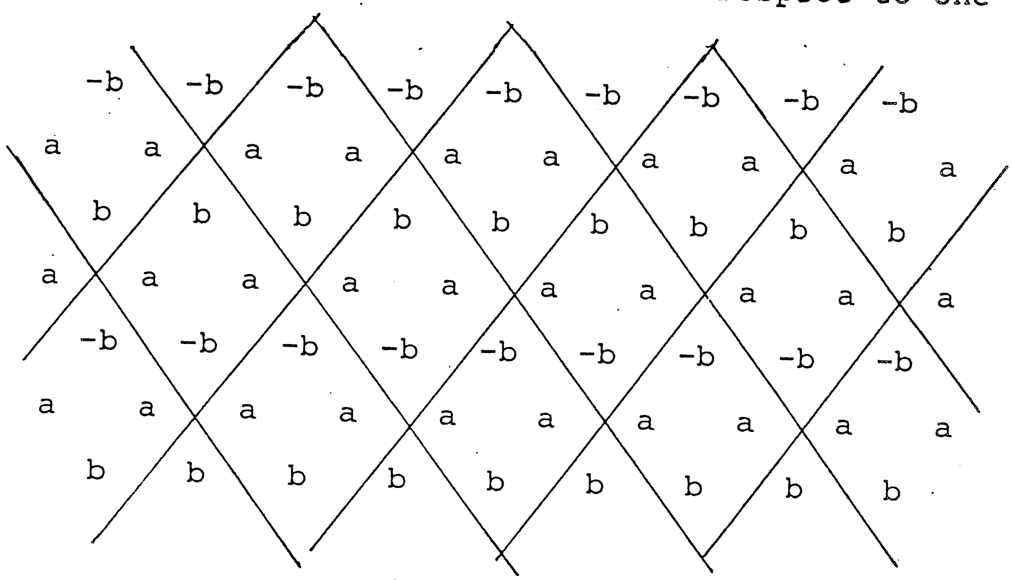
Hence one can write

$$i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and complex numbers are represented in the form

$$a + bi = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Compare this matrix representation with the waveform presentation of section 7. In the matrix the real and imaginary waveforms are presented separately and rotated 90° with respect to one another!



Each complex number creates its own checkerboard plane. Moreover, *there is a formal similarity between matrix multiplicaton and wave-combination as given by the formula* $\alpha\beta = R(\alpha) * \beta + I(\alpha) * \overline{\beta}$. Observe:

$$M\gamma = \begin{bmatrix} a & c \\ d & b \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ae + cf \\ de + bf \end{bmatrix}$$

$$= \begin{bmatrix} ae \\ bf \end{bmatrix} + \begin{bmatrix} cf \\ de \end{bmatrix}$$

$$= \begin{bmatrix} a \\ b \end{bmatrix} * \begin{bmatrix} e \\ f \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} * \overline{\begin{bmatrix} e \\ f \end{bmatrix}}$$

$$\left(\overline{\begin{bmatrix} e \\ f \end{bmatrix}} = \begin{bmatrix} f \\ e \end{bmatrix} \right).$$

By defining the real and imaginary parts of the matrix

$$\text{Real}(M) = R(M) = R \begin{bmatrix} a & c \\ d & b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix},$$

$$\text{Imag}(M) = I(M) = I \begin{bmatrix} a & c \\ d & b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix},$$

we have $M\gamma = R(M) * \gamma + I(M) * \overline{\gamma}$. Matrix multiplication conforms to the same pattern as wave combination.

We shall write $M = R(M) + \sqrt{\ } I(M)$,

$$\begin{bmatrix} a & c \\ d & b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} + \sqrt{\ } \begin{bmatrix} c \\ d \end{bmatrix},$$

where $\sqrt{\ }$ is a formal symbol designating the imaginary part of

the matrix. Let * act linearly on the formal sum and define
$\overline{\beta + \sqrt{\ }\,\gamma} = \overline{\gamma} + \sqrt{\ }\,\overline{\beta}$.

