MthT 430 Spivak Problem 1.21

21. Assume that if

$$
\begin{aligned}
|x - x_0| &< \min\left(\frac{\epsilon}{2\left(|y_0| + 1\right)}, 1\right), \\
|y - y_0| &< \frac{\epsilon}{2\left(|x_0| + 1\right)}.
\end{aligned}
$$

then

$$|xy - x_0 y_0| < \epsilon.$$

The *trick* is to write $xy - x_0y_0$ in terms of $x - x_0$ and $y - y_0$. There are several ways to do this, but the one which works best is

$$
\begin{aligned}
xy - x_0y_0 &= x\left(y - y_0 + y_0\right) - x_0y_0 \\
&= x\left(y - y_0\right) + \left(x - x_0\right)y_0 \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}
$$

Then

$$
\begin{aligned}
|\mathrm{I}| &\le |x|\,|y - y_0| \\
&\le \left(|x_0| + |x - x_0|\right)|y - y_0| \\
&\le \left(|x_0| + 1\right)|y - y_0| \\
&< \epsilon/2, \\
|\mathrm{II}| &\le |x - x_0|\,|y_0| \\
&< \epsilon/2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|xy - x_0y_0| &\le |\mathrm{I}| + |\mathrm{II}| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}
$$