Take Home Exam - Math 310 - Linear Algebra - Spring 2008

1. Let $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \end{bmatrix}$. Consider the system $A[x_1, x_2, x_3, x_4]^T = [a, b, c]^T$.

   (a) Use row reduction to determine the general solution for this system. For what values of a, b, c do there exist solutions to this system of equations? Give the vector form of the solution to

   $$A[x_1, x_2, x_3, x_4]^T = [1, 1, 1]^T.$$

   Find a basis for the null space $N(A)$.

   (b) Using your work in part (a), give the row reduced echelon form R for the matrix A. What is the rank of A? What is the dimension of the column space of A? Give a basis for the column space of A that is a subset of the columns of A.

   (c) Let $Col(A)$ denote the column space of A. Find a basis for $Col(A)^{\perp}$. (Hint: Use the basis for $Col(A)$ that you found in part (b).)

2. (a) Let

   $$M = \begin{bmatrix} t & 1 \\ 0 & t \end{bmatrix}.$$

   Prove by induction that

   $$M^n = \begin{bmatrix} t^n & nt^{n-1} \\ 0 & t^n \end{bmatrix}.$$

   Thinking of each entry of M^n as a function of t, show that

   $$d(M^n)/dt = nM^{n-1}$$

   for $n \geq 1$ where $M^0 = I$ denotes the 2×2 identity matrix.

   (b) Consider the oriented graph G whose incidence matrix is $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. This means that there are two vertices in the graph, labeled 1 and 2. There is an oriented edge from 1 to 2 and each vertex has two oriented loops from itself, to itself. Draw a diagram of the graph G. How many oriented paths of length 137 are there in G starting at vertex 1 and ending at vertex 2?

3. Let A be an $n \times n$ matrix. Define the trace of A by the formula $tr(A) = \Sigma_{i=1}^{n} a_{ii}$. That is, the trace of a matrix is the sum of the diagonal entries of the matrix. Recall that for $n \times n$ matrices A and B, $tr(AB) = tr(BA)$.

(a) Prove that for two $n \times n$ matrices A and B, $tr(A+B) = tr(A)+tr(B)$.

(b) Prove that for any $n \times n$ matrix A, $tr(A) = tr(A^T)$.

(c) Show that there do not exist $n \times n$ $(n \neq 0)$matrices A and B such that $AB - BA = I_n$ where I_n is the $n \times n$ identity matrix. (Hint, take the trace of the left-hand side and take the trace of the right-hand side. Show that they cannot be equal.)

4. Let $S = [1/\sqrt{2}, cos(x), cos(2x), \cdots, cos(nx), sin(x), sin(2x), \cdots, sin(nx)]$. We know that S is an orthonormal set of vectors in $C[-\pi, \pi]$ with inner product defined by $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$. Thus S can be taken as the basis of its span $W = Span(S)$.

   (a) What is the dimension of the subspace W (defined above) of $C[-\pi, \pi]$?

   (b) Determine the best least squares approximation to $h(x) = x$ by a function from the subspace W. Hint: You will need to know the integrals $\int xsin(kx)dx$ and $\int xcos(kx)dx$. You can look these up, or do them by integration by parts.

5. In this problem and the subsequent problems we are concerned with the following question: Given an $n \times n$ matrix A, does there exist a non-zero vector $v \in R^n$ such that $Av = \lambda v$ for some real number λ? If

$$Av = \lambda v$$

   then

$$Av - \lambda v = 0.$$

   Hence

$$Av - \lambda Iv = 0$$

   where I is the $n \times n$ identity matrix. Hence

$$(A - \lambda I)v = 0.$$

   In order for this system to have a non-zero solution v (for some choice of value for λ) we need that the determinant

$$Det(A - \lambda I) = 0.$$

   We say that

$$C_A(x) = Det(A - xI)$$

   is the *characteristic polynomial* for A and we call solutions λ to the equation $C_A(x) = 0$ the *eigenvalues* of the matrix A. A non-zero vector such that $Av = \lambda v$ is said to be an *eigenvector* of A belonging to the eigenvalue λ.

(a) Let $L : R^2 \longrightarrow R^2$ be a linear transformation whose matrix in the standard basis is

$$A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}.$$

Show that $C_A(x) = x^2 - 5x + 6$, and find the roots of $C_A(x) = 0$. Let λ_1 and λ_2 denote these two roots. Find an eigenvector v_1 belonging to λ_1. Find an eigenvector v_2 belonging to λ_2. Show that v_1 and v_2 are linearly independent. Thus $E = [v_1, v_2]$ is a basis for R^2. Find B, the matrix of the linear transformation L in the basis E. Find an invertible matrix P such that $B = P^{-1}AP$.

(b) Let

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that $C_M(x) = x^2 - (a + d)x + (ad - bc)$.

(c) Give an example of a 2×2 matrix that has no real eigenvalues.

(d) Give an example of a 2×2 matrix whose characteristic polynomial is $(\lambda - 5)^2$, and such that the space of eigenvectors with eigenvalue 5 is one-dimensional. Give a second example with the same characteristic polynomial, and such that the space of eigenvectors with eigenvalue 5 is two dimensional. Hint: Consider the properties of the following two matrices.

$$M = \begin{bmatrix} 7 & 1 \\ 0 & 7 \end{bmatrix},$$

and

$$M = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}.$$

6. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

(a) Compute the characteristic polynomial $C_A(x)$.

(b) Find the eigenvalues of A and determine an eigenvector for each eigenvalue.

(c) Use the eigenvectors in part (b) to form a basis for R^3 with respect to which the linear transformation corresponding to A is diagonal. Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

7. In this problem we apply eigenvectors and eigenvalues to the solution of a system of differential equations. Suppose you are asked to solve

$$y_1' = ay_1 + by_2$$

$$y_2' = cy_1 + dy_2$$

where y_1 and y_2 are differentiable functions of a variable t and y' denotes dy/dt. Then try solutions in the form

$$y_1 = x_1 e^{\lambda t},$$

$$y_2 = x_2 e^{\lambda t},$$

where x_1 and x_2 are constant real numbers and λ is also a constant real number. Note that

$$y_1' = \lambda x_1 e^{\lambda t},$$
$$y_2' = \lambda x_2 e^{\lambda t}.$$

Thus we are attempting to solve

$$\lambda x_1 e^{\lambda t} = a x_1 e^{\lambda t} + b x_2 e^{\lambda t},$$

$$\lambda x_2 e^{\lambda t} = c x_1 e^{\lambda t} + d x_2 e^{\lambda t}.$$

But since $e^{\lambda t} \neq 0$, this is the same as trying to solve

$$\lambda x_1 = a x_1 + b x_2,$$

$$\lambda x_2 = c x_1 + d x_2.$$

With $v = (x_1, x_2)^T$, this is the eigenvalue problem for the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

That is, we are looking for the eigenvalues of this matrix and for eigenvectors that belong to them.

Find solutions to the differential system

$$y_1' = y_1 + y_2$$

$$y_2' = -2y_1 + 4y_2.$$

by using the method described above. You should find two different fundamental solutions to the system corresponding to two distinct eigenvalues of the matrix

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$