Wang Algebra and the Spanning Trees of a Graph

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In these notes we will look at graphs. A *graph* is a collection of *nodes* or *vertices*, usually depicted as dark spots or points, and a collection of *edges* that can connect two nodes or connect a node with itself. For example, the graph below has four nodes and five edges. It is a c*onnected* graph in the sense that there is a pathway along the edges between any two nodes.

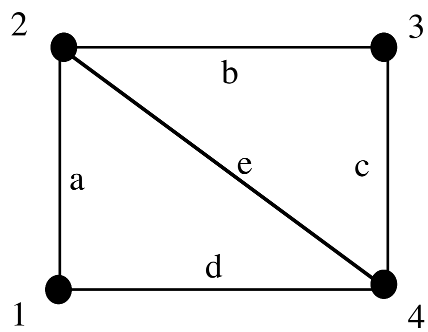


Figure 1 - A connected graph G.

Graphs are fundamental mathematical structures and they have lots of applications. We are all familiar with the graphical notation for electrical circuits. Subway system maps are graphs with special decorations. In general, when we want to desribe engineering systems, economic systems, and other systems of relationship, we can start with a collection of definite entities (the nodes) and the information about how they are connected with one another (the edges).

A *cycle* in a graph is a collection of nodes and edges that have a circular form as in the graph above where a,b,c,d form a cycle. Note that a is an edge from 1 to 2, b is an edge from 2 to 3, c is an edge from 3 to 4, d is an edge from 4 to 1. The collection of nodes and edges closes on itself in this way in a cycle. Exercise: Find all the cycles in the graph in Figure 1.

In these notes we will explore the spanning trees in a graph G.
A *tree* is a graph that is conncected and has no cycles. A *spanning tree* of G is a tree inside G that uses all the nodes of G. For example, the tree below (with edges a, b,c) is a spanning tree of the graph G of Figure 1.

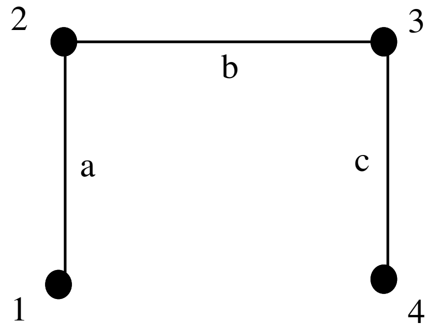


Figure 2 - A spanning tree for G of Figure 1.

We will use an algebraic system devised by Wang (See [1,2]). In this system there are generatiing symbols a,b,c, ... that will correspond to edges of a graph. We will write commutative products of these symbols. For example we will write, for {a,b,c}: a,b,c,ab,ac,bc, abc. These correspond to the follwing graphs related to the triangle graph T:

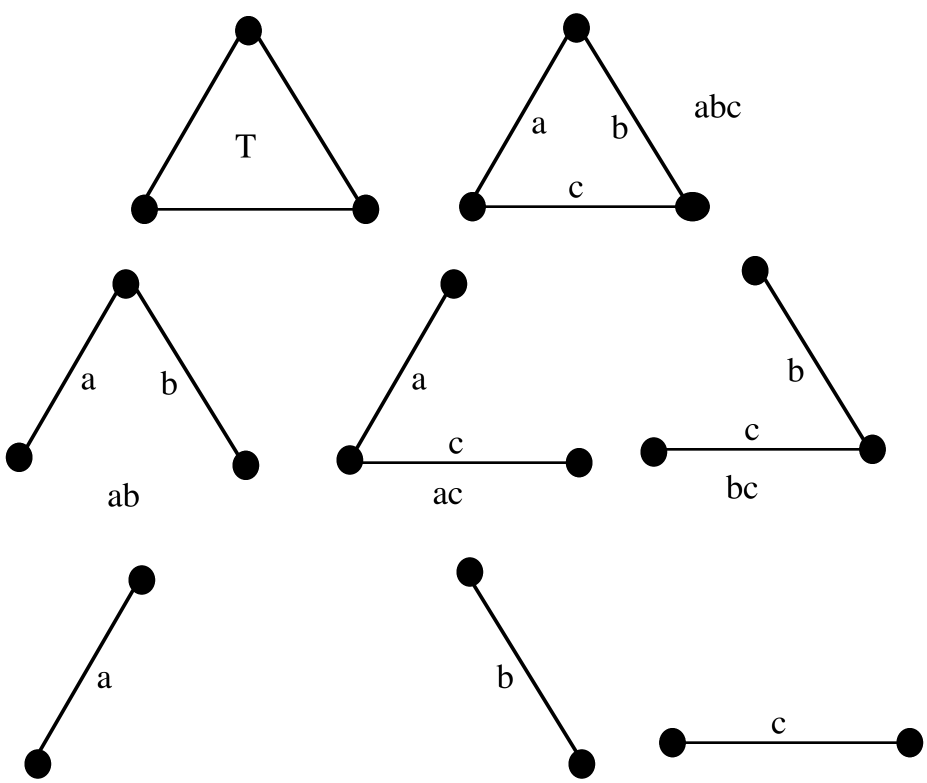


Figure 3 - Triangle graph T and its connected subgraphs.

As you can see from this figure, abc corresponds to the whole triangle graph T, while ab,ac and bc correspond to the spanning subtrees of T, and a,b,c correspond to the individual edges of T.

In the Wang algebra,
xy = yx,
x+ x = 0
and
xx = 0
for any x and y.

Thus we have (a +b)(a + c) = aa + ac + ba + bc = ac + ab + ac in the Wang algebra.

With these rules for handling the algebra, Wang tells us how to find the spanning trees:

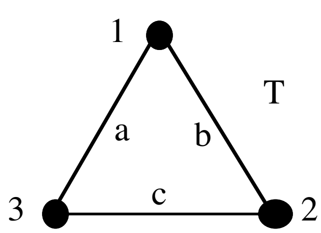
The Wang Rules for Finding all Spanning Trees of a Graph G
1. For each node write the sum of all the edge-labels that meet that node.

2. Leave out one node and take the product of the sums of labels for all the remaining nodes.

3. Expand the product in 2. using the Wang algebra.

4. The terms in the sum of the expansion obtained in 3. are in 1-1 correspondence with the spanning trees in the graph.

For example, in the triangle graph T we have the sums corresponding to nodes as shown in the Figure below.



1: a + b
2: b + c
3: a + c

Thus using just nodes 2 and 3, we get the product
(b+c)(a+c) = ba + bc + ca + cc = ab + bc + ac.
(Remember that cc = 0 in the Wang algebra!)
Certainly ab, bc and ac correspond to the spanning trees of T.

Exercises:

1. Check that the products obtained by using nodes 1 and 2 and by using nodes 1 and 3 also give the spanning trees of T.

2. Use the Wang method to find the spanning trees of the graph G of Figure 1. Draw pictures of all the trees that you get from this graph.

References

1. R. J. Duffin. **An Analysis of the Wang Algebra of Networks** *Transactions of the American Mathematical Society*, Vol. 93, No. 1 (Oct., 1959), pp. 114-131.

2. R. J. Duffin. **Some problems of mathematics and science.** Bulletin of the American Mathematical Society, Vol. 80, No. 6, November 1974, pp. 1053- 1069.