Logarithms and Powers

arguments and Arguments

If $z \neq 0$, z has the polar coordinate representation

$$z = re^{i\theta},$$
$$r = |z| > 0,$$
$$\theta = \arg(z).$$

The angle $\theta = \arg(z)$ is determined only modulo 2π.

If for every $z \neq 0$ we make a particular choice $\mathrm{Arg}(z)$, then $\mathrm{Arg}(z)$ is called a *Principle Argument* function. For such a Principal Argument function, $\mathrm{Arg}(z)$ is continuous at $z = z_0$ iff for all z near z_0,

$$|\mathrm{Arg}(z) - \mathrm{Arg}(z_0)| < \pi.$$

Common choices for $\mathrm{Arg}(z)$ are

$$\begin{aligned}\mathrm{Arg}(z) &= \mathrm{Arg}_{[0,2\pi)}(z) \\ &= \arg(z), 0 \leq \arg(z) < 2\pi, \\ \mathrm{Arg}(z) &= \mathrm{Arg}_{(-\pi,pi]}(z) \\ &= \arg(z), -\pi < \arg(z) \leq \pi.\end{aligned}$$

The first choice gives a Principal Argument Function which is continuous everywhere *except along the positive x–axis*. The second choice gives a Principal Argument Function which is continuous everywhere *except along the negative x–axis*.

logarithms and Logarithms

Definition. *If z is a nonzero complex number then* a logarithm *of z is any complex number w such that*

$$\exp(w) = z.$$

Any logarithm *is of the form*

$$\log(z) = \ln|z| + i\arg(z).$$

If $\mathrm{Arg}(z)$ *is a Principal Argument Function, the function*

$$Log(z) = \ln|z| + i\mathrm{Arg}(z)$$

is called a Principal Branch of the Logarithm.

Exercise. Choose

$$\mathrm{Arg}(z) = \mathrm{Arg}_{(-\pi,\pi]}(z) = \arg(z), -\pi < \arg(z) \leq \pi.$$

Show that

$$\mathrm{Log}(z) \equiv \ln|z| + i\mathrm{Arg}(z)$$

is analytic *except along the negative x–axis*.

Hint: For $\Re z > 0$, draw a picture to show that $\mathrm{Arg}(x+iy) = \arctan\left(\frac{y}{x}\right)$ and verify the Cauchy–Riemann equations. Then give similar representations for $\mathrm{Arg}(z)$ in the regions $\Im z > 0$ and $\Im z < 0$.

powers and Powers

If $z \neq 0$ and a is *any complex number*, z^a is any complex number of the form

$$\begin{aligned} z^a &= e^{a\log(z)} \\ &= e^{a\cdot(\ln|z|+i\arg(z))}. \end{aligned}$$

In general z^a has more than one possible value. Choosing a Principle Argument Function $\mathrm{Arg}(z)$ give a Principal Branch of the Power Function, which is analytic at the places where the corresponding Principal Branch Logarithm Function $\mathrm{Log}(z)$ is analytic (and perhaps elsewhere in special cases). The corresponding function $z^a \equiv e^{a\mathrm{Log}(z)}$ is called a *branch* of the power function z^a.

Note that at any point where $\mathrm{Arg}(z)$ is continuous, $\mathrm{Log}(z)$ and the Branch $f(z) \equiv e^{a\mathrm{Log}(z)}$ of the power function z^a are analytic, and

$$\frac{dz^a}{dz} = az^{a-1} \quad \text{(same branches)}.$$

Exercises

1. Show that i^i is real and find the values of all of its branches.
2. For $z \neq 0$, how many values are there for $z^{\frac{1}{2}}$?
3. Show that for any branch, $\lim_{z\to 0} z^{\frac{1}{2}} = 0$.

   If $f(z) = e^{\frac{1}{2}\mathrm{Log}(z)}$, is continuous and analytic at z, then

   $$f'(z) = \frac{1}{2f(z)}.$$

4. Show that it is NOT possible to define $\mathrm{Arg}(z)$ in such a way that

   $$\begin{aligned} f(z) &= z^{\frac{1}{2}} \\ &= e^{\frac{1}{2}\mathrm{Log}(z)}, \\ &= e^{\frac{1}{2}((\ln|z|+i\mathrm{Arg}(z)))} \end{aligned}$$

   is analytic in the region $\{z \mid 0 < |z| < R\}$.

Choices of a Principal Argument Function

If D is a simply connected region not containing $z = 0$, the function $f(\zeta) = \dfrac{1}{\zeta}$ is analytic in D and for any path $C_{z_0 \to z}$ in D from z_0 to z,

$$\int_{z_0}^{z} \frac{1}{\zeta}\, d\zeta \equiv \int_{C_{z_0 \to z}} \frac{1}{\zeta}\, d\zeta,$$

the integral being independent of the particular path chosen. Fixing $z_0 \in D$ and a fixed choice for $\mathrm{Arg}(z_0)$, we can define on D:

$$\begin{aligned}
\mathrm{Log}(z_0) &= \ln|z_0| + i\mathrm{Arg}(z_0),\\
\mathrm{Log}(z) &= \mathrm{Log}\,(z_0) + \int_{z_0}^{z} \frac{1}{\zeta}\, d\zeta\\
&= \ln|z| + i\mathrm{Arg}(z),\\
\mathrm{Arg}(z) &= \mathrm{Arg}(z_0) + \Im \int_{z_0}^{z} \frac{1}{\zeta}\, d\zeta.
\end{aligned}$$

A simply connected region D which does not contain $z = 0$ can be constructed as the complement of a *branch cut* which consists of any simple curve Cwhich has 0 as an initial point and extends to $z = \infty$.