Then

$$\overline{\begin{bmatrix} a & c \\ d & b \end{bmatrix}} = \overline{\begin{bmatrix} a \\ b \end{bmatrix} + \sqrt{\ }\begin{bmatrix} c \\ d \end{bmatrix}} = \overline{\begin{bmatrix} c \\ d \end{bmatrix}} + \sqrt{\ }\,\overline{\begin{bmatrix} a \\ b \end{bmatrix}} = \begin{bmatrix} d & b \\ a & c \end{bmatrix}.$$

Proposition 8.1. Let M and N be 2 x 2 matrices. Then their matrix product is given by the formula

$$MN = R(M) * N + I(M) * \overline{N}.$$

Proof. Let

$$M = \begin{bmatrix} a & c \\ d & b \end{bmatrix}, \quad N = \begin{bmatrix} e & g \\ h & f \end{bmatrix}.$$

Then

$$R(M) * N + I(M) * \overline{N} = \begin{bmatrix} a \\ b \end{bmatrix} * \begin{bmatrix} e & g \\ h & f \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} * \begin{bmatrix} h & f \\ e & g \end{bmatrix}$$

$$= \begin{bmatrix} ae & ag \\ bh & bf \end{bmatrix} + \begin{bmatrix} ch & cf \\ de & dg \end{bmatrix} = \begin{bmatrix} ae + ch & ag + cf \\ bh + de & bh + dg \end{bmatrix}$$

$$= MN.$$

The formal similarity between matrix multiplication and waveform combination ties the usual mathematics of matrices into the web of pattern.

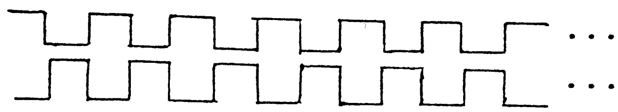
9. *The Wave-Structure of an Event*

Right now is an event. Observer and observed occur, each supporting the other. Their difference is imaginary. The observer *is* the observed. Time is *now* and this happens in its presentness.

The primitive event of self-looking, self-producing-self-producing-self-producing-self-producing- ... is symbolized by the reentering mark, ⅃ , and its associated waveforms

I = ...

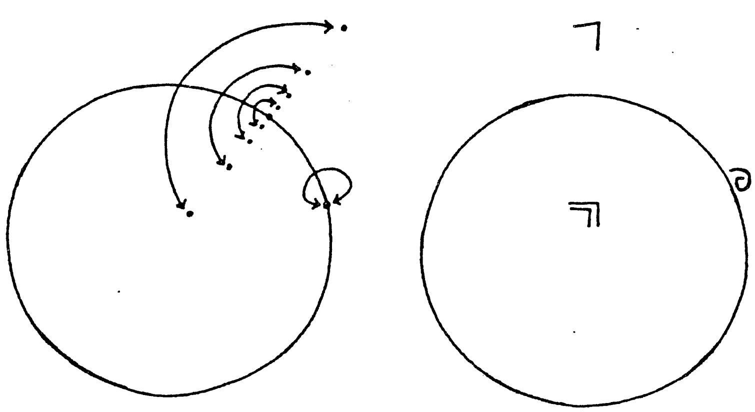
J = ...



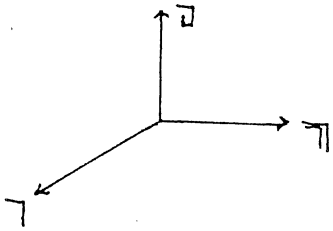
with IJ = ⏋. Taken singly, the waveforms represent the observer's oscillation I - not I - I - not I - I - not I - .... Together they form a conversation and a taking of turns. Taken individually the two waveforms are indistinguishable. When together, the *phase difference* suffices to tell them apart.

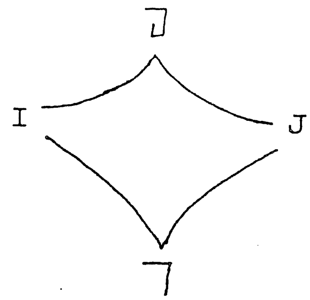
The closing of the circuit, the inturning gesture, the act of oscillation are all forms of the establishment of boundary. This first circularity ⅃ , circular return, is not quite a boundary *in* anything. But with it come the remarkable simulacra we call scenes, spaces, sides, pattern and place.

For a geometrical sense of ⅃ as boundary, think of crossing as an act of reflection. The boundary then takes the role of mirror, and is invariant under crossing: ⅃⏋ = ⅃.

