Analysis Fact Sheet

Completeness of Real Numbers

The following are equivalent statements of the *completeness* of the real numbers.

- Every decimal expansion represents a real number x:

$$x = \pm N.d_1 d_2 \ldots ,$$
$$d_k \in \{0, 1, ..., 9\} .$$

This is the statement that every infinite series of the form

$$d_1 10^{-1} + d_2 10^{-2} + \ldots , \quad d_k \in \{0, 1, ..., 9\} ,$$

converges.

- Every binary expansion represents a real number x:

$$x = \pm N.b_1 b_2 \ldots ,$$
$$b_k \in \{0, 1\} .$$

This is the statement that every infinite series of the form

$$b_1 2^{-1} + b_2 2^{-2} + \ldots , \quad b_k \in \{0, 1\} ,$$

converges.

- **Bolzano–Weierstraß Property:** Every bounded sequence $\{x_k\}$ has an accumulation point: there is a real number x such that every neighborhood of x contains x_k for infinitely many k.

Informally we shall say that infinitely many terms of the sequence are *close* to x. An *accumulation point* of the sequence is also called a *limit point* of the sequence.

- Every bounded sequence has a convergent subsequence.

- Bounded increasing sequences have limits.

- Every nonempty set of real numbers which is bounded above has a *least upper bound.*

- Every Cauchy sequence has a limit.

  In the language of Knopp: A sequence $\{x_k\}$ is a Cauchy sequence if *almost all the terms are close together* in the precise sense:

  > For every $\epsilon > 0$, there is a a finite set F_ϵ such that if $j, k \notin F_\epsilon$, then $|x_j - x_k| < \epsilon$.

Useful Theorems about Continuous Functions

The following results about continuous functions are actually equivalent to any of the above characterizations of the completeness of real numbers. Fairly straightforward proofs can be done by using the Bolzano–Weierstraß property and a proof by contradiction.

- If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ is bounded on $[a, b]$.

- If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ is uniformly continuous on $[a, b]$.

  Uniformly continuity on $[a, b]$ means that there a function $\phi(\Delta x)$ which is $o(1)$ as $\Delta x \to 0$ such that

$$|f(x + \Delta x) - f(x)| \leq \phi(\Delta x)$$

  for all $x, x + \Delta x \in [a, b]$.

- If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ assumes its maximum value at some point in $[a, b]$.

Completeness of Complex Numbers

The above statements have consequences for complex numbers:

- The real and imaginary parts of complex numbers have decimal [binary] expansions.

- **Bolzano–Weierstraß Property:** Every bounded sequence $\{z_k\}$ of complex numbers has an accumulation point: there is a complex number z such that every neighborhood of z contains z_k for infinitely many k.

  Informally we shall say that infinitely many terms of the sequence are *close* to z. An *accumulation point* of the sequence is also called a *limit point* of the sequence.

- Every bounded sequence of complex numbers has a convergent subsequence.

- Every Cauchy sequence of complex numbers has a limit.

Useful Theorems about Continuous Functions of a Complex Variable

The statements for complex numbers require the concept of *closed* set.

> A set A of complex numbers is *closed* if it contains all its accumulation points[1]: if $\{z_k\}$ is a sequence of points in A, and $\lim_{k\to\infty} z_k = z$, then $z \in A$.

Fairly straightforward proofs of the following can be done by using a proof by contradiction and the Bolzano–Weierstraß property.

- If $f(z)$ is continuous on a closed and bounded set[2] A, then $f(z)$ is bounded on A.

- If $f(z)$ is continuous on a closed and bounded set A, then $f(z)$ is uniformly continuous on A.

Uniformly continuity on A means that there a function $\phi(\Delta z)$ which is $o(1)$ as $\Delta z \to 0$ such that

$$|f(z + \Delta z) - f(z)| \leq \phi(\Delta z)$$

for all $z, z + \Delta z \in A$.

- If $f(z)$ is continuous on a closed and bounded set A, then $|f(z)|$ assumes its maximum value at some point in A.

[1] A peculiar viewpoint of closed would be: A set A of complex numbers is closed if it has the Bolzano–Weierstraß property when considered as an entity in itself.

[2] A closed and bounded set is a *compact* set