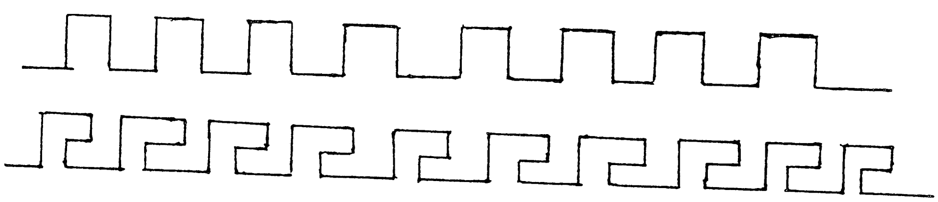


Each distinction gives rise to a trinity: ⏋ , ⏋| , ⊔ . Inversely, the reentering mark as imaginary value gives rise to the (individually indistinguishable) waveforms I and J that distinguish one another and hence can be distinguished.





The symbols have deep roots.



Spirals, gestures of reference, repeating pattern.

Frieze patterns that decorate walls.

Walls and symbols for walls.

Waveforms delineating boundaries that separate imaginary space.

Spaces of the imagination.

Inside and outside dancing out of an

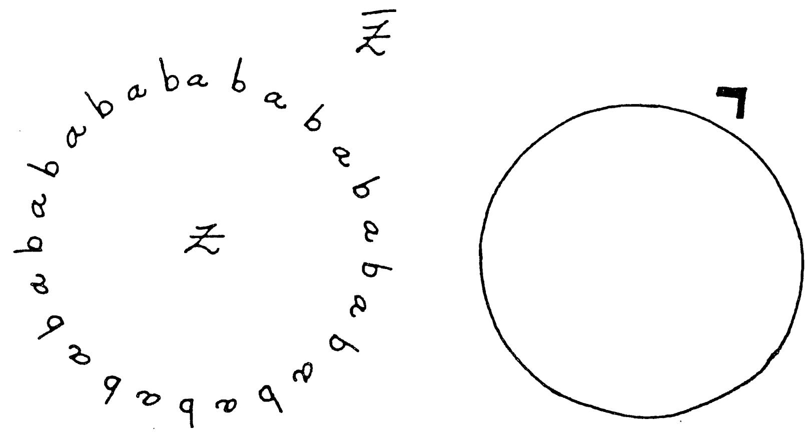
Ever-vanishing I.

$$E = \ldots \overset{z}{\underset{z}{ab}}\,\overset{z}{\underset{z}{ab}}\,\overset{z}{\underset{z}{ab}}\,\overset{z}{\underset{z}{ab}}\,\overset{z}{\underset{z}{ab}}\,\overset{z}{\underset{z}{ab}}\,\overset{z}{\underset{z}{ab}}\,\overset{z}{\underset{z}{ab}}\,\overset{z}{\underset{z}{ab}}\,\overset{z}{\underset{z}{ab}}\,\overset{z}{\underset{z}{ab}}\,\overset{z}{\underset{z}{ab}} \ldots$$

The real waveform becomes boundary between sides labelled z, $\bar{z}$.

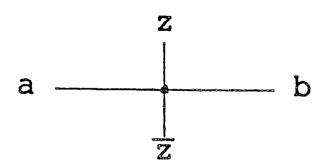
$$\frac{z}{\bar{z}}$$

Temporality is the tempo of pattern.



The complex part of the event-wave occurs at 90° to the real part forming a matrix or crossing.

$$E = \begin{bmatrix} a & z \\ \bar{z} & b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} + \sqrt{}\begin{bmatrix} z \\ \bar{z} \end{bmatrix},$$



$$E = \begin{array}{ccccccccc} \ddots & & & & & & & & \\ & a & z & & & & & & \\ & \bar{z} & b & & & & & & \\ & & & a & z & & & & \\ & & & \bar{z} & b & & & & \\ & & & & & a & z & & \\ & & & & & \bar{z} & b & & \\ & & & & & & & a & z \\ & & & & & & & \bar{z} & b \\ & & & & & & & & \ddots \end{array}$$

Matrix represents view, window, frame, freeze of event. Hence the matrix is the form of *operator*. One waveform expanded ... ababaca ... . The other folded to the two sides $z|\bar{z}$. Folding both waveforms, we obtain

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} [a,b] \\ z \end{bmatrix} \in \mathbb{D} \times \mathbb{C},$$

a point in Minkowski spacetime (see the end of section 7). In time-space coordinates the correspondence is as follows:

$$(T, X, Y, Z) \leftrightarrow \begin{bmatrix} T + iX \\ Y + iZ \end{bmatrix} \in \mathbb{D} \times \mathbb{C}$$

$$\updownarrow$$

$$\begin{bmatrix} T + X & Y + iZ \\ Y - iZ & T - X \end{bmatrix} = \begin{bmatrix} a & z \\ z & b \end{bmatrix} = E\ .$$

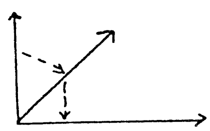
The determinant of the hermitian matrix E gives the spacetime interval. At this point we make contact not only with relativity, but also with quantum mechanics, for hermitian matrices (E above) are quantum mechanical operators. They correspond to observations of a two-state system.

Digression. Quantum Mechanics of a Two-State System

A system that has two ostensible physical states (such as a particle with two spin states) is represented mathematically by a continuum of superpositions of these states with relative complex probability amplitudes. Thus if the two states are labelled U and D, then one considers superpositions $e = zU + wD$, where z and w are complex numbers with $|z|^2 + |w|^2 = 1$ $(|z| = z\bar{z})$. Observation is modelled by the algebraic situation $He = \lambda e$, where

λ is a real number, e is a state superposition, and the operator H corresponds to some observable quantity. In order to obtain real λ (the measurement), the matrix H must be hermitian; thus hermitian matrices correspond not only to points of spacetime, but also to the observables of a two-state system!

Some sense of the logic of this model is obtained by considering an experiment with light-polarizing material. Two polarizing filters rotated at 90° with one another will not pass any light. But if a third filter is placed between them at an angle of 45°, then light will pas through the triplet. Thus an appropriate mathematical model for the filtering process must be something like projection of one direction upon another direction. Two directions at 90° to one another give a null projection, while two sucessive 45° projections can be non-null.



The quantum mechanical model needs complex probabilities in order to explain interference phenomena.

Lemma 9.1. Let

$$E = \begin{bmatrix} a & z \\ \bar{z} & b \end{bmatrix},$$

then $Ee = \lambda e$ has solutions e if and only if $\lambda = \lambda_+$ or λ_-, where

$$\lambda_+ = \left(\frac{a+b}{2}\right) + \sqrt{\left(\frac{a-b}{2}\right)^2 + z\bar{z}}\ ,$$

$$\lambda_- = \left(\frac{a+b}{2}\right) - \sqrt{\left(\frac{a-b}{2}\right)^2 + z\bar{z}}\ .$$

Hence if

$$E = \begin{bmatrix} T + Y & Y + iZ \\ Y - iZ & T - X \end{bmatrix},$$

then

$$\lambda = T \pm \sqrt{X^2 + Y^2 + Z^2}$$

and

$$T = \frac{1}{2}(\lambda_+ + \lambda_-) \ , \quad \sqrt{X^2 + Y^2 + Z^2} = \frac{1}{2}(\lambda_+ - \lambda_-).$$

Hence the event E, viewed as a quantum mechanical operator, can observe its own time and length.

Proof. $Ee = \lambda e \iff (E - \lambda)e = 0.$

This has solutions if and only if the determinant of $(E - \lambda)$ is zero. Determinant $(E - \lambda) = \lambda^2 - (a + b)\lambda + (ab - z\bar{z})$.

$$\lambda^2 - (a + b)\lambda + (ab - z\bar{z}) = 0 \iff \lambda = \lambda_+ \text{ or } \lambda_-$$

as above.

Observation involves the split into event as operator E, and event as operand e. We should, at least, consider what may lie below such a split. Then there are only events. But not all products will remain in the same pattern. That is, the product of two hernitian matrices is not always hermitian. However, powers of a hermitian matrix are hermitian, and powers of an event on the light-cone remain on the light-cone.

Recall that

$$E = \begin{bmatrix} T + X & Y + iZ \\ Y - iZ & T - X \end{bmatrix}$$

is said to be on the light-cone if $T^2 = X^2 + Y^2 + Z^2$. Let E be denoted by the pair $(T, \vec{R})$ where $\vec{R} = (X, Y, Z)$.

Proposition 9.2. Let $E = (T, \vec{R})$ be a point on the light-cone. Then $E^{n+1} = \left((2T)^{g(n)}T, (2T)^{g(n)}\vec{R} \right)$, where $g(n) = f_n - 1$ and f_n is the n^{th} Fibonacci number.

Proof. It is easy to verify that

$$\begin{bmatrix} T + X & Y + iZ \\ Y - iZ & T - X \end{bmatrix} \begin{bmatrix} T' + X' & Y' + iZ' \\ Y' - iZ' & T'' - X' \end{bmatrix} = \begin{bmatrix} T'' + X'' & Y'' + iZ'' \\ Y'' - iZ'' & T'' - X'' \end{bmatrix},$$

where

$$\begin{aligned} T'' &= TT' + XX' + YY' + ZZ' \pm i\,(ZY' - YZ'), \\ X'' &= TX' + XT', \\ Y'' &= YT' + TY' + (XY' - YX'), \\ Z'' &= ZT' + TZ' + (XZ' - ZX'). \end{aligned}$$

In the case at hand we have $ZY' - YZ' = XY' - YX' = XZ' - ZX' = 0$. Hence $T'' = TT' + \vec{R} \cdot \vec{R}'$, $\vec{R}'' = T\vec{R}' + T'\vec{R}$, where $(a,b,c)(d,e,f) = (ad,be,cf)$ is the usual inner product, and $t(a,b,c) = (ta,tb,tc)$ is scalar multiplication. Futhermore , when (T', R') is a power of E, then $\vec{R} \cdot \vec{R} = TT'$. Hence, $T'' = 2TT'$, $\vec{R}'' = T\vec{R}' + T'\vec{R}$. The result now follows by induction.

Expanding the event.

Given that $z = c + di$ corresponds to the matrix

$$\begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

we can expand the event thus:

$$E = \begin{bmatrix} a & z \\ z & b \end{bmatrix} = \begin{bmatrix} a & 0 & c & d \\ 0 & a & -d & c \\ c & -d & b & 0 \\ d & c & 0 & b \end{bmatrix}.$$

Now the real wave form acts precisely as notational mirror for the two sides. Each side $z, \overline{z}$ is seen to be *itself* an event, and there is a hint of the recursive concatenating formal structure that will continue to decompose closer and closer views of the event structure into more copies of itself. The event becomes fractal form.

Quaternions Again.

$$\begin{bmatrix} T + X & Y + iZ \\ Y - iZ & T - X \end{bmatrix} = T + X\sigma_X + Y\sigma_Y + Z\sigma_Z ,$$

$$\sigma_X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \quad \sigma_Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma_Z = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} .$$

These are called the Pauli Spin Matrices.[10] They satisfy the identities

$$\sigma_X{}^2 = \sigma_Y{}^2 = \sigma_Z{}^2 = 1,$$

$$\sigma_X\sigma_Y = -i\sigma_Z ,$$

$$\sigma_Y\sigma_Z = -i\sigma_X ,$$

$$\sigma_Z\sigma_X = -i\sigma_Y .$$

Hence, letting

$$I = i\sigma_X ,$$

$$J = i\sigma_Y ,$$

$$K = i\sigma_Z$$

the quaternions appear once again.

Pattern of Observation.

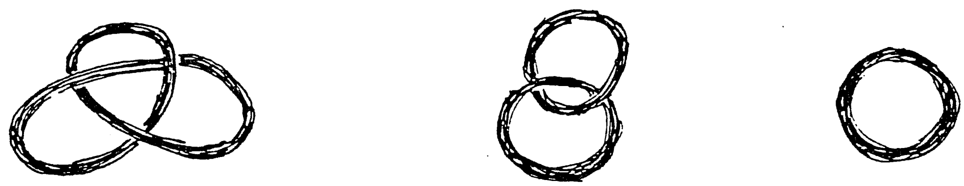
We have seen that Ee = λe is the physicist's model for observation. E is the operator and stands for an observer. It operates on the vector e, changing it. The "ratio" Ee : e is λ, a real number, the measurement.

This model should be compared with the formal stabilizations such as ⅃| = ⅃. Here the operator, ⌉ , by acting upon itself, ⌉⌉, *creates* the operand. It also gives rise to the act of measurement, when it is seen as separate from the process ⌉, ⌉⌉, ⌉⌉⌉ , ... that builds ⅃. Our discussion of ⅃ is purely formal, without the scaling of numerical measurement.

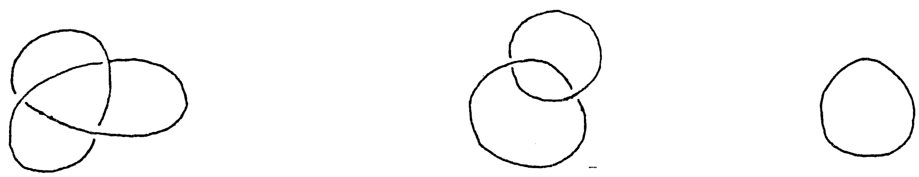
It is possible to produce examples that exhibit various mixtures of the formal and the numerical. Investigation along these lines should help to clarify the meaning and extent of applicability of the quantum mechanical ($Ee = \lambda e$) model.

10. *Projection - Coding - Description*

I want to discuss wholes and parts. We start with the notion of a knot or a link in apparent three-dimensional space.



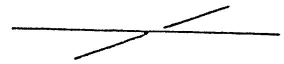
Knot Link Unknot

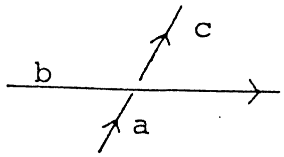


Schematic Projections

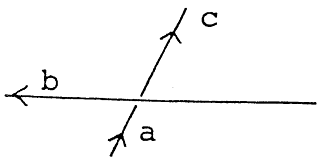
A knot is an endless loop of rope embedded in the three-space. It is knotted if there is no spacial deformation, short of tearing the rope, that transforms the knot to the unknot (a simple loop). A link consists of many loops, possibly entangled with one another.

A knot is a fine example of a whole form. It has no particular (special) decomposition into parts. The knot's properties are determined through its (contextual) relation to the space in which it is embedded.

Nevertheless, any picture or projection of the knot invariably cuts it up into a group of interrelated pieces. In a schematic projection of the knot, these pieces are arcs in the plane, meeting at the crossings:  . The interrelationship of the parts may then be indicated by labelling each arc, and by describing the situation at each crossing:

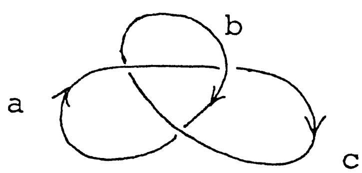


$$c = a \triangleright b$$



$$c = a \,\bar{\triangleright}\, b$$

[c is obtained by crossing b from a (see note 11).] A *code* is then obtained that describes the knot.



$$\left.\begin{array}{l} c = a \triangleright b \\ b = c \triangleright a \\ a = b \triangleright c \end{array}\right\} \text{Code}$$

The code (description) is self-referential. Each term is defined via the other terms. Self-referential linguistic form codes circularly-interconnected spatial form.

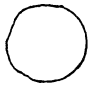
Many different projections (hence many different decompositions into parts, and many different codes) may represent the same knot up to topological equivalence. Topological invariants (invariants of topological form) are obtained by examining how the descriptions change under spatial deformation of the knot.

To return to a sense of the whole from the parts requires an integration that is somehow independent of the particular decomposition. This requires blindness (but systematic blindness!) to certain details. The parts must condense into the whole.

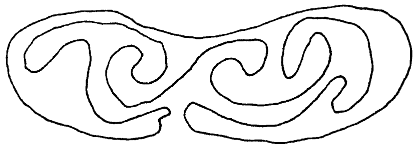
Sign and Space -- A Summary

1. *The form we take to exist arises from framing nothing.*

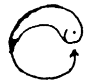
2. *A circle makes a distinction in the plane.*



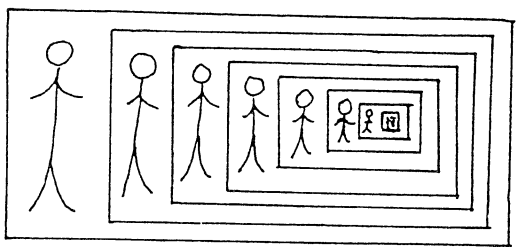
3. *You could get lost!*



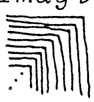
4. *The snake eats its tail.*



5. *The form reenters its own indicational space.*



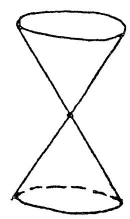
6. *Imaginary value.*

 $= \rceil \leftrightarrow \sqrt{-1}$

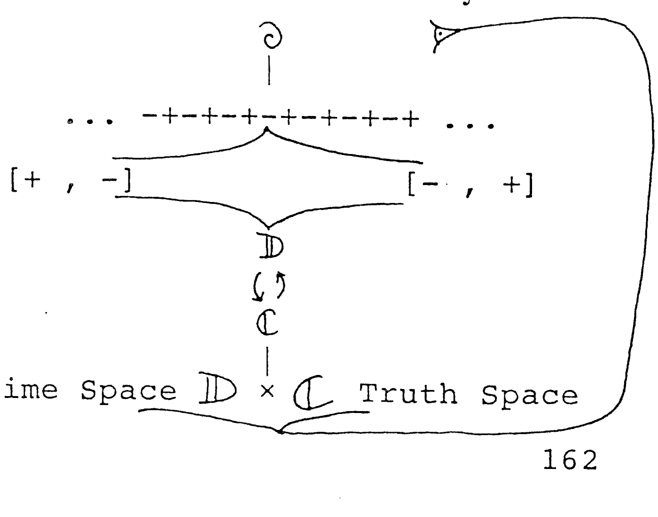
7. *Agreement is the place of light.*

$$\alpha = T + \sqrt{-1}\,X\,, \quad \beta = Y + \sqrt{-1}\,Z$$

$$\Rightarrow \alpha * \bar{\alpha} - \beta\bar{\beta} = T^2 - X^2 - Y^2 - Z^2.$$



8. *The event observes itself.*



$$\sqrt{-1}^{\sqrt{-1}} = e^{-\pi/2}$$

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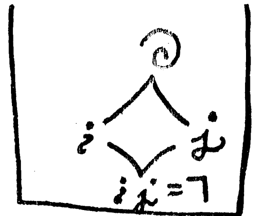
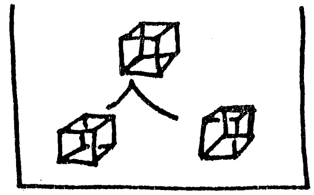
$\boxed{\sqrt{-1}}$

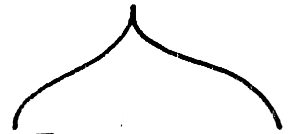




Constructing Space-Time

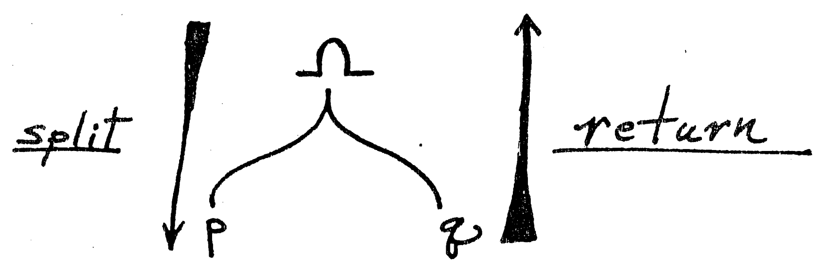
The filter undergoes process of
Evolution
Mitigated by observations upon itself.
Procession of levels of
Ordering and re-ordering
Create
Complex web of differentiations.
A prototype: the UR-sound.

$\sqcap$: ···0101010101010···



p = [1,0] [0,1] = q

Two complementary views.



Process is observed.

$$\boxed{p - q = i}$$

The difference is imaginary

$p = [1,0]$, $q = [0,1]$

$p - q = [1,-1] = i$

A second splitting.
Structural (left) versus
Metaphorical (right).
And then a
Weaving of
Metaphorical interpretive
Strands with
Literal formal strands.
Interwoven nexus.

$$p - q = i$$

$$[a,b] - [b,a] = (a-b)i \quad .